Recovering cylindric and conic surfaces from contours and reflections

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Received 27 May 1998; received in revised form 18 December 1998

Abstract

Cylindric and conic metallic parts are considered, which are subject to surface treatments (such as, e.g., lathe finishing or grinding) that make both Lambertian reflection models and purely specular ones inadequate. A new reflection model is introduced for such surface treatments, and a method is derived for the determination of the pose and the intrinsic geometric parameter of a cylindric surface or a conic one starting from a single view. This method uses both the projection of the outline of the surface and the reflection of known light sources onto it. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Reflection models; Reflection; Cylinder; Cone; Outline; Localization; Surface recovery

1. Introduction

In manipulation tasks, metallic parts having cylindric or conic shapes are often encountered. Cylindric parts are geometrically characterized by the position of the axis and the radius. Conic parts are characterized by the position of the vertex, the axis orientation and the aperture angle. The determination of the surface geometry for cones or cylinders (CSG problem) is addressed in this letter. It is known, that the CSG problem cannot be solved starting from a single view of the outline of the lateral surface of the object. However, if the object surface is light reflecting (e.g., because it is metallic), then sometimes the knowledge of the position of the light sources can help in solving the problem.

Some research has been conducted on the interpretation of the projection of the outlines of revolution surfaces (Nalwa, 1989; Glachet et al., 1991) or of straight homogeneous generalized cylinders (SHGCs) (see Ponce et al., 1989). However, the outline information is not sufficient for fully recovering the geometry of the viewed surface. This also applies to the cylindric and conic surfaces referred to in this letter, even though they are a particularization of the above-mentioned revolution surfaces or of the SHGCs. In particular, Nalwa (1989) and Glachet et al. (1991) provide methods to compute the projection of the revolution axis, since its 3D pose cannot be determined from the contours.

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If the contour of a cross section is entirely visible in the image, then the shape of an SHGC can be recovered from a single, orthographic view (Ulupinar and Nevatia, 1995). Puech et al. (1997) use the image of two cross sections to locate a Straight Uniform Generalized Cylinder in the 3D space. A cylindrical label of known height, pasted on the cylindrical surface, also provides sufficient information for the localization of a cylinder from a single perspective view (You et al., 1992). In the approach proposed in this letter, we handle those cases, where the contour of any cross section is either not visible or it cannot be extracted reliably. We also suppose, that no contour line, internal to the cylindrical surface, can be extracted.

Sanderson et al. (1988) first proposed to use the reflection from a set of known light sources onto a specular surface to determine its shape. Gross and Boult (1996) recover the shape of an SHGC from both contour and reflection: this method assumes a Lambertian reflection model and it is based on the intensity values along the surface image. Wink et al. (1994) use the radius lighting direction to locate a cylinder.

However, very often metallic surfaces are subject to a treatment, as, e.g., lathe turning or grinding, that introduces a raggedness on the surfaces. For these kind of surfaces, which are considered in this letter, neither a purely specular reflection model nor a Lambertian one can be adopted (see, e.g., the lathe-finished objects shown in Fig. 8).

Therefore a new reflection model, called axial symmetric reflection model, is proposed: this model is qualitative, in that it only specifies the shape of the reflection region without specifying the intensity values along the reflecting surface. The axial symmetric model (Section 2) is based on the assumption that the surface treatment, although introducing a raggedness, preserves the axial symmetry of the ideal conic or cylindrical surface: this model is especially appropriate for surface treatments, such as lathe turning or grinding. The reflection equations based on this reflection model are derived in Section 3.

Thereafter (Section 4), a method is derived for the determination of both the pose and the geometric parameter of a cylindric surface from both its contour and its reflections from known point sources. There, it is shown that for an orthoperspective projection, two point sources are needed. In Section 5, a solution to the CSG problem for conic surfaces is presented: also in this case two point sources are needed for orthoperspective projection. The implementation of the solution to both CSG problems is described in Section 6, where experimental results are also reported. Concluding remarks are left to Section 7.

2. The axial symmetric reflection model

Consider a point light source $L$, a viewpoint $O$ and a reflecting surface $S$. While we indicate by $S$ the actual surface, let us indicate by $S_0$ the ideal (i.e., perfectly polished) surface: the ideal surface $S_0$ is conic (or cylindrical).

A point $B$ on the surface is called reflection point from $L$ to $O$ if a nonnull part of the light intensity from $L$, that incides onto $B$, is reflected towards $O$. The locus of the reflection points from $L$ to $O$ on $S$ is called reflection region from $L$ to $O$.

Due to the surface raggedness, the reflection at a surface point $B$ cannot be analyzed simply by applying a purely specular reflection model to the ideal surface $S_0$. On the other hand, a simple Lambertian reflection model cannot be adopted, since (i) the diffusion component for a metallic surface is small, compared with the specular reflection component and (ii) the surface treatment does not maintain isotropy. Therefore a qualitative reflection model is adopted in this work, which only specifies the position and the shape of the reflecting region, and not the intensity function along it.

The main hypothesis, underlying the reflection model, is that the surface treatment, although introducing a surface raggedness, preserves the symmetry axis of the ideal conic or cylindrical surface $S_0$: i.e., the symmetry axis of $S$ coincides with the symmetry axis $a$ of $S_0$. In other words, the function $r = r(z)$ expressing the radius of the circular cross section in terms of the abscissa along the symmetry axis, consists of
two superimposed contributions: a linear part $r_0(z)$ (constant in the cylinder) plus a rapidly varying contribution due to the raggedness $\delta r(z)$.

A further hypothesis concerns the typical dimension $s$ of the raggedness. This dimension can be approximated by the inverse of the spatial frequency mostly represented in $\delta r(z)$. We suppose that the typical dimension of the surface raggedness is much smaller than (i) the light-surface distance, (ii) the surface-viewpoint distance and (iii) the radius of the cross section containing $B$. In this case, the direction of the normal varies slowly (as $1/r$) when moving along the circular cross section (see Fig. 1), while it rapidly varies when moving along the surface meridian (i.e., the direction normal to the cross section).

Therefore, if we consider a point $B$ on the surface, the reflection normal at $B$, i.e., the normal to the actual surface $S$ at $B$, does not coincide with the normal to the ideal surface $S_0$: nevertheless, because of the axial symmetry, the reflection normal goes through the symmetry axis $a$. In other words, the reflection normal at $B$ is contained in the plane defined by the surface point $B$ and $a$.

Now consider a generic surface slice, obtained by cutting the surface $S$ by means of two parallel planes, both perpendicular to the symmetry axis $a$, separated by a distance $s$: the set of normal orientations at the points of this slice is 2D, and it constitutes a significant subset of all the possible orientations. Therefore, given a point light source $L$ and a viewpoint $O$, in general there will be one of these normals, say at the point $B$ on the slice, which bisects the directions $LB$ and $OB$: thus there will be a point $B$ on the considered slice, which is a reflection point from $L$ to $O$.

Since almost all slices will contain one reflection point from $L$ to $O$, the locus of the reflection points from $L$ to $O$ will be a discrete set. However, since the normal orientation varies slowly along the direction of the cross section, the reflection points belonging to two neighbouring slices will be close to each other.

Now we consider the image of this locus. Due to the restricted camera resolution, often two neighbouring reflection points cannot be distinguished within the digitized image. Thus the image of the locus of the reflection points from $L$ to $O$ will appear as the image of a continuous line (see Fig. 2). This line is called reflection curve from $L$ to $O$.

![Fig. 1. Variability of the normal direction.](image-url)
Notice that the present reflection model only specifies the position of the reflection points, avoiding to rely on assumptions, adopted in other approaches (Nalwa, 1989), about the quantitative behaviour of the light intensity characterizing the reflection.

**Observation 1.** Now suppose that the surface is far enough from both (i) the light source $L$ and (ii) the viewpoint $O$ (i.e., the above distances are much larger than the size of the surface $S$): in this case all the rays coming from $L$ which incide on the surface $S$ can be considered parallel, as well as all the rays reflected from $S$ to $O$. As a consequence, (i) all the rays inciding on $S$ have a common direction and (ii) all the rays reflected from $S$ to $O$ have a common direction. Therefore, all the reflection normals must have a common direction: namely, the direction bisecting the angle formed by the incidence direction and the reflection direction. Since each reflection normal passes through the symmetry axis $a$, the requirement that all reflection normals be parallel implies that all the reflection points from $L$ to $O$ must lie on a same plane through $a$: as a consequence, they lie on a same meridian of the surface.

### 3. Reflection equations

Now we consider the situation where a surface reflects a light ray, coming from a point light source $L$, towards an observation point $O$. Let $B$ be the reflection point on the surface: given a reflection normal $n$ at $B$, the reflection condition is that $n$ bisects the angle formed by $OB$ and $LB$. In other words, $n$ is on the plane $(O, B, L)$ and

$$\langle n, \widehat{OB} \rangle = \langle n, \widehat{LB} \rangle.$$  

Often, the reflection normal $n$ is orthogonal to the surface at the reflection point $B$: this standard situation is modified in case the axial symmetric reflection model is adopted (see Section 2).

To develop reflection equations, useful for the axial symmetric reflection model, we consider a modified version of the above equation.

**Property 1.** Let $u$ be any vector orthogonal to the reflection normal $n$ for rays $\widehat{LB}$ and $\widehat{OB}$: then

$$\cos(u, \widehat{LB}) = -\cos(u, \widehat{OB}).$$

**Proof.** Consider a plane perpendicular to $n$ and two oriented lines parallel, respectively, to the two vectors $\widehat{OB}$ and $\widehat{LB}$. Let $O'$ and $L'$ indicate their respective intersection points with the plane. Given the direction of
a vector \( u \) on the plane normal to \( n \), let us construct two oriented lines parallel, respectively to \( u \) and to \(-u\), passing through, respectively \( O' \) and \( L' \). The angle \( \langle \overrightarrow{OB}, u \rangle \) can be obtained by rigidly rotating the angle \( \langle \overrightarrow{LB}, -u \rangle \) about the normal \( n \) (see Fig. 3). Therefore these angles are equal, and hence \( \langle \overrightarrow{LB}, u \rangle = \pi - \langle \overrightarrow{OB}, u \rangle \) whence the thesis. □

Applying the above property, one can write

\[
|\overrightarrow{LB}| = \frac{\overrightarrow{LB} \cdot u}{\cos \langle u, \overrightarrow{LB} \rangle} = \frac{\overrightarrow{LB} \cdot u}{\cos \langle u, \overrightarrow{OB} \rangle}.
\]

Substituting \( \overrightarrow{LB} \) with \( \overrightarrow{OB} - \overrightarrow{OL} \) in the above equation, we have

\[
|\overrightarrow{OB} - \overrightarrow{OL}| = \frac{\overrightarrow{OL} \cdot u}{\cos \langle \overrightarrow{OB}, u \rangle} - \frac{\overrightarrow{OB} \cdot u}{\cos \langle \overrightarrow{OB}, u \rangle} = \frac{\overrightarrow{OL} \cdot u}{\cos \langle \overrightarrow{OB}, u \rangle} - |\overrightarrow{OB}|.
\]

Squaring both members of this equation, and simplifying, we derive

\[
|\overrightarrow{OB}| = |\overrightarrow{OL}| \frac{\cos^2 \langle \overrightarrow{OL}, u \rangle - \cos^2 \langle \overrightarrow{OB}, u \rangle}{2 \cos \langle \overrightarrow{OB}, u \rangle (\cos \langle \overrightarrow{OL}, u \rangle - \cos \langle \overrightarrow{OB}, \overrightarrow{OL} \rangle \cos \langle \overrightarrow{OB}, u \rangle)}.
\] (1)

It is recalled, that in the present approach we do not assume that the contour of a cross section is (even partially) visible. Therefore the recovery of the surface will only be based on the occluding contours of the lateral surface, and on the reflection of known light sources.

4. The cylinder

Under orthoperspective projection, the two lines constituting the image of the lateral contour of the cylinder appear approximately parallel, whatever be the actual orientation of the cylinder axis: the parallelism of the image lines is due to the fact that interpretation planes of the lateral contour lines are nearly parallel.

The projection of the cylinder axis is given by the intersection between the image plane and the plane, that bisects the two above interpretation planes.
The knowledge of the two interpretation planes of the lateral contours allows to determine: (i) one of the two parameters representing the axis orientation and (ii) the relationships between the distance of the cylinder axis from O and the cylinder radius. Two further parameters need to be determined, namely the second parameter representing the axis orientation and, e.g., the distance between the cylinder axis and O. Since the reflection curve is a straight line parallel to the cylinder axis (Observation 1), each point light source corresponds to a single equation. Therefore, the reflection from two light point sources \( L_1 \) and \( L_2 \) is needed to solve the CSG problem.

A frame reference is adopted, such that the projection of the cylinder axis is along the \( z \)-axis (notice that the interpretation plane of this projection bisects the two interpretation planes of the contour lines). The \( y \)-axis is selected such that the \((x, y)\)-plane intersects both reflection curves: let \( B_1 \) and \( B_2 \) be these intersection points.

Let \( r \) be the unknown radius of the observed cylinder, and let \( \phi \) be the unknown inclination angle of the cylinder axis with respect to the \( z \)-axis of the reference: among the infinite normals to the cylinder axis, the one contained in the plane through \( O \) and the axis is \( \mathbf{n}_0 = [0 \ -\cos\phi \ -\sin\phi]^T \). Let \( A \) be the intersection point between the \( y \)-axis and the cylinder axis, and let \( d \) be the unknown length of the segment \( OA \). Let \( A' \) be the point of the cylinder axis, whose distance \( d' \) from \( O \) is minimum. Let \( C' \) and \( C \) be the points on the image of the cylinder contour, at the same \( z \)-value as \( A' \) and \( A \), respectively. Let \( \psi' \) and \( \psi \) be the angle which \( OA' \) and \( OA \) form with \( OC' \) and \( OC \), respectively (see Fig. 4).

Given the two reflection points \( B_1 \) and \( B_2 \) on the same \( z \) as \( A \), the angle formed by \( OA \) and the segment \( OB_1 \) (\( OB_2 \)) is indicated by \( \beta_1 (\beta_2) \). Let \( \theta_1 (\theta_2) \) be the angle formed by \( \mathbf{n} \) and the normal to the cylinder surface through \( B_1 (B_2) \). The coordinates of the point light sources, with respect to the adopted reference, are \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\).

The problem is to determine \( d, \phi \) and \( r \), given \((x_1, y_1, z_1), (x_2, y_2, z_2), \psi, \beta_1 \) and \( \beta_2 \).

First, let us consider a single reflection, say \( B_1 \), and let us use, for simplicity, \((x, y, z), \theta \) and \( \beta \) in place of \((x_1, y_1, z_1), \theta_1 \) and \( \beta_1 \). It is \( d' = d \cos \phi \) and \( \tan \psi' = \tan \psi / \cos \phi \), from which \( r = d' \sin \psi' = d \tan \psi \cos \psi' \), and hence

\[
r = d \frac{\tan \psi \cos \phi}{\sqrt{\tan^2 \psi + \cos^2 \phi}}. \tag{2}
\]

In addition, from Fig. 5, it is \( d = r (\cos \theta / \cos \phi + \sin \theta / \tan \beta) \), and therefore

\[
r = d \frac{\sin \beta \cos \phi}{\sin \beta \cos \theta + \cos \beta \sin \theta \cos \phi}, \tag{3}
\]

and, from \( d \beta \sin \beta = r \sin \theta \),

\[
d = d \beta \frac{\sin \beta + \cos \beta \tan \theta \cos \phi}{\tan \theta \cos \phi}. \tag{4}
\]

Combining Eqs. (2) and (3), we obtain

\[
1 + \cot \beta \tan \theta \cos \phi = \frac{\sqrt{\tan^2 \psi + \cos^2 \phi}}{\tan \psi} \sqrt{1 + \tan^2 \theta},
\]

which leads to a two-degree equation on \( \tan \theta \), whose solutions are expressed in terms of the unknown \( \phi \):

\[
\tan \theta = \frac{-\tan^2 \psi \cot \beta \pm \sqrt{\cos^2 \phi + \tan^2 \psi \sqrt{\cot^2 \beta \tan^2 \psi - 1}}}{\cot^2 \beta \tan^2 \psi \cos \phi - \cos \phi - \tan^2 \psi / \cos \phi}. \tag{5a}
\]

The value of \( \tan \theta \) can be approximated by the following, simpler expression:
\[ \tan \theta \approx \frac{\tan \beta}{\sqrt{\sin^2 \psi - \tan^2 \beta}} , \]  

which can be derived from \( d \tan \beta \approx r \sin \theta \approx d \sin \psi \sin \theta \).

To apply the reflection equation (1), the vector \( \mathbf{u} \) orthogonal to the normal of the cylinder surface through \( B \) has to be expressed as a function of \( \phi \) and \( \theta \):

\[ \mathbf{u} = \begin{bmatrix} 1 & 0 & \cos \theta \\ 0 & \cos \phi & \sin \theta \cos \phi \\ 0 & \sin \phi & \sin \theta \sin \phi \end{bmatrix} . \]
The direction of the vector \( \overrightarrow{OB} \) is \([\sin \beta \cos \beta 0]^T\). Now the reflection equation can be applied:

\[
\begin{align*}
\frac{d_B}{\overrightarrow{OL}_1} &= \frac{\overrightarrow{L}_1^2 - (x + y \cos \phi \tan \theta + z \sin \phi \tan \theta)^2}{2(x \sin \beta + y \cos \beta - (x + y \cos \phi \tan \theta + z \sin \phi \tan \theta) / (\sin \beta + \cos \beta \cos \phi \tan \theta)).}
\end{align*}
\]

From Eq. (4), by recalling that \(1 / \cos \phi = \sqrt{1 + \tan^2 \phi}\), we can find the expression of \(d\) in terms of the unknown \(t = \tan \phi\):

\[
\begin{align*}
\frac{d}{\overrightarrow{OL}_1} &= \frac{\overrightarrow{L}_1^2}{(x_1 \sin \beta_1 + y_1 \cos \beta_1)(1 + z_1 t)} - (x_1 \cot \theta_1 \sqrt{1 + t^2 + y_1 + z_1 t})^2
\end{align*}
\]

and equating the right-hand sides of the two, an equation in the unknown \(t\) is obtained.

If an approximate value of \(\theta_1\) and \(\theta_2\) is used, which are independent of the unknown \(\phi\), then \(d_{B_1} = d_{B_2}\) yields a six-degree equation on \(\tan \phi\). This condition can be satisfied by using an iterative method in which, at each step, the expression (5a) of \(\theta_i\) in terms of the unknown \(\phi\) is substituted by evaluating it at the previously calculated value of \(\phi\). At the first step, where no current value of \(\phi\) is available yet, the simplified relation (5b) is used.

**Cylinder-orthoperspective.**

1. for \(i = 1 \) to \(2\) do \(\cot \theta_i \leftarrow \sqrt{\sin^2 \psi_i - \tan^2 \beta_i} / \tan \beta_i;\)
2. solve system (7a) and (7b) wrt \(t\) and select the best solution for \(\phi\);
3. repeat
4. substitute the value of $\phi$ in (5a) to calculate $\tan \theta_1$ and $\tan \theta_2$;
5. find a new solution of (7a) and (7b) for $r$;
6. until the new solution differs from the previous one by less than a given threshold;
7. find $d$ from (5a);
8. find $r$ from (7a) and (7b);
9. find $\kappa$ from (2);
10. end.

5. The cone

As usual, the projection of the cone axis is given by the intersection between the image plane and the plane, that bisects the two interpretation planes of the lateral contour lines. This plane through $O$ contains the cone axis.

The knowledge of the two interpretation planes of the lateral contours allows to determine: (i) one of the two parameters representing the axis orientation, (ii) the direction of the line joining the viewpoint $O$ to the cone vertex $A$ and (iii) the relationships between the distance $d$ between $A$ and $O$ and the cone semiaperture $\alpha$. Two further parameters need to be determined, namely the second parameter representing the axis orientation and, e.g., the distance $d$. Since the reflection curve from a point light source is a straight line constrained to go through the cone vertex $A$ (Observation 1), each reflection yields one equation. Therefore, the reflection from two light point sources is needed to determine the position of $A$, the axis orientation and the aperture angle.

A reference is adopted, specified as follows: the $y$-axis joins the viewpoint $O$ and (the projection of) the cone vertex $A$, the $z$-axis is in the plane containing (the projection of) the cone axis. Let $\psi$ the angle between this plane and one of the two interpretation planes of the lateral contour lines.

The orientation of the cone axis is specified by the angle $\phi$ it forms with the $z$-axis. Among the infinite normals to the cone axis, the one contained in the plane containing $O$ and the axis is given by $n = [0 \hspace{5pt} -\cos \phi \hspace{5pt} -\sin \phi]^T$.

Since (under orthoperspective projection) any reflection line goes through the vertex $A$, we can use a reflection point $B$ at an infinitesimal distance from $A$ along the reflection line. Let $[\lambda \hspace{5pt} 1 \hspace{5pt} \lambda/\mu]^T$ be a vector along the direction $OB$. This direction is known from the image of $B$: $\lambda$ is infinitesimal, while $\mu$ represents the direction of the image of the reflection line. The distance $d_B$ between $O$ and $B$ coincides with the distance $d$ between $O$ and the vertex $A$ (see Fig. 6(a)).

Let $n$ be the normal to the cone axis, joining $B$ to the axis, and let $\theta$ the angle it forms with $n_a$:

$$n = \cos \theta \begin{bmatrix} 0 \\ -\cos \phi \\ -\sin \phi \end{bmatrix} + \sin \theta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin \theta \\ -\cos \theta \cos \phi \\ -\cos \theta \sin \phi \end{bmatrix}.$$  

The direction orthogonal to $n$ and to the cylinder axis is given by

$$u = \cos \theta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \sin \theta \begin{bmatrix} 0 \\ \cos \phi \\ \sin \phi \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \cos \phi \\ \sin \theta \sin \phi \end{bmatrix}.$$  

The problem is to determine $d$, $r$ and the cone semi-aperture $\alpha$ from the apparent semi-aperture $\psi$, the position of the point light sources $L_1 = (x_1, y_1, z_1)$ and $L_2 = (x_2, y_2, z_2)$, and the directions of the images of the reflection lines. These directions are characterized by $\mu_1$ and $\mu_2$.

First, we express the unknown semi-aperture $\alpha$ as a function of the unknown $\phi$. The normal to the interpretation plane of the image of the (right-hand side) occluding contour is (see Fig. 6(b))
\[ n_r = \begin{bmatrix} \cos \psi & 0 & \sin \psi \end{bmatrix}^T. \] The plane containing the axis and the tangent to the cone through the above occluding contour has its normal given by

\[ n_{al} = a \times n_r = \begin{bmatrix} -\sin \psi \sin \phi \\ \cos \psi \cos \phi \\ \cos \psi \sin \phi \end{bmatrix}. \]

The direction of the above occluding contour is given by the intersection between its interpretation plane and the plane normal to \( n_{al} \); therefore it can be found by the vector product of the normals to these planes.
The semi-aperture \( \alpha \) is given by
\[
\cos \alpha = \frac{a \cdot l}{|l|} = \sqrt{\sin^2 \phi + \cos^2 \phi \cos^2 \phi} = \sqrt{\sin^2 \phi \sin^2 \phi + \cos^2 \phi},
\]
and therefore \( \sin \alpha = \sin \psi \cos \phi \), from which
\[
\tan \alpha = \frac{\cos \phi}{\sqrt{\sin^2 \phi + \cot^2 \phi}}.
\] (8)

The unit vector along the cone axis is \( \mathbf{a} = [0 \quad \sin \phi \quad -\cos \phi]^T \). The vector from \( O \) to the reflection point \( B \) is given by
\[
\mathbf{OB} = d \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + ka + k \tan \alpha \mathbf{n} \right) = d \begin{bmatrix} k \tan \alpha \sin \theta \\ 1 + k(\sin \phi - \tan \alpha \cos \phi \cos \theta) \\ -k(\cos \phi + \tan \alpha \sin \phi \cos \theta) \end{bmatrix},
\] (9)
where \( k \) is a constant. The determination of \( k \) is not necessary, since the reflection point \( B \) is at an infinitesimal distance from the vertex \( A \).

Now, using Eq. (8), \( \theta \) is expressed in terms of the unknown \( \phi \), by equating \( \mu \) to the ratio between the first and third component of \( \mathbf{OB} \):
\[
\mu = -\frac{\sin \theta}{\sin \phi \cos \theta + \sqrt{\sin^2 \phi + \cot^2 \phi}} = -\frac{\tan \theta}{\sin \phi + \sqrt{\sin^2 \phi + \cot^2 \phi} \sqrt{1 + \tan^2 \theta}}.
\]

Solving this equation with respect to \( \tan \theta \) yields two solutions:
\[
\tan \theta = -\mu \frac{\sin \phi \pm \sqrt{(\sin^2 \phi + \cot^2 \phi)(1 - \mu^2 \cot^2 \phi)}}{1 - \mu^2 (\sin^2 \phi + \cot^2 \phi)}.
\] (10)

Now the reflection equation (1) can be applied. Let \( [0, 1, 0]^T \) be the unit vector along the direction \( \mathbf{OB} \). From the reflection equation:
\[
d = d_B = \frac{\mathbf{OB}^2 - (x \cot \theta + y \cos \phi + z \sin \phi)^2}{2(y - (x \cot \theta + y \cos \phi + z \sin \phi)/\cos \phi^2}.
\]
Simplifying it and rewriting it for both reflections reduces to
\[
d = d_{B_i} = \frac{(x_i \cot \theta_i \sqrt{1 + \tan^2 \phi + y_i + z_i \tan \phi})^2 - \mathbf{OB}^2}{2(x_i \cot \theta_i \sqrt{1 + \tan^2 \phi + z_i \tan \phi})}, \quad i = 1, 2.
\] (11)

If an approximate value of \( \theta_i \) is used, which is independent of the unknown \( \phi \), then \( d_{B_1} = d_{B_2} = d_B \) yields a six-degree equation on \( \tan \phi \). This condition can be satisfied by using an iterative method in which, at each step, the (inverse of) expression (10) of \( \theta_i \) in terms of the unknown \( \phi \) is substituted by evaluating it at the previously calculated value of \( \phi \). At the first step, where no current value of \( \phi \) is available yet, the following simplified relation is used:
\[
\cot \theta_i \simeq -\frac{\sqrt{1 - \mu_i^2 \cot^2 \psi}}{\mu_i \cot \psi}, \quad i = 1, 2.
\]
This approximate relation has been derived by rewriting (10) for $\phi = 0$.

Cone-orthoperspective.
1. for $i = 1$ to 2 do $\cot \theta_i = -\sqrt{1 - \mu_i^2 \cot^2 \psi_0} / (\mu_i \cot \psi_0)$;
2. solve system $d_{B_1} = d_{B_2}$ (11) wrt $\tan \phi$ and select the best solution for $\phi$;
3. repeat
4. substitute the value of $\phi$ in (10) to calculate $\cot \theta_1$ and $\cot \theta_2$;
5. find a new solution of $d_{B_1} = d_{B_2}$ (11) for $\tan \phi$;
6. until the new solution differs from the previous one by less than a given threshold;
7. calculate $a$ from $\sin a = \sin \psi_0 \cos \phi$;
8. end.

6. Implementation and experimental results

We implemented a system aimed at recovering pose and geometry of metallic conic and cylindric parts, which have been subject to lathe turning or grinding. The experimental setup is shown in Fig. 7. Each of the two point light sources is realized by putting a lamp in an empty cylinder made of heavy paper. The cylinder height is 45 cm. This cylinder presents eight small holes (diameter 5 mm) at different heights (one hole each 5 cm): only one of these holes is open, while the other seven are closed. In this way a single point source is obtained for each lamp. By changing the open hole, the height of the light source is varied.

The acquired image is processed in order to extract both the outlines of the lateral surface of the object and the reflections of the point light sources on the object surface. The obtained edges, extracted by means of a Canny edge detector, are then segmented into straight line segments. This process can produce many lines, which do neither belong to the object outline nor constitute the reflecting line of any light point source (think of, e.g., shadows). In order to eliminate such spurious lines, first the parallelism of the line segments is analyzed: if a sufficient number of parallel lines is found, these are supposed either to be part of the image of the cylinder outline or to constitute the reflecting lines. In this case, those lines which are not parallel to the above lines are removed from further consideration, and the two lines whose distance inbetween is
maximum are retained as those constituting the image of the cylinder outline, while the other parallel lines are retained as associated to the reflection lines.

Fig. 8. Some real objects.
If a sufficient number of parallel lines is not found, then the concurrence of the line segments is examined through a voting scheme. If a sufficient number of concurrent lines is found, the remaining lines are re-
moved from further consideration. Among the concurrent lines, those two which form the largest angle are retained as those constituting the image of the object outline. The residual lines are retained as associated to the reflection lines.

Often, a pair of neighbouring edges is associated with each reflection line. In order to locate the reflecting line with good accuracy, a nonmaxima suppression is performed within the image region between the two neighbouring edges, based on the intensity level: a pixel is retained as belonging to the reflection line if its intensity is a relative maximum along the direction perpendicular to the direction of the line bisecting the two neighbouring edges.

Once the lateral contour lines and the reflection lines have been determined, one of the two methods presented in Sections 4 and 5 is applied, according to whether the four lines are parallel or not.

Observation 2. If the application of the method in Section 5 to a set of four concurrent lines results in an aperture angle $\alpha \simeq 0$, then the analyzed object is recognized as a cylinder, whose view has been taken according to a perspective projection (instead of an orthoperspective one).

In the sequel some experimental results are reported, showing the performance of the implemented system on real images.

A set of real images is shown in Fig. 8. The detected edges, including contours and reflection lines, are shown in Fig. 9. The estimated geometric parameters of the objects shown, together with their position and orientation, are reported in Table 1, where they are compared to the real parameters.

Notice that the error in the estimation of the features are within 5%. This shows the validity of the axial symmetric reflection model for real objects as those used in the experiments.

### 7. Conclusions

A new reflection model is introduced to account for surface treatments, that introduce a raggedness which preserves the pre-existing axial symmetry. The reflection equation according to the introduced model are exploited to derive a method for recovering the geometry of cylindrical and conic surfaces, which underwent the above described surface treatments, from a single view.

This method uses the image of the lateral occluding contours of the surface, and the reflection from two point light sources in known positions.

---

**Table 1**

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<th>Zenith</th>
<th>$x_A$</th>
<th>$y_A$</th>
<th>$z_A$</th>
<th>$d$</th>
<th>$r$</th>
<th>$z$</th>
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<td>18</td>
<td>95</td>
<td>1.5</td>
</tr>
<tr>
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<td>30</td>
<td>$-0.1$</td>
<td>81</td>
<td>$18.6$</td>
<td>94</td>
<td>1.9</td>
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<tr>
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<td>70</td>
<td>0</td>
<td>85</td>
<td>18</td>
<td>90.9</td>
<td>1.83</td>
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<td>76</td>
<td>16</td>
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<td>1.83</td>
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</table>
Some experimental results on real images are reported, showing the accuracy of the method and the appropriateness of the introduced reflection model.

Future research directions involve the application of extended versions of the qualitative reflection model to recover (metallic) solids of revolutions from single views.

References


