Generation of Pseudo-Random Power-Law Noise Sequences by Spectral Shaping

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Abstract: - Since the beginning, simulation has been an essential tool for assessing the performance of synchronization networks and of stand-alone clocks. In this paper, the issue of simulating the clock noise process in the time domain, by generating pseudo-random sequences having specified power spectrum, is studied. In particular, the so-called power-law noise is addressed, which is one of most adopted models used to characterize clock noise, for example in simulating the dynamics of SDH/SONET pointer adjustments or the slip occurrence in digital switching exchanges. Therefore, an effective algorithm to this purpose is provided. Moreover, some results of simulations of all kinds of power-law noise are provided. The algorithm described is general and may be applied also in other contexts, to simulate noise of any kind.

Key-Words: - digital communication, jitter, SDH, simulation, SONET, synchronization, wander.

1 Introduction

Designing complex systems of clocks as synchronization networks [1]—[4] is not an obvious task. In order to describe the behavior of clocks in synchronization networks and to accurately specify their characteristics, first it has been necessary to identify a proper mathematical model of the clock and of the timing signals generated and distributed. Since the beginning, simulation has been an essential tool for assessing the performance of synchronization networks and of stand-alone clocks.

There is a huge literature on time and frequency stability characterization for precision oscillators [5]—[20]. Historically, a dichotomy between the characterization of oscillators in the *Fourier-frequency domain* and in the *time domain* was established. Examples of stability measures in the frequency domain are the *Power Spectral Densities* (PSDs, or simply spectra) of the phase, time and frequency fluctuations, since they are functions of the Fourier frequency *f*. On the other hand, variances of the same fluctuations, averaged over a given observation interval, are examples of stability measures in the time domain, since they are functions of the observation interval t (time).

In this paper, the issue of simulating the clock noise process in the time domain, by generating pseudo-random sequences having specified PSD, is studied. In particular, the so-called power-law noise is addressed, which is one of most adopted models used to characterize clock noise.

Being able to generate pseudo-random sequences having specified PSD is essential, for example, when aiming at simulating the dynamics of SDH/SONET pointer adjustments [21], the slip occurrence in digital switching exchanges and, more in general, the performance of clock chains in synchronization networks.

In Sec. II, fundamentals on clock noise characterization are given, the basic quantities are defined and the power-law model is introduced. Then, in Sec. III, the issue of generating pseudo-random sequences having specified power spectrum is studied, providing an effective algorithm to this purpose. Finally, in Sec. IV, some results of simulations of all kinds of power-law noise are provided.

2 Clock Noise Characterization: the Power-Law Model

In telecommunications, a *clock* is a device able to supply a timing signal, ideally periodic, usable for the control of telecommunication systems. A mathematical model describing an actual timing signal $s(t)$ is given by [5][6]

$$s(t) = A \sin \Phi(t) \qquad (1)$$

where A is a constant amplitude coefficient and $\Phi(t)$ is the *total instantaneous phase* expressed by

$$\Phi(t) = 2p\,(n_{\mathrm{nom}} + \Delta n)t + pDn_{\mathrm{nom}}t^2 + j\,(t) + \Phi_0 \qquad (2)$$

where Δn represents the *frequency offset* of the actual clock from the *nominal frequency* n_{nom}, D is the *linear fractional frequency drift* rate, mainly describing oscillator ageing effects, $j(t)$ is the *random phase deviation,* modelling oscillator intrinsic phase noise sources, and Φ_0 is the initial phase offset.

The generated *Time* function $\mathrm{T}(t)$ of a clock is defined, in terms of its total instantaneous phase, as

$$\mathrm{T}(t) = \frac{\Phi(t)}{2pn_{\mathrm{nom}}} \tag{3}$$

It is worthwhile noticing that for an ideal clock $\mathrm{T}_{\mathrm{id}}(t)=t$ holds, as expected. Also the random phase deviation $j(t)$ is often expressed in terms of time, as

$$x(t) = \frac{j(t)}{2pn_{\mathrm{nom}}} \tag{4}$$

Moreover, for a given clock, the *Time Error* function TE(t) between its time T(t) and a reference time $\mathrm{T}_{\mathrm{ref}}(t)$ is defined as

$$\mathrm{TE}(t) = \mathrm{T}(t)\text{-}\mathrm{T}_{\mathrm{ref}}(t) \tag{5}$$

For a clock slaved to the reference timing signal, $x(t)$=TE(t) holds.

In the frequency domain, the model most frequently used to represent the clock output phase noise is the so-called *power-law model* [6]. In terms of the one-sided Power Spectral Density (PSD) of $x(t)$ such model is expressed by

$$S_x(f) = \begin{cases} \dfrac{1}{(2p)^2} \sum_{a=-4}^{0} h_{a+2} f^{a} & 0 \le f \le f_h \\ 0 & f > f_h \end{cases} \tag{6}$$

where the h_{-2}, h_{-1}, h_0, h_{+1} and h_{+2} coefficients are device-dependent parameters and f_{h} is an upper cut-off frequency, mainly depending on low-pass filtering in the oscillator and in its output buffer amplifier. This clock upper cut-off frequency is usually in the range 10-100 kHz in precision frequency sources.

The five types of the model (6) are: *White Phase Modulation* (WPM) for a=0, *Flicker Phase Modulation* (FPM) for a=-1, *White Frequency Modulation* (WFM) for a=-2, *Flicker Frequency Modulation* (FFM) for a=-3 and *Random Walk Frequency Modulation* (RWFM) for a=-4.

3 Generation of Pseudo-Random Sequences by Spectral Shaping

In this section, the issue of generating pseudo-random sequences having specified PSD is studied. This is the main issue, in order to simulate clock phase instabilities according to some suitable model such as the power law (6) for the phase noise spectrum. Therefore, the task of the procedure outlined in this section is to generate a pseudo-random sequence $\{x_i\}$ of N samples with custom power spectrum $S_x(f)$. Obviously, samples may represent TE or frequency samples.

The generation algorithm is outlined in Fig. 1. First, two independent sequences $\{a_i\}$ and $\{b_i\}$ of N random numbers uniformly distributed and with negligible correlation between them (i.e., with white spectrum) are generated (*white uniform deviates*).

Then, one sequence $\{c_i\}$ of N white-spectrum numbers with Gaussian distribution (*white Gaussian deviates*) is computed through the following transformation (Box-Muller method [22][23])

$$\begin{cases} y_{i1} = \sqrt{-2\ln a_i}\cos 2pb_i \\ y_{i2} = \sqrt{-2\ln a_i}\sin 2pb_i \end{cases} \tag{7}$$

by choosing any of the two sequences $\{y_{i1}\}$ and $\{y_{i2}\}$ as $\{c_i\}$.

Spectral shaping of the white-spectrum sequence $\{c_i\}$ is then accomplished through Fast Fourier Transform (FFT), by filtering with suitable transfer function $\mathrm{H}(f_n)$ so that

$$\left|\mathrm{H}(f_n)\right|^2 = \frac{\breve{S}_x(f_n)}{K} \tag{8}$$

where $\breve{S}_x(f_n)$ denotes the two-sided PSD and K is a normalization factor, to finally yield the sequence $\{x_i\}$.

In this case, the burden of convolution algorithms such as overlap-and-add and zero-padding methods [22] is unnecessary, since the only constraint is to get a sequence just having the custom power spectrum $S_x(f)$. The resulting procedure can be thus straightforward indeed: the $\{c_i\}$ data set is just crammed into computer memory, FFTed, multiplied sample-by-sample by $\{\mathrm{H}_n\}$ and inversely FFTed back to yield $\{x_i\}$.

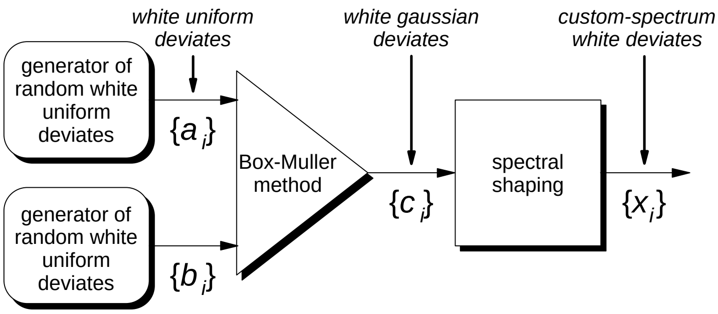

Fig. 1: Generation of the pseudo-random sequence of TE samples $\{x_i\}$.

4 Simulation of Power-Law Noise

This section shows some results of simulations of all five kinds of power-law noise (6), obtained by applying the generation algorithm outlined in Fig. 1.

First, in order to simulate WPM (a=0) noise, two white and uniformly distributed pseudo-random sequences of length $N=2^{18}=262144$ were generated.

Then, applying the transformation formula (7), one white Gaussian pseudo-random sequence of the same length was obtained, thus simulating Gaussian WPM noise.

Spectral shaping was then accomplished by filtering in the Fourier domain the WPM (a=0) noise sequence through integrators of fractional order $-a/2$ [24], having transfer function $\mathrm{H}_{-a/2}(f)=K(j2\pi f)^{a/2}$, to generate the FPM ($a$=-1), WFM ($a$=-2), FFM ($a$=-3) and RWFM ($a$=-4) noise sequences according to the power-law model (6).

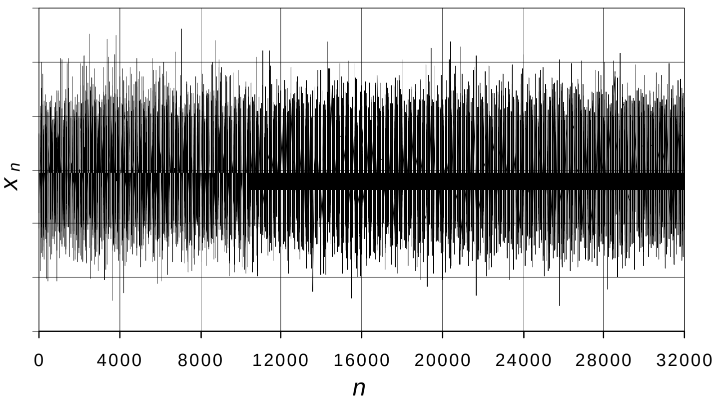

Fig. 2:
First eighth of the generated WPM noise sequence (a=0).

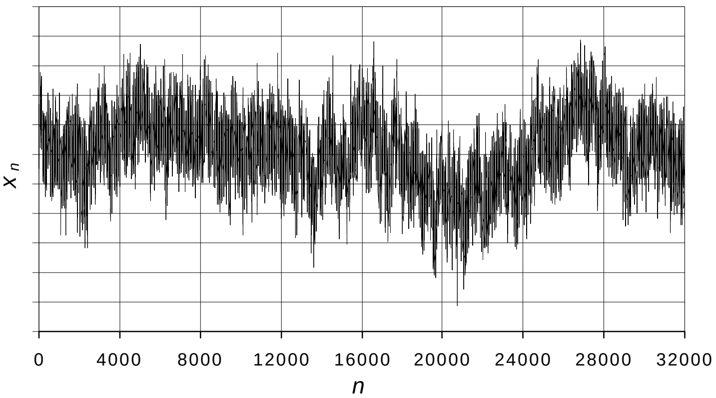

Fig. 3:
First eighth of the generated FPM noise sequence (a=-1).

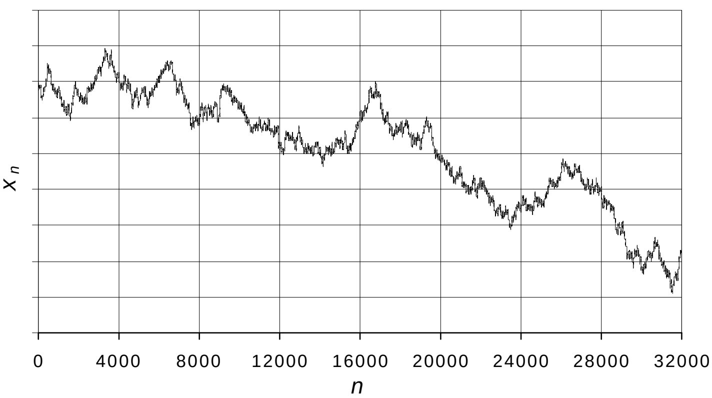

Fig. 4:
First eighth of the generated WFM noise sequence (a=-2).

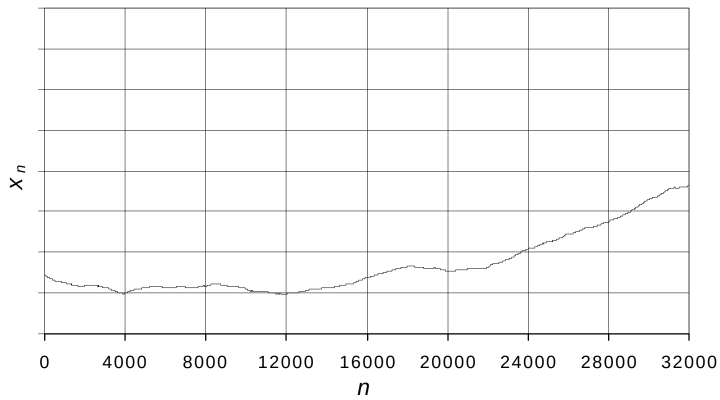

Fig. 5:
First eighth of the generated FFM noise sequence (a=-3).

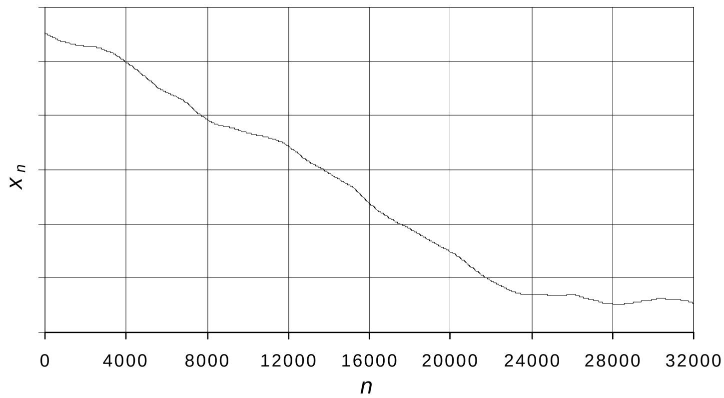

Fig. 6:
First eighth of the generated RWFM noise sequence (a=-4).

For ease of graphical representation, it was decided to plot only the first 32000 samples of each sequence, since most common spreadsheet software packages do not allow the graphical plotting of longer data sequences. In conclusion, Figs. 2 through 6 show the first segment (i.e., one eighth of the overall length) of the resulting five TE sequences $\{x_n\}$, respectively affected by WPM, FPM, WFM, FFM and RWFM noise.

By visual inspection of Figs. 2 through 6, it is quite interesting to point out that the realizations of the process TE(t) are as smoother as the parameter a in (6) is decreased. In fact, any f^{α} noise can be viewed as a $f^{\alpha+1}$ noise filtered through the half-order integrator with transfer function $\mathrm{H}_{1/2}(f) = 1/\sqrt{j2pf}$. Therefore, Figs. from 2 to 6 show indeed what happens by repeatedly integrating a white noise.

5 Conclusions

In this paper, the issue of simulating the clock noise process in the time domain, by generating pseudo-random sequences having specified power spectrum, was studied. In particular, the so-called power-law noise was addressed. An effective algorithm to this purpose was provided. Some results of simulations of all kinds of power-law noise were also provided.

The algorithm described is general and may be applied in several contexts, to simulate noise of any kind. Applied to simulate clock noise, for example, it has been used by the author of this paper to simulate the dynamics of SDH/SONET pointer adjustments [21] or the slip occurrence in digital switching exchanges. The performance of synchronization networks can be also assessed by simulation.

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