Shape and orientation of revolution surfaces from contours and reflections

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Abstract

The reconstruction of shape and pose of a solid of revolution from a single image is addressed. When there is no cross section, whose contour can be extracted from the image, this problem is underdeterminate; therefore, a reflection from a point light source is used in addition to the contour information. Under the orthoerpective hypothesis, when axial-symmetric reflection model is applicable, the reflection appears along a meridian of the revolution surface. This fact is exploited in order to determine both the orientation of the revolution axis and the shape of the profile. Promising experimental results have been obtained. © 2002 Pattern Recognition Society. Published by Elsevier Science Ltd. All rights reserved.

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1. Introduction

The 3D interpretation from 2D data is one of the main source of problems in Computer Vision. This paper addresses the recovery of 3D shape and orientation of an observed object from a single view. The considered object class is that of revolution surfaces. If the contour of a (top or base) circular cross section is not detected from the image, the problem is underdeterminate. To cope with this underdeterminacy, the reflection of a point light source is used.

The analysis of generalized cylinders, and in particular of solids of revolutions, has already been addressed in the literature. In Ref. [1] it was shown that the orthogonal projection of the contour of a revolution surface is symmetric with respect to the projection of the axis of revolution. In Ref. [2], the perspective projection of an axis of revolution is determined given the image of its external contour. Approaches to the model-based localization of generalized cylinders and, respectively, general curved objects are presented in Refs. [3,4]. In Ref. [5], the projection of the axis of a straight homogeneous generalized cylinder (SHGC) was determined; in that work, it was recognized that the determination of the orientation of the SHGC is an underdeterminate problem. In Ref. [6], the shape of a SHGC from a single image was recovered. This method requires that a cross section is entirely visible. In real images, however, the cross section is often occluded or it can hardly be extracted by edge detection. One of the first works, reporting the use of an ad hoc illumination system for recovering the shape of a curved surface from a single view, is due to Sanderson et al. [7]. In Ref. [8], the 3D shape of a SHGC is recovered—without the need to extract the image of a cross...
2. Problem formulation

2.1. Surfaces of revolution

A surface of revolution is obtained by rotating a planar curve, called profile, about an axis, called revolution axis. This surface can be regarded as a generalized cylinder in which (i) the spine is rectilinear, (ii) the cross section is circular, and (iii) the cross section is orthogonal to the revolution axis.

A revolution surface can be represented by writing the radius of the cross section as a function of the abscissa along the revolution axis: \[ X^2 + Y^2 = R^2 \quad \text{with} \quad R = R(Z). \]

The profile of the revolution surface is represented by the function \( R(Z) \).

The observation of a (revolution) object from a single viewpoint \( O \) only allows to determine the shape of the profile, i.e., the profile modulo a scale factor \( d \), given by the distance between the viewpoint and the revolution axis. Therefore, the shape of the profile is represented by the function \( R/d(Z/d) \).

2.2. The axial-symmetric reflection model

The axial-symmetric reflection model has been introduced in Ref. [9] to account for the reflecting behavior observed in metallic revolution solids, subject to certain surface treatments such as, e.g., lathe-finishing. These treatments introduce a surface raggedness, whose typical size \( s \) can be of the order of \( 10^{-2} \) or \( 10^{-1} \) mm. Often, the ragged surface maintains the axial symmetry of the ideal surface. As a consequence (i) the surface normal at any point goes through the symmetry axis and (ii) the direction of the surface normal rapidly varies along a meridian line (a meridian line is the intersection between the revolution surface and a plane through the revolution axis).

Let us consider a slice of the surface, obtained by cutting it along two parallel planes, both perpendicular to the revolution axis, separated by a distance \( s \). In this way, the thickness of the slice is given by the typical raggedness size \( s \). Now, consider a meridian line through a point \( B' \), and consider the portion of this meridian line within a distance \( s \) from \( B \); due to (i) and (ii), the set of surface normals at points within this portion is a significant subset of the set of directions pointing from \( B' \) towards the revolution axis. The set of the surface normal directions at the points of the slice is a significant part of the 2D set of all the possible directions.

As a consequence, given a viewpoint \( O \) and a point light source \( L \)—not occluded by the object—there will be a point \( B' \) on the slice, where the surface normal bisects the angle formed by the segments \( OB' \) and \( LB' \). This point is a reflection point from \( L \) to \( O \).

Since this is true for (almost) any slice of the surface, the locus of the reflection points from \( L \) to \( O \) will be a discrete set of surface points. The distance between two neighboring reflection points will be of the order of the thickness \( s \). At the typical viewing distances, this distance is lower than the camera resolution; therefore, the locus of the reflection points from \( L \) to \( O \) appears as a continuous line in the image. This line on the image plane is called reflection. Notice that its appearance is very different from what would happen with a purely specular surface, where the reflection would appear as a (small set of) point(s).
2.3. The addressed problem

Given an image of a revolution surface, whose light reflecting behavior is in accordance with the axial-symmetric model of Section 2.2, and a point light source \( L \), the addressed problem is to determine the orientation of the revolution axis and a representation of the shape of the surface profile. The used data consist in contours extracted from a single image; they include the external contour relative to the limbs of the surface, and the image of the reflection line. In the determination of the orientation and the shape of the profile, there is no need to know the position of the point light source \( L \); if the position of \( L \) is known, then one can also determine the distance \( d \) between the revolution axis and \( O \) and the absolute size of the profile \( R(Z) \).

Notice that the present approach (i) does not use intensity data, which are used in Ref. [8], and (ii) does not need the extraction of a cross-section contour, as is done in Ref. [6].

For simplicity, we assume the following hypothesis, called orthoperspective hypothesis: the revolution surface \( \mathcal{S} \) is far enough both from the light source \( L \) and from the viewpoint \( O \), so that (i) the light rays incident from \( L \) onto the different points of \( \mathcal{S} \) are approximately parallel and (ii) the light rays reflected from the different points of \( \mathcal{S} \) to \( O \) are approximately parallel.

3. Contours of solids of revolution: properties

A single perspective image of the occluding contour of a revolution surface is not sufficient to determine the profile of this surface (as can be seen by considering a cone). Therefore, in order to extract information about the shape and the orientation of the revolution surface, a reflection is used originated by a point light source.

Two points \( P_1 \) and \( P_2 \) on the image contour are said to be paired if they are image of two points on the revolution surface, which belong to the same circular cross section.

The following lemma, proved in Ref. [5], is useful for finding the projection of the revolution axis:

**Lemma 1** (Ponce et al. [5]). The two straight lines, tangent to the contour at two paired points \( P_1 \) and \( P_2 \), intersect on the projection of the revolution axis.

In the sequel, some basic properties are presented relative to reflections of point light sources, according to the axial-symmetric reflection model, under the orthoperspective hypothesis.

**Lemma 2.** If a revolution surface \( \mathcal{S} \) is compatible with the axial-symmetric reflection model, then the locus of the reflecting points from \( L \) to \( O \) on \( \mathcal{S} \) is a contained in a meridian of the surface.

**Proof.** The reflection normal at each point \( B' \) on \( \mathcal{S} \) must bisect the angle between the directions \( \overline{LB'} \) and \( \overline{OB'} \). From the orthoperspective hypothesis, all points \( B' \in \mathcal{S} \) identify a common direction \( \overline{OB'} \) and a common direction \( \overline{LB'} \); let \( n \) be the direction bisecting the above two directions. Any surface point, for which there exists a reflecting normal parallel to \( n \), is a reflection point from \( L \) to \( O \) (i.e., it belongs to the reflecting curve from \( L \) to \( O \)).

Now, according to the axial-symmetric reflection model, the set of reflection normals at any surface point \( B' \) is the set of directions pointing from \( B' \) towards the revolution axis. Since the condition for a generic point \( B' \in \mathcal{S} \) to be a reflection point is that \( n \) is one of the reflection normals, then any reflection point must lie on the plane containing the revolution axis and the direction \( n \). Thus, any reflection point from \( L \) to \( O \), must be contained in the above plane; since it also must belong to the surface \( \mathcal{S} \), it must belong to the a meridian line of the revolution surface.

An image point \( B \) on the reflection is said to be associated to a contour point \( P \) if \( B \) is the image of a surface point \( B' \) which belongs to the same cross section of the surface point \( P' \) projecting on \( P \).

In view of the lemmas, the image point \( B \) on the reflection, which is associated to two paired points \( P_1 \) and \( P_2 \), satisfies the following property (under orthoperspective assumption and axial-symmetric reflection model).

**Theorem 1.** Let \( P_1 \) and \( P_2 \) be two paired contour points, and let the tangent lines to the contour at \( P_1 \) and \( P_2 \) intersect at the image point \( A \); the line tangent to the reflection at a point \( B \) associated to \( P_1 \) and \( P_2 \) intersects the projection of the symmetry axis at \( A \).

**Proof.** Let the points \( P_1 \) and \( P_2 \) be two paired points, and let \( P'_1 \) and \( P'_2 \) be the two points on the revolution surface, having \( P_1 \) and, respectively, \( P_2 \) as image. The image point \( B \) associated to the paired points \( P_1 \) and \( P_2 \) is the image of a surface point \( B' \) on the same circular cross section containing the points \( P'_1 \) and \( P'_2 \). Therefore, the tangent line to the meridian through \( B' \) at \( B' \) intersects the axis at the same point \( A' \) where the tangent lines to the meridians at \( P'_1 \) and \( P'_2 \) intersect. Since the reflection through \( B \) is the projection of the meridian through \( B' \) from Lemma 1, then the thesis follows since (orthop-)perspective projections preserve incidence relations.

This theorem provides a necessary condition for a point on a reflection to be associated to two paired points \( P_1 \) and \( P_2 \) on the contour. However, since this condition is not sufficient, a cumulative approach for the determination of the orientation of the revolution axis is proposed in the next section.
4. The orientation of the revolution axis

In Ref. [2], Glachet et al. present a cumulative method for finding the perspective projection of a revolution axis from its outline. This method is based on the fact that the plane bisecting the interpretation planes of the image lines tangent to the contour at the paired points \( P' \) and \( P'' \) coincides with the interpretation plane of the projection of the revolution axis.

The first step towards the determination of the revolution axis orientation is the application of the cumulative method [2] to determine the perspective projection of the revolution axis.

Observe that, once the projection of the revolution axis has been determined, the orientation of the revolution axis is characterized by one residual degree of freedom representing the angle, which the revolution axis forms with the direction of its projection onto the image plane.

Once the projection of the revolution axis is known, the contours and the reflection are re-projected onto a transformed image plane, which is perpendicular to the plane containing the viewpoint \( O \) and the (projection of the) revolution axis. Any of the infinite planes perpendicular to the plane through \( O \) and the revolution axis would be appropriate; for simplicity, one could select, among these planes, the one containing the projection of the revolution axis onto the previous image plane.

Notice that, within this transformed image plane, the contour is symmetric with respect to the projection of the revolution axis. Therefore, the pairs of paired points can be easily determined by finding, for any point \( P' \) on the image contour, the contour point \( P'' \) symmetrical to \( P' \) with respect to the projection of the revolution axis.

Given a pair \((P', P'')\) of paired points, the lines tangent to the contour at \( P' \) and \( P'' \) intersect at a point \( A \) on the projection of the revolution axis. It would be desirable to find the point \( B \), associated to \( P' \) and \( P'' \), on the reflection; from Theorem 1, the tangent to the reflection at \( B \) must go through \( A \). However, this condition may be satisfied by more than one reflection point. Therefore, for each pair \((P', P'')\) of paired points on the contour, a set of candidate points on the reflection can be constructed, that satisfy the tangency condition of Theorem 1.

A set of candidate triples \((P', P'', B)\) are generated, where \( P' \) and \( P'' \) are paired contour points, while \( B \) is a point on the reflection that satisfies the tangency condition of Theorem 1.

Starting from this set of triples, a cumulative method is used to determine the orientation of the revolution axis. Based on two algorithms described in the appendices, each triple of the set is transformed into one candidate inclination angle for the cross section; recall that any cross section is orthogonal to the revolution axis. Therefore, a voting approach, based on a 1D Hough transform, is applied to determine a sufficiently voted value of the inclination angle. Starting from the inclination angle of the cross section, the orientation angle of the revolution axis is directly obtained by adding \( \pi/2 \) to the inclination angle.

Specifically, the method consists of the following steps:

1. Given the contour and the reflection, a set of contour points is selected, and for each of them the paired contour point is determined, based on the symmetry with respect to the projection of the revolution axis.
2. For each pair of paired points \((P'_i, P''_i)\)
   (a) the point \( A_i \) is determined, where the tangents to the contour at \( P'_i \) and \( P''_i \) intersect; and
   (b) the points \( B_{ij} \) on the reflection are determined, for which a tangent exists, which goes through \( A_i \).
3. The generated triples are collected into a set.
4. The range of the possible orientation angles of the revolution axis is subdivided into a number \( N \) of sub-ranges.
5. For each triple \((P'_i, P''_i, B_{ij})\)
   (a) the algorithm presented in Appendix A is applied in order to construct the ellipse, which (i) goes through \( P'_i \), \( P''_i \), and \( B_{ij} \) and (ii) is tangent to the contour at \( P'_i \) and at \( P''_i \);
   (b) the inclination angle of the circumference, whose projection is the constructed ellipse, is calculated by the algorithm in Appendix B (this provides up to three solutions for the inclination angle), and the corresponding candidate orientation angles of the revolution axis are determined by adding \( \pi/2 \) to the inclination angles of the circumference; and
   (c) a vote is assigned to each of the cells containing the value(s) of the calculated orientation angle.
6. The cell with the maximum score is determined, and the central value of this cell is retained as the orientation angle of the revolution axis.

The selection of the pairs of paired contour points (Step 1) is based on the local curvature of the contour. In fact, given two paired contour points \( P' \) and \( P'' \), a candidate associated point \( B \) on the reflection is determined by imposing that the tangent to the reflection at \( B \) is concurrent with the tangent lines to the contour at \( P' \) and \( P'' \). The lower the local curvature at a contour point \( P' \), the lower the local curvature of the reflection line at the associated reflection point \( B \), and the larger the error in the determination of the position of \( B \). Therefore, the contour points have to be selected where the local curvature of the contour is sufficiently high.

In order to determine the tangent to the contour (Step 2(a)) or to the reflection (Step 2(b)), the median filtered differencing method [10] is applied.

In the appendices, two basic algorithms are reviewed, for the reader’s convenience.

Appendix A presents the construction of an ellipse, given a symmetry axis, an incidence point, and a
tangency point. This algorithm can be used to determine the ellipse through a point \( P \) and tangent to the contour at two paired points \( P' \) and \( P'' \) (Step 5(a)), since these two points, and their tangents, are symmetric with respect to the projection of the revolution axis; therefore, this projection is a symmetry axis of the ellipse. Given this symmetry axis, it is sufficient to add only the incidence point \( B \) and one of the two tangency points (say, \( P' \)) to determine the ellipse.

Appendix B illustrates the determination of the orientation of a circumference, given the ellipse representing its perspective projection. This algorithm is relative to particular symmetry conditions; the orthogonal projection of the viewpoint \( O \) onto the image plane is contained in one of the symmetry axes of the ellipse. This is the case of Step 5(b), since the orthogonal projection of \( O \) onto the (transformed) image plane belongs to the projection of the revolution axis, and this projection is a symmetry axis of the ellipse. In this case, for symmetry, one diameter of the circumference is parallel to the image plane and perpendicular to the projection of the revolution axis. A second diameter, perpendicular to the above one, lies on the plane through the viewpoint \( O \) and the projection of the revolution axis; the orientation of the circumference is characterized by the inclination angle between this diameter and the projection of the revolution axis onto the image plane. This method reduces to a three-degree algebraic equation, which yields up to three real solutions for the inclination angle.

**Observation.** The present method cannot be applied to cones, which are solids of revolution with linear profile. In fact, the problem of determining a point \( B \) on the reflection, associated to two given paired contour points \((P', P'')\) is underdetermined; the contour tangents at any two contour points intersect at the projection of the cone vertex. Under the orthoperspective hypothesis, the tangent to the reflection at any point on the reflection goes through the projection of the vertex; therefore, any point on the reflection is a good candidate for being associated to two given paired contour points \((P', P'')\) (see Step 2(b) of the algorithm).

5. The profile

Once the inclination angle of the revolution axis has been determined, the profile can be constructed based on the outline. If the position of the point light source \( L \) is not known, the profile can be constructed modulo a scale factor; in other words, only the shape of the profile can be determined. The scale factor is proportional to the distance \( d \) between the viewpoint \( O \) and the revolution axis.

In Section 5.2, it is shown how the knowledge of the position of the light source \( L \) can be used to determine the scale factor and, hence, the actual size of the profile and the distance between its revolution axis and \( O \).

In Section 4, a transformed image plane was considered, such that the orthogonal projection of the viewpoint \( O \) onto the image plane belongs to the projection of the revolution axis. Now, having determined the orientation of the revolution axis, a new image plane is considered together with a new reference relative to the scene. The new scene reference is centered at \( O \), and its \( Z \)-axis is parallel to the revolution axis. Let the point \( C' \) be the orthogonal projection of \( O \) onto the revolution axis; the \( Y \)-axis of the new reference is parallel to the segment \( OC' \). Let the point \( C \) be the perspective projection of \( C' \) onto the new image plane; the new image plane is defined such that it goes through \( C \) and it is perpendicular to the segment \( OC \). The contour is perspectively projected onto the new image plane. Notice that, within this new image plane, the projection of the revolution axis is still a symmetry axis of the contour and, in addition, the projection of the revolution axis is parallel to the revolution axis. A reference frame is also considered within the new image plane, which is centered on \( C \) and whose \( z \)-axis is given by the projection of the revolution axis.

5.1. The shape of the profile

In general, the line on a revolution surface, whose perspective projection is an image contour, does not coincide with a meridian of the surface.

Given a contour point \( P \), let \( P' \) be the surface point whose image is \( P \); let us indicate by \( Z \) the \( z \) coordinate of \( P' \) and by \( R \) the radius of the circular cross section of the revolution surface at \( P' \).

The construction of the profile is performed by associating, to each contour point \( P \), the ratios \( Z/d \) and \( R/d \), where \( d \) is the distance between the viewpoint \( O \) and the revolution axis. In the sequel, the expressions of \( Z/d \) and \( R/d \) for a given contour point \( P \) are derived.

Within the image, let us indicate by \( z \) the \( z \) coordinate of the contour point \( P \), and let \( r \) be the distance between \( P \) and the \( z \)-axis (recall that the \( z \)-axis is the projection of the revolution axis). Let \( f \) indicate the focal distance of the new image plane, and let \( \alpha \) be the inclination angle, with respect to the axis projection, of the tangent to the contour at \( P \).

Now let us consider the cone tangent to the revolution surface at the cross section through the point \( P' \), and let \( \beta \) be its (unknown) semi-aperture angle. We consider two cross sections of this cone: the first, through \( P' \), at the (unknown) height \( Z \), the second at the height \( 0 \). The projection of this cone has an external contour, which is tangential to the contour at \( P \). This external contour is constituted by two straight lines intersecting at the image \( A \) of the cone apex; the angle formed by any of these two lines and the \( z \)-axis is \( \alpha \). The image point \( Q \) on the
cone contour at height 0 is at a distance \( r + z \tan \alpha \) from the z-axis, by triangle similarity. The image point \( Q \) is also the projection of a point \( Q' \) on the cone. The segments joining \( P' \) and \( Q' \) to the centers of their respective cross sections are parallel; the angle they form with the direction of the X-axis is called \( \theta \). It can be shown (see Fig. 1) that the cone semi-aperture \( \beta \) is given by

\[
\tan \beta = \tan \alpha \cos \theta.
\]

Let \( R' \) be the radius of the cross section of the tangent cone at \( Q' \). From Fig. 1,

\[
\sin \theta = \frac{R'}{d},
\]

but also, from being \( Q \) the perspective projection of \( Q' \):

\[
\tan \theta = \frac{r + z \tan \alpha}{f}.
\]

From \( R/d = R'/d - Z/d \tan \beta \),

\[
\frac{R}{d} = \left( \frac{r + z \tan \alpha}{f} - \frac{Z}{d} \tan \alpha \right) \cos \theta.
\]

The ratio between the x coordinate and the z coordinate of \( P \), is given by

\[
\frac{r}{z} = \frac{R}{Z} \cos \theta = \frac{R/d}{Z/d} \cos \theta
\]

and hence

\[
\frac{R}{d} = \frac{r}{z} \frac{1}{\cos \theta} \frac{Z}{d}.
\]

Equating the two expressions of \( R/d \) results in a linear equation on \( Z/d \), recalling that

\[
\frac{1}{(\cos^2 \theta)} = 1 + \tan^2 \theta = 1 + \frac{(r + z \tan \alpha)^2}{f^2},
\]

the expression of \( Z/d \) is obtained:

\[
\frac{Z}{d} = \frac{fz}{f^2 + r(r + z \tan \alpha)}
\]

and thus

\[
\frac{R}{d} = \frac{r \sqrt{f^2 + (r + z \tan \alpha)^2}}{f^2 + r(r + z \tan \alpha)}.
\]

For each contour point \( P \), the inclination angle \( \alpha \) of the tangent to the contour is determined, together with its image coordinate \( z \) and its distance \( r \) from the z-axis; the last two expressions give the scaled cylindric coordinates \( Z/d \) and \( R/d \) of the cross section at the surface point \( P' \) corresponding to \( P \).

5.2. The size of the profile

Up to now, the orientation of a solid of revolution and the shape of its profile has been determined without using the information about the position of the point light source \( L \). If this information is known, also the distance \( d \) between the viewpoint and the revolution axis can be calculated, and hence the absolute size of the profile can be determined.

First the coordinates of the point light source are expressed relative to the reference introduced at the beginning of Section 5. Then a point \( B' \) on the revolution surface is considered, projecting onto a reflection point \( B \); the segments \( OB \) and \( OB' \) have the same direction. To determine the scale factor, the length of the segment \( OB' \) must be calculated. The orientation of the plane containing the revolution axis and the reflection line can easily be calculated. The reflection normal, which is contained in the above plane, is also contained in the plane through \( O, B, L \). Intersecting these two planes, the direction of this reflection normal is determined. The length of \( OB' \) is calculated by imposing that the direction of the above reflection normal bisects the direction of \( OB' \) and the direction of \( LB' \); in this way the direction of \( LB' \) is determined, and hence the length of \( OB' \) is straightforwardly derived.

6. Experimental results

The described technique has been implemented and tested on a set of 25 real images. The various phases of the technique are articulated as follows. The contours are extracted by a Canny edge detector [11]. Linear-quadratic interpolation [12] is applied to determine the position of the contour points with subpixel accuracy. A reflection is characterized by two-faced contours with nearly opposite gradients; this characterization is used to distinguish occluding contours from reflections. The tangents to contours and reflections are calculated as described in Ref. [10]. The projection of the revolution axis is determined by the method of Ref. [2]. The space orientation of the
revolution axis is determined by the technique described in Section 4, while the shape of the profile is computed as described in Section 5.

The results of the described technique on two real images of the test set are illustrated in Figs. 2 and 3 and in Table 1. Each figure shows the original image, the gradient intensity, the detected edges, the reflection, the external contours, the projection of the revolution axis, the re-projection of the edges onto a plane parallel to the revolution axis (Section 5), and the determined profile shape.

Over the tested images, the errors in the axis orientation are within two degrees. As expected, the error in the profile shape is almost negligible; in fact, it mainly consists of an aspect error amounting to the cosine of the orientation error (which differs from one by second-order terms).

7. Conclusions

In Ref. [9] it was shown that, at least under orthoperspective projection, two reflections (from point light sources) are needed to recover the orientation and the shape of a conic surface. Although the recovery of shape and orientation of a solid of revolution is a generalization of the above problem, it has been shown that only one reflection is needed to solve this problem. This derives from the fact that the conic surface is a degeneration of a revolution surface where, due to the null curvature of the contour, the ellipse constituting the projection of a given cross section cannot be constructed by using only one reflection.

Appendix A. Construction of an ellipse given a symmetry axis, an incidence point, and a tangency point

This appendix addresses the following problem: find the ellipse that is symmetric with respect to a given axis, contains the points B and P, and its tangent at P has a given direction. (In our problem, the symmetry axis is given by the projection of the revolution axis.)

A reference frame is defined on the image plane, centered on the point of the plane nearest to the viewpoint O, and such that the z-axis is along the projection of the revolution axis. With respect to this reference, the equation of the ellipse is

\[ \frac{x^2}{a^2} + \frac{(z - z_E)^2}{b^2} = 1, \]
Fig. 3. (a) Original image, (b) gradient intensity, (c) detected edges, (d) reflection, (e) external contour, (f) revolution axis projection, (g) rectified view, and (h) profile shape.

Table 1

| Measured and real orientation of the revolution axis |
|---------------------------------------------|----------------|---------------|
| Real angles (degrees) | Measured angles (degrees) | Angle error (degrees) |
| With x-axis | With image plane | With x-axis | With image plane | |
| Image in Fig. 2 | 90 | 28 | 88.9 | 26.4 | 1.87 |
| Image in Fig. 3 | 80 | 15 | 81.6 | 13.8 | 1.96 |

where \( a \) and \( b \) are the lengths of the principal semi-axes, while \((0, z_E)\) are the coordinates of the center \( E \) of the ellipse; \( a, b, \) and \( z_E \) are the unknown parameters to be determined. To determine these parameters, we impose that (i) the ellipse equation is satisfied for both \((x, z) = (x_P, z_P)\) and \((x, z) = (x_B, z_B)\) and (ii) the angular coefficient of the tangent at \( P \) is equal to a given value \( m_P \).

It can easily be shown that, for an ellipse with vertical and horizontal principal axes, the angular coefficient \( m_P \) of the tangent at an ellipse point \( P = (x_P, z_P) \) is given by

\[
m_P = \frac{b^2}{a^2} \frac{x_P}{z_P - z_E}.
\]  

\[(A.1)\]

The difference between the equations, which express that both \( P \) and \( B \) belong to the ellipse, can be rewritten as

\[
\frac{b^2}{a^2}(x_P^2 - x_B^2) = -(z_P^2 - z_B^2) + 2(z_P - z_B)z_E.
\]  

\[(A.2)\]

Writing \( b^2/a^2 \) as a function of \( z_E \) (Eq. (A.1)), and substituting this expression to \( b^2/a^2 \) in Eq. (A.2) yields

\[
z_E = \frac{(z_P^2 - z_B^2) - (x_P^2 - x_B^2)(z_P/x_P)m_P}{2(z_P - z_B) - (x_P^2 - x_B^2)(m_P/x_P)}.
\]  

\[(A.3)\]

the ratio \( b^2/a^2 \) can be found from Eq. (A.1). Once both \( z_E \) and \( b^2/a^2 \) are known, \( a^2 \) can directly be found from
the ellipse equation
\[ a^2 = x_P^2 + \frac{(z_P - z_E)^2}{(b^2/a^2)}. \]

**Appendix B. Orientation of a circumference from its perspective projection**

Let us consider the circumference, whose perspective projection is a given ellipse, whose construction has been illustrated in the previous appendix. We use the same reference frame, defined in Appendix A, for the image plane. The ellipse is symmetric with respect to the \( z \)-axis, and the orthogonal projection of the viewpoint \( O \) onto the image plane belongs to this symmetry axis. The orientation of the circumference has to be determined. Instead of using a general method (see, e.g., Ref. [13]), an ad hoc method is adopted, exploiting this symmetry condition.

Under perspective projection, the projection of the center \( C \) of the circumference does not coincide with the center \( E \) of the ellipse. The orientation of the circumference is represented by two degrees of freedom; because of symmetry, there exists one diameter of the circumference, which is parallel to the second symmetry axis of the ellipse (which in turn is parallel to the \( x \)-axis). The second degree of freedom represents the orientation of the diameter, perpendicular to the above one. The projection of this diameter is contained in the projection of the revolution axis. Since the center of the circumference belongs to this diameter, its projection \( C \) is on the projection of the revolution axis. The orientation angle of the circumference is therefore defined as the angle formed by the normal to the circumference and the direction of the projection of the revolution axis.

To determine the orientation of this diameter, the tangent to the ellipse through a contour point \( P \) on the outline is considered. This tangent is the projection of a tangent to the circumference at the point \( P' \), whose image is \( P \). The diameter of the circumference through \( P' \) is perpendicular to this tangent. Therefore, the vanishing point of the direction of this tangent is seen from \( O \) along a viewing direction perpendicular to the viewing direction of the vanishing point of the diameter. Since the horizon line of the plane containing the circumference is parallel to the horizontal diameter, only the \( z \) (vertical) coordinate of this horizon line must be determined. This can be done by imposing the perpendicularity of the above viewing directions, where the diameter projects onto the line through \( P \) and the projection \( C \) of the circumference center. The position of \( C \) in the image is not known. However, due to the cross ratio invariance under perspective, the vertical coordinate \( z_C \) of \( C \) is related to the vertical coordinate \( z \) of the horizon line by the following relation:

\[ z_C - z_E = \frac{b^2}{z - z_E}. \]

where \( b \) is the length of the vertical semi-axis of the ellipse.

Applying the perpendicularity constraint, an algebraic three-degree equation is obtained. In the sequel, the details of the derivation of the three-degree equation is illustrated.

Two straight lines are considered in the perpendicularity constraint: (i) the tangent to the ellipse at \( P \) and (ii) the line through \( C \) and \( P \). Both lines are expressed in the form \( x = x(z) \). In particular, the tangent to the ellipse at \( P \) is expressed by

\[ x_1 = x_P + \frac{z - z_P}{m_P}, \]

where \((x_P, z_P)\) are the coordinates of \( P \), and \( m_P \) is the angular coefficient of the tangent at \( P \), while the straight line through \( C \) and \( P \) is expressed by

\[ x_2 = x_P \left( 1 - \frac{z - z_P}{z_C - z_P} \right) = x_P \left( 1 - \frac{z - z_P}{(z_C - z_E) - (z_P - z_E)} \right) \]

and, rewriting \( z_C - z_E \) in terms of \( z \),

\[ x_2 = x_P \frac{b^2 - (z - z_E)^2}{b^2 - (z_P - z_E)(z - z_E)}. \]

The viewing direction of a point on the first line is constrained to be orthogonal to the viewing direction of a point on the second line:

\[ [x_1 \ f \ z] \begin{bmatrix} x_2 \\ f \\ z \end{bmatrix} = 0. \]

Substituting to \( x_1 \) and \( x_2 \) their expressions in terms of \( z \) yields the following algebraic three-degree equation in \( z \):

\[ \left( f^2 + z^2 \right) \left( b^2 - (z_P - z_E)(z - z_E) \right) + x_P \left( x_P + \frac{z - z_P}{m_P} \right) \left( b^2 - (z - z_E)^2 \right) = 0. \]

Either one or three real solutions exist for the \( z \) coordinate of the horizon line. The inclination angle of the diameter of the cross section is then \( \arctan(z/f) \).

**References**


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