Mode determination in noisy bimodal images by histogram comparison

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Abstract

This letter introduces the “mode determination” problem relative to a bimodal image corrupted by noise, i.e., the problem of estimating the mean intensity level of each of the two homogeneous regions constituting a bimodal image. The mode determination problem can be seen as a generalization of the threshold determination problem. This generalization is introduced to cope with the low performances of thresholding techniques when dealing with highly noisy images. A mode determination method is proposed, based on the comparison of the histogram of different (averaging) convolutions of the given image. Experimental results are presented showing the performance of the proposed method.

1. Introduction

Suppose that an ideal image is partitioned into two regions $R_1$ and $R_2$, in which the grey-level is constant. The modes of this image are defined by the grey-level values $L_1$ and $L_2$ which are assumed, respectively, in $R_1$ and $R_2$. Consider an image, obtained by corrupting the ideal image with additive noise. The segmentation problem consists in determining the regions $R_1$ and $R_2$ starting from the corrupted image.

In this letter a different problem is addressed, namely the mode determination, i.e., the determination of the modes $L_1$ and $L_2$ given the corrupted image. Mode determination is easy once the image has been segmented. In fact once the regions $R_1$ and $R_2$ have been determined, their modes can be estimated by averaging the grey-levels within each region. However, the mode determination problem is relevant per se, since in a number of applications (such as, e.g., in the analysis of biomedical imagery) the determination of the shape of the regions $R_1$ and $R_2$ is not required, while the determination of the modes $L_1$ and $L_2$ is required together with the determination of the area $M_1$ and $M_2$ of the regions. In this letter, we propose a simple mode determination method which does not require segmentation as an intermediate step.

For instance, this problem is relevant in many applications of Nuclear Medicine, such as, e.g., in the analysis of noisy images of symmetric organs (brain, lungs, lower limbs). Anomalies in one of the two parts of a symmetric organ can be detected by comparing the average grey-levels of the two parts of the organ. Notice, also, that the determination of the shape of the regions is not required in this problem.

The mode determination problem can be seen as a generalization of the threshold determination problem for the high noise case. Infact, for high noise levels no thresholded image can be found, in which the two
component regions are appropriately discriminated. A thorough survey of approaches to image thresholding, can be found in (Sahoo et al., 1988). Many approaches to threshold determination are based on the analysis of the first order grey-level statistics (Kittler and Illingworth, 1986; Cho et al., 1989; Boukharouba et al., 1985; Snyder et al., 1990; Kapur et al., 1985; Pun, 1980; Huang and Wang, 1995; Chang et al., 1994). These methods allow to determine the modes from the histograms, only for values of the signal to noise ratio (SNR) well above one. None of the above approaches exploits the spatial correlation between the pixels. The spatial correlation is exploited in (Abutaleb, 1989; Haralick et al., 1973; Kirby and Rosenfeld, 1979; Chen et al., 1994), where a two-dimensional histogram is analyzed: one dimension represents the pixel grey-levels, another dimension represents the average grey-levels within windows centered at the pixels. This last dimension is intended to account for the spatial correlation among the pixels. These approaches allow to handle images characterized by SNR values slightly above one, but their performances rapidly degrade when the SNR drops below one.

The approach proposed in this letter is based on the comparison of the histograms of different averaging convolutions of the given image. A simple relationship is derived between the intersections between the histograms of different convolutions and the modes of a bimodal image corrupted with gaussian additive noise. This relationship is used to estimate the modes.

As for any technique based on grey-level statistics, the proposed method relies on the assumption that the areas of the two regions do not differ by a too large factor.

Some preliminary experimental results are presented both on synthetic images and on real ones, characterized by SNR values well below one, to show the performance of the proposed technique.

2. Problem formulation

An ideal (uncorrupted) image is considered, described by an intensity function $I_0(x, y)$ defined in a square domain $R$ contained in the image plane: $(x, y)$ represent the pixel coordinates within the image plane. Let $M = n \times n$ be the number of pixels contained in $R$. It is supposed that the uncorrupted image is characterized by only two values $L_1$ and $L_2$ (also called “modes”) of the intensity $I_0$. In addition, let $L_1 < L_2$,

$$I_0(x, y) = \begin{cases} L_1, & \text{if } (x, y) \in R_1, \\ L_2, & \text{if } (x, y) \in R_2. \end{cases}$$

The two regions $R_1$ and $R_2 = R - R_1$ contain respectively $M_1$ and $M_2 = M - M_1$ pixels.

Here the “smoothness” of the borderline $B$ between the two regions $R_1$ and $R_2$ is characterized by the number $N_\rho$ of pixels whose distance from $B$ is less than (or equal to) $\rho$, where $\rho$ is a variable, nonnegative parameter. These pixels are said to be external, while the pixels internal to the region $R_1$ ($R_2$) are the pixels of $R_1$ ($R_2$) whose distance from the borderline is at least $\rho$. The number of the pixels internal to $R_1$ is indicated by $M_\rho_1$. It holds that

$$\forall \rho \quad M_\rho_1 + M_\rho_2 + N_\rho = M,$$

and it is supposed for simplicity that the external pixels are equally partitioned between $R_1$ and $R_2$:

$$M_\rho_1 \simeq \frac{M_2}{2}, \quad M_\rho_2 \simeq \frac{M_1}{2} - \frac{N_\rho}{2}.$$  

The real (given) image $l(x, y)$ is corrupted with an additive noise $w$, which is supposed to be white and gaussian:

$$l(x, y) = I_0(x, y) + w(x, y),$$

where

$$E[n(x, y)] = 0,$$

and

$$E[w(x_1, y_1)w(x_2, y_2)] = \begin{cases} \sigma^2, & \text{if } (x_1, y_1) = (x_2, y_2), \\ 0, & \text{if } (x_1, y_1) \neq (x_2, y_2). \end{cases}$$

The signal to noise ratio is defined as the ratio between the difference between the modes and the standard deviation $\sigma$ of the additive noise:

$$\text{SNR} = \frac{\Delta L}{\sigma} = \frac{L_2 - L_1}{\sigma}.$$

The mode determination problem consist in determining the modes $L_1$ and $L_2$ given the corrupted image.
The mode determination method proposed in Section 4 relies on the "noninteracting modes" assumption discussed in Section 3.1: this assumption is formulated through expression (5) in terms of (i) the signal-to-noise ratio, (ii) the areas $M_1$ and $M_2$ of the two regions, and (iii) the smoothness parameter $N_p$.

The following notation is introduced to indicate the gaussian distributions with standard deviation $\sigma$, its integral, and its derivative:

$$G(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}, \quad G_\sigma(z) = \frac{G(z/\sigma)}{\sigma},$$

$$E(z) = \int_{-\infty}^{z} G(t) \, dt, \quad E_\sigma(z) = E\left(\frac{z}{\sigma}\right),$$

$$G'(z) = -zG(z).$$

The histogram of the given image is therefore approximated by

$$f_\sigma(z) = M_1G_\sigma(z - L_1) + M_2G_\sigma(z - L_2). \quad (1)$$

### 3. Comparison of probability density functions

In this section we introduce the comparative analysis of the histograms of different convolutions of a bimodal image. In this analysis, we suppose the two modes to be "noninteracting" (Section 3.1). The non-interacting modes conditions will be discussed in Section 3.2.

We consider a class of averaging convolutions of the given image $I(x,y)$, $(x,y) \in \mathbb{R}$. Each averaging convolution is characterized by the two parameters $p$ and $\lambda$ (with $\lambda \leq \rho$): $\rho \times \rho$ is the size of the averaging window centered at the pixel $(x,y)$, while $\lambda^2$ is the number of the pixels in the window, which are weighted in the average. A convolution $I_{p,\lambda}(x,y)$ of the given image $I(x,y)$ is defined as

$$I_{p,\lambda}(x,y) = I(x,y) \ast W_{p,\lambda}(x,y), \quad \lambda \leq \rho,$$

where $W_{p,\lambda}(x,y)$ assumes a nonzero value (namely, $1/\lambda^2$), on only $\lambda^2$ pixels out of the $\rho^2$ pixels in the square window centered at $(0,0)$. The above pixels are uniformly distributed within the window (see Fig. 1). An averaging window is said to be full density if $\lambda = \rho$.

The full density, nonsquare window $2 \times 1$ (see Section 3.2) is indicated by $W_{\sqrt{2},\sqrt{2}}$.

Before proceeding along the letter, some further notation is introduced: $\Delta f(z) = f''(z) - f'(z)$ denotes the difference between two histograms; $L$ denotes the mean intensity level of the image; $L^2$ denotes the mean squared intensity of the image; $r = M_1/M_2$ denotes the ratio between the areas of the regions $R_1$ and $R_2$; $L^2_0 = (M_1L_1^2 + M_2L_2^2)/(M_1 + M_2)$ denotes the mean squared intensity of the uncorrupted image; $F(z) = \sum_{t=0}^{z} f(t)$ denotes the "integral" of a histogram.

### 3.1. The "noninteracting modes" case

In this section a bimodal image is considered, constituted by two regions $R_1$ and $R_2$. Within the region $R_1$ ($R_2$), the intensity level is given by two contributions: one constant contribution given by $L_1$ ($L_2$) and one additional contribution given by a gaussian white noise. Therefore the grey-level distribution is given by the sum of two gaussian contributions: the first one is centered at $L_1$, and its area is $M_1$, while the second one is centered at $L_2$ and its area is $M_2$. Their common variance is $\sigma^2$. The histogram of the given bimodal image is a discrete approximation of the above distribution.

When the convolution of a bimodal image with a function $W_{p,\lambda}(x,y)$ is considered, a third contribution arises: namely, the contribution produced by the external pixels, i.e., the zone within a distance $\rho$ from the borderline $B$ between $R_1$ and $R_2$.

The histogram contribution produced by the external pixels is the convolution between a difference of step (or Heaviside) functions $H(-)$, centered at $L_1$ and $L_2$ respectively, and a Gaussian whose standard deviation is $\sigma/\lambda$.
\[ f'_{\sigma/\lambda}(z) = \frac{N_p}{\Delta L} (H(z - L_1) - H(z - L_2)) * \Gamma_{\sigma/\lambda}(z) \]
\[ = \frac{N_p}{\Delta L} (\Gamma_{\sigma/\lambda}(z - L_1) - \Gamma_{\sigma/\lambda}(z - L_2)), \]

where \( \Gamma_{\sigma/\lambda}(z) \) indicates the error function.

The two modes contribute with two Gaussians centered at \( L_1 \) and \( L_2 \), whose standard deviation is \( \sigma/\lambda \), and whose areas are \( M_{p1} \) and \( M_{p2} \). The histogram of the convolved bimodal image can be therefore approximated by

\[ f_{\sigma/\lambda}(z) = f'_{\sigma/\lambda}(z) + M_{p1} \Gamma_{\sigma/\lambda}(z - L_1) + M_{p2} \Gamma_{\sigma/\lambda}(z - L_2). \]  

We now consider the intersections between the histograms of two different convolutions of the given image: one with the averaging function \( W_{p,\lambda'} \) and the other one with the averaging function \( W_{p,\lambda''} \). Let be \( \beta = \lambda''/\lambda' \). We consider the two most distant intersections between the two histograms. Notice that the abscessae of these two intersections are, respectively, lower than \( L_1 \) and higher than \( L_2 \); we therefore distinguish among a low intersection and a high intersection (which correspond to a low and a high mode).

The two modes are said to be noninteracting if the low intersection is not affected by the high mode and the high intersection is not affected by the low mode: i.e., if one can neglect the difference between the largest value of \( z \) satisfying

\[ f_{\sigma/\lambda'}(z) = f_{\sigma/\lambda''}(z) \]

and the largest value of \( z \) satisfying

\[ f'_{\sigma/\lambda'}(z) + M_{p2} \Gamma_{\sigma/\lambda'}(z - L_2) = f'_{\sigma/\lambda''}(z) + M_{p2} \Gamma_{\sigma/\lambda''}(z - L_2). \]

The mode determination method proposed in this letter relies on the fact that some convolutions can be constructed, for which the noninteracting modes conditions are satisfied (see Section 3.2).

Now we derive an approximation of the abscessa of one of the intersections (say, the high one) of the histograms of two different convolutions of the given image.

In the noninteracting modes hypothesis, the high intersection is obtained by solving the following equation in the abscessa \( z \).

\[ \Delta f(z) = \frac{M_p}{(\sigma/\lambda')} \left( G\left( \frac{z - L_2}{(\sigma/\lambda')} \right) - \beta G\left( \frac{z - L_2}{(\sigma/\beta \lambda')} \right) \right) \]
\[ - \frac{N_p}{\Delta L} \left( E\left( \frac{z - L_2}{(\sigma/\lambda')} \right) - E\left( \frac{z - L_2}{(\sigma/\beta \lambda')} \right) \right) = 0. \]

Clearly, no closed form exists for the solution of this equation. We derive an approximate solution by applying the first iteration of the Newton method, starting from an abscessa \( z^{(0)} \) obtained by neglecting the contribution of the external pixels. Imposing

\[ G\left( \frac{z - L_2}{(\sigma/\lambda')} \right) - \beta G\left( \frac{z - L_2}{(\sigma/\beta \lambda')} \right) = 0, \]

we obtain

\[ z^{(0)} = L_2 + \mu \frac{\sigma}{\lambda'}, \quad \mu = \sqrt{\frac{2 \ln \beta}{\beta^2 - 1}}. \]  

To simplify further notation, we define

\[ \gamma(\alpha) \triangleq E(\alpha) - E(\alpha \beta), \quad \text{and} \]
\[ \delta(\alpha) \triangleq G'(\alpha) - \beta^2 G'(\alpha \beta), \]

and we omit the index “2” from \( M_{p2} \) and \( N_{p2} \). It can be shown that

\[ \Delta f(z^{(0)}) = -\gamma(z^{(0)}) \frac{M_p}{\Delta L} \]

and

\[ \frac{\partial \Delta f(z)}{\partial z} \bigg|_{z^{(0)}} = \delta(z^{(0)}) \frac{M_p}{(\sigma/\lambda')^2}. \]

Applying the first iteration of the Newton method starting from \( z^{(0)} \), and approximating the high intersection \( \tilde{z} \) with the obtained result yields

\[ \tilde{z} \approx z^{(1)} = z^{(0)} - \frac{\Delta f(z^{(0)})}{\delta(z^{(0)})} \bigg|_{z^{(0)}} = \mu \frac{\sigma}{\lambda'} + \frac{\gamma(z^{(0)})}{\delta(z^{(0))}} \frac{M_p}{\Delta L} \frac{\sigma^2}{\lambda'^2} + L_2. \]  

Once the abscessa \( \tilde{z} \) of the high intersection between the two histograms has been detected, an equation is obtained in three unknowns \( L_2, \sigma, N_p/(M_p \Delta L) \). If three pairs of histograms are intersected, while maintaining a constant value of \( \beta = \lambda''/\lambda' \), the values of the three unknowns can be determined.
3.2. On the noninteracting modes conditions

In general, the two modes of the given bimodal image are not noninteracting. The, say, high intersection $z_2$ of the histograms of two different convolutions of the given image is higher than the high mode $L_2$: $(z_2 \geq L_2)$. Therefore, the low mode does not affect this intersection only if its contribution at $L_2$ is negligible with respect to the contribution of the high mode:

$$M_{p1} \frac{1}{\sigma/\lambda} G\left(\frac{L_2 - L_1}{\sigma/\lambda}\right) \ll M_{p2} \frac{1}{\sigma/\lambda} G\left(\frac{L_2 - L_2}{\sigma/\lambda}\right) = \frac{M_{p2}}{\sqrt{2\pi\sigma/\lambda}}.$$  

In this relation the contribution of the external pixels has been neglected, since it is positive. This condition reduces to

$$\lambda \geq \frac{\sigma}{\Delta L} \sqrt{2(\ln \frac{M_{p1}}{M_{p2}} + \ln K)}, \tag{5}$$

where $K$ is a constant much larger than 1 (e.g., such that $\ln K = 4$). Notice that the first factor in the right-hand side coincides with the noise to signal ratio.

Relation (5) refers to $\lambda$ and $\Delta L$ which are still unknown. A rough estimate of $\sigma$ and a (rough) lower bound to $\Delta L$ can however be easily determined.

To estimate $\sigma$, the expression of the average intensity value is approximated by

$$L = \frac{M_1L_1 + M_2L_2}{M_1 + M_2},$$

and the mean square value of the intensity in the given image is approximated by

$$\overline{L^2} = \frac{M_1L_1^2 + M_2L_2^2}{M_1 + M_2} + \sigma^2.$$  

The mean square value of the intensity in the convolution between the given image and the averaging function $W_{\sqrt{2},\sqrt{2}}$ (i.e., a full density averaging window $2 \times 1$) is approximated by

$$\overline{L^2} = \frac{M_1L_1^2 + M_2L_2^2}{M_1 + M_2} + \frac{\sigma^2}{2},$$

where the contribution of the external pixels has been neglected, since a small averaging window ($2 \times 1$) has been used. Notice that the two left-hand sides of the last two expressions can be extracted from the given image. Combining these two expressions, a rough estimate of $\sigma^2$ can be obtained:

$$\sigma^2 = 2(\overline{L^2} - \overline{L^2}). \tag{6}$$

A lower bound on $\Delta L$ is now derived. Supposing that $r = (M_1/M_2) \geq 1$, from the (approximate) expressions of $\overline{L^2}$ and $\overline{L^2}$ the following equation is obtained:

$$\overline{L^2} = \frac{M_1L_1^2 + M_2L_2^2}{M_1 + M_2} = 2\overline{L^2} - \overline{L^2}. \tag{6'}$$

$L_1$ and $L_2$ can be written in terms of the mean level $\overline{L}$ and the ratio $r = M_1/M_2$:

$$L_1 = \overline{L} - \frac{\Delta L}{r + 1}, \quad L_2 = \overline{L} + \frac{r\Delta L}{r + 1}.$$  

Combining the two last expressions,

$$\Delta L^2 = \frac{(r + 1)^2}{r}(\overline{L^2} - \overline{L^2}). \tag{7}$$

A lower bound on $\Delta L$ can be derived from $r \geq 1$:

$$\Delta L \geq 2\sqrt{\overline{L^2} - \overline{L^2}}. \tag{8}$$

Combining this inequality with condition (5), we obtain

$$\lambda \geq \frac{\sigma}{\sqrt{2(\overline{L^2} - \overline{L^2})}} \sqrt{\frac{M_{p1}}{M_{p2}}} + \ln K. \tag{9}$$

For instance, if it is supposed that the ratio between $M_{p1}$ and $M_{p2}$ is at most $\exp(4)$, then it is sufficient (5) for $\lambda$ to be at least $4\sigma/\Delta L$.

Combining this with (9) and (6), a condition is derived for $\lambda$ in terms of “measurable” quantities, i.e., quantities which can be extracted from the image:

$$\lambda \geq 2\sqrt{2}\sqrt{(\overline{L^2} - \overline{L^2})/(\overline{L^2} - \overline{L^2})}. \tag{10}$$

4. The mode determination method

In this section we propose a mode determination method based on the comparison of several histograms relative to different convolutions of the given image. A feature which can often be extracted reliably from a pair of histograms, is the position of their intersections. In practice, it has been observed that this feature can
be reliably extracted if the ratio $\beta$ between the two values of $\lambda$ is at least 2.

In Section 4.1, the solution to a system of three equations in the form of (4) is derived. In Section 4.2, the determination of the "main" intersection between two histograms is discussed. In Section 4.3, the proposed method is formulated.

4.1. Solving the system (4)

By substituting $\lambda'$ to, respectively, three different values $\lambda_1, \lambda_2$ and $\lambda_3$, the following system of equations is obtained:

$$\tilde{z}_k = \mu \frac{\sigma}{\lambda_k} + \gamma \frac{N_p}{\delta M_p} \frac{\sigma^2}{\Delta L} \lambda_k^2 + L_2, \quad k = 1, 2, 3. \quad (4')$$

This system is only apparently nonlinear. In fact, the nonlinear term can be eliminated from the three equations, obtaining two linear equations in $\sigma$ and $L_2$. By solving these equations, the following expressions are obtained:

$$L_2 = \frac{\lambda_1^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \tilde{z}_1$$
$$- \frac{\lambda_2^2}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)} \tilde{z}_2$$
$$+ \frac{\lambda_3^2}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \tilde{z}_3$$

and

$$\sigma = \frac{1}{\mu} \left( \frac{\lambda_1^2(\lambda_2 + \lambda_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \tilde{z}_1 - \frac{\lambda_2^2(\lambda_1 + \lambda_3)}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)} \tilde{z}_2 + \frac{\lambda_3^2(\lambda_1 + \lambda_3)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \tilde{z}_3 \right) \quad (11)$$

where $\tilde{z}_k$ ($k = 1, 2, 3$) indicates the abscissa of the high intersection between the intensity histogram of the image convolution with an averaging function $W_{p, \lambda_k}$ and the intensity histogram of the image convolution with averaging function $W_{p, \beta_k}$.

Notice that the solution to the approximate equations (4') depends on $N_p$ and $M_p$ only through the dependence of the intersections $\tilde{z}_k$ on them.

4.2. Determining the two "main" intersections between two histograms

The mode determination method proposed in this letter is based on the comparison between the histogram of different convolutions of the given image. Let $f''(z)$ be the histogram of the convolution with $W_{p, \lambda''}$, and let $f'''(z)$ be the histogram of the convolution with $W_{p, \lambda'}$. Let $\lambda' < \lambda''$. The features needed for this comparison are the abscissae of the intersections between two histograms. There exist either two or four intersections between the intensity probability density functions of two different averaging convolutions of a same bimodal image. Among them, the two most distant intersections are relevant to the method. However, when the intensity histograms are considered, more than four intersections can often be detected due to the particular "realization" of the noise process which is superimposed to the uncorrupted image. The problem arises of extracting the two most distant "main" intersections between two histograms.

In Section 4.2.1, a particular case is considered, where only two intersections exist between the probability density functions of two different convolutions of the corrupted image. A definition of "main" intersections between their two histograms is proposed. In Section 4.2.2, the definition is extended to the case where four intersections exist.

4.2.1. First case: two "main" intersections

The difference $f(z) = f'''(z) - f''(z)$ between two histograms goes through zero in correspondence of their intersections. The integral $F(z) = \sum_{i=0}^{z} f(t)$ of $f$ goes through a (local) minimum or a (local) maximum in correspondence to each intersection. Since $\lambda' < \lambda''$, then the abscissa $\tilde{z}_1$ of the intersection corresponding to a minimum should be lower than the abscissa $\tilde{z}_2$ corresponding to a maximum.

We define the two "main" intersections between the two histograms $f''(z)$ and $f'''(z)$ as the pair $(\tilde{z}_1, \tilde{z}_2)$ such that (i) $F(z)$ has a local minimum in $\tilde{z}_1$ and a local maximum in $\tilde{z}_2 > \tilde{z}_1$, and (ii) the excursion $F(\tilde{z}_2) - F(\tilde{z}_1)$ is maximal, i.e. for no other pair $(z_1, z_2)$ satisfying (i) is the excursion $F(z_2) - F(z_1) > F(\tilde{z}_2) - F(\tilde{z}_1)$.

The two main intersections may be found by a single-scan of $F(z)$.!
A lower bound (8) \( \Delta L' = 2\sqrt{L_0^2 - L^2} \) is known for \( \Delta L \). Since the adopted convolutions satisfy the non-interacting modes condition (10), then \( \sigma / \lambda \ll \Delta L' \). In addition, for \( \beta = 2 \) is \( \mu < 1 \). Therefore it is possible to discriminate between the first and the second case, namely \( z_1 - z_2 > \Delta L' \) or \( z_1 - z_2 < 2\mu \sigma / \lambda \).

In the second case, there are four main intersections, and the two most distant ones can be selected as indicated in the first paragraph of this subsection.

4.3. Formulation of the method

The mode determination procedure can now be summarized:

1. procedure mode determination
2. compute \( L, L^2 \) and \( L_2 \) from the image;
3. compute \( L_2^2 \) and \( \sigma \) from (6) and (6')
4. let \( \lambda_{\min} \leftarrow \) right-hand side of (10);
5. let \( \rho \leftarrow 4\lambda \);
6. let \( \beta \leftarrow 2 \);
7. let \( (\lambda_1, \lambda_2, \lambda_3) \leftarrow (\lambda, 1.5\lambda, 2\lambda) \);
8. for \( k = 1 \) to 3 do
9. convolve image with \( W_{\rho, \lambda_k} \) and with \( W_{\rho, \beta \lambda_i} \);
10. construct histogram of both convolutions;
11. \( (z''_k, z''_{k'}) \leftarrow \) main intersections
   \( f_{\rho, \sigma / \lambda_k}, f_{\rho, \sigma / \beta \lambda_i} \)
12. enddo
13. compute \( L_2 \) (or \( L_1 \)) from (11)
14. end.

Since this method is based on cumulative information, namely on histograms, the area of the smaller region may not be too smaller than that of the larger one. This motivates the requirement (10) that, for the adopted \( \rho \), the ratio \( M_{\rho_1} / M_{\rho_2} \) is below \( \exp(4) \).

Indicating by \( n \times n \) the image size and by \( I \) the number of grey-levels, the time complexity of the mode determination procedure is \( T = O(n^2) + O(I) \). Other methods exploiting spatial correlation show higher complexities: for instance for the method in (Chen et al., 1994) is \( T = O(n^2) + O(I^{8/3}) \). For the relative-entropy approach [13] is \( T = O(n^2) + O(I^2) \). The present method, however, has a large multiplicative factor for the \( O(n^2) \) term, since five convolutions are needed, whereas, e.g., only one convolution is needed in (Chen et al., 1994).
Fig. 3. (a) The maximum excursion intersections coincide with the most distant ones. (b) The maximum excursion intersections do not coincide with the most distant ones.
5. Experimental results

The mode determination method has been tested both on synthetic and real images.

A set of synthetic images have been considered, characterized by different values of the SNR. These images have been constructed by superimposing a white, gaussian noise of increasing variance to an original bimodal image. The original image is illustrated in Fig. 4. The modes of this image (i.e., the mean grey-levels of the two regions constituting the image) are $L_1 = 112$ and $L_2 = 142$. The three images shown in Figs. 5–7 are obtained by adding to the original image a gaussian noise, whose standard deviation has been set to, respectively, $\sigma = 30$ (corresponding to SNR = 1), $\sigma = 60$ (SNR = 0.5) and $\sigma = 90$ (SNR = 0.33). The technique described in Section 4 has been applied to each of the three images of Figs. 5–7, and the obtained estimates have been reported in Table 1. Notice that the estimated modes differ from the true ones only slightly. The estimate of the standard
deviation $\sigma$ of the additive noise by means of expression (6) is also quite accurate.

To visualize the results of the mode determination, a rough segmentation of the image has been performed, based on the obtained mode estimates $\hat{L}_1$ and $\hat{L}_2$. This segmentation has been performed in the following way. Starting from the given image, the smallest averaging convolution is considered, such that the non-interacting conditions formulated by expression (7) are satisfied. Two regions of this convolution are defined: (i) the "grey" region, which contains all the pixels whose level is $\geq \hat{L}_1$, and (ii) the "white" region, which contains the pixels whose level is $\geq \hat{L}_2$. The region $R_2$ should be somehow "midway" between the grey region and the white region. This region is constructed by simultaneously applying an operator called "constrained erosion" to the grey region and an operator called "constrained dilation" to the white region. The constrained erosion recursively modifies the grey region. Each step of this process consists in removing from the current grey region any non-white pixel, whose neighbors are not all within the (current) grey region. The constrained dilation recursively modifies the white region. Each step of this process consists in adding to the current white region any grey pixel, having at least one white neighbor. Notice that during this process, the white region is still contained into the grey region. The process ends as soon as the white and the grey regions coincide. The resulting white (or grey) region is retained as the approximation of the region $R_2$, while the rest of the image is retained as the approximation of the region $R_1$. The results of the appli-
cation of this rough segmentation to the three images of Figs. 5–7, are reported in Figs. 8–10. It is worth noting the performance of the procedure on the image of Fig. 7 (SNR = 1/3), from which the two component regions are very hard to discriminate by eye.

To illustrate the application of the method to real images, a noisy lung image has been considered (see Fig. 11). The modes of the image are not known a priori. The same steps described in the previous paragraph are performed, obtaining the rough segmentation shown in Fig. 12. The estimated modes are $\hat{L}_1 = 109$ and $\hat{L}_2 = 139$. The estimated standard deviation is $\hat{\sigma} = 98.15$ (thus the estimated SNR is 0.31). The value of $\lambda_{\min}$ suggested by the method is 9. The results of the rough segmentation can be used to comparatively analyze the two parts of the symmetric organ in order to detect differences in the average grey-level.

6. Conclusions

A technique for the mode determination in noise corrupted bimodal images has been presented. It is based on the intersection between different averaging convolutions of the given image. This technique estimates the variance of the additive noise, and it uses this estimate to determine the appropriate size of the convolution window. The proposed technique has been experimented both on synthetic images and on real
ones: the results show that it can deal with high noise levels (SNR < 1).

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References


