Solutions of Problems on Probability theory

Chapter 1

P.1.1 Rolling three dices, evaluate the probability of having k equal faces, with $k \in [0;2;3]$.

Solution.

$$P(0) = \frac{(6)_3}{6^3} = 0,\overline{5} \qquad P(2) = \frac{6 \cdot 3 \cdot 5}{6^3} = 0,41\overline{6} \qquad P(3) = \frac{6}{6^3} = 0,02\overline{7}$$

P.1.2 Rolling a dice three times, evaluate the probability of having at least one 6.

Solution.

$$p = 1 - \frac{5^3}{6^3} = 0,42129..$$

P.1.3 Assuming women and men exist in equal number, and assuming that 5% of the men are colour blind and that 0,25% of the women are colour blind, evaluate the probability that a person drawn at random is colour blind. Then evaluate the probability that, having drawn a colour-blind person, this is a male.

Solution. By the total probability theorem

$$P(D) = \frac{1}{2}\frac{5}{100} + \frac{1}{2}\frac{0.25}{100} = 0.02625$$

Using Bayes Theorem

$$P(M|D) = P(D|M)\frac{P(M)}{P(D)} = \frac{5}{100}\,\frac{1/2}{0.02625} = 0.952$$

P.1.4 Drawn a card from a deck of 52 cards, verify wheter the following events are statistically independent:

a) $A = \{drawing\ of\ a\ picture\ card\};\qquad B = \{drawing\ of\ a\ hearth\ card\}$

b) What if the king of hearths is missing from the deck of cards?
c) What if a card, at random, is missing?

Solution. a) independent b) not independent c) of the women

P.1.5 A dice A has four red faces and two white faces. A dice B, vice-versa, has two red faces and four white faces. You flip a coin once, if heads the game continues with dice$A$, otherwise it continues with dice B. a) On rolling the dice, what is the probability that a red face appears on the dice? b) and at the second rolling of the same dice? c) If the first two rollings show a red face, what is the probability that also on the third rolling is red? d) If the first n rollings show a red face, what is the probability that you are using dice A?

Solution.

$$a)\ \frac{1}{2} \qquad b)\ \frac{1}{2} \qquad c)\ \frac{3}{5} \qquad d)\ \frac{2^n}{1+2^n}$$

P.1.6 An urn contains two white balls and two black. A ball is drawn and replaced with a ball of a different colour. Then a second ball is drawn. Calculate the probability p that the first extracted was white, when the second is white.

Solution: 1/4

P.1.7 The probabilities that three different archers, A, B hit the mark, independently of one another, are respectively 1/6, 1/4 and 1/3. Everyone shoots an arrow. a) Find the probability that only one hits the mark. b) If only one hits the mark, what is the probability he is archer A?

Solution:

$$a)\quad \frac{31}{72} \qquad b)\quad \frac{6}{31}$$

P.1.8 A duel among three people A, B and C is carried out according to the Russian roulette. A six round revolver is loaded with two cartridges. The duelists pass cyclically the weapon, spinning the cylinder every time (so that each duelist has 1/3 probability of being on a loaded chamber) and shooting themselves as long as only one remains alive. Assuming that A is the first, what is the probability that each duelist is the first to die? b) and to win?

Solution:

1. $$p_A = \frac{9}{19} \qquad p_B = \frac{6}{19} \qquad p_C = \frac{4}{19}$$
2. $$p_A = \frac{56}{209} \qquad p_B = \frac{69}{209} \qquad p_C = \frac{84}{209}$$

   You have at least two ways to get at the result. The one is to evaluate all the sequence. For example, if p_A is the probability that A dies first we have

   $$p_A = 1/3 + (2/3)^3(1/3) + (2/3)^6(1/3) + ... = (1/3)\sum_{i=0}^{\infty}(8/27)^i = 9/19$$

   The second one is to observe that

   $$p_B = (2/3)p_A; \qquad p_C = (2/3)^2 p_A.$$

   Since we have also

   $$p_A + p_B + p_C = 1$$

   this provide an equation whose solution is p_A, and the others follow.

P.1.9 From a deck of 52 cards we draw two cards. Find the probabilities of the following events $A = \{$ first card is a King; the second figure$\} = \{K_1; F_2\}$ $\quad B = \{$ at least one figure$\}$

Solution: We can count favorable outcomes in the space of outcomes. Alternatively, we can use the conditional probability definition:

$$p(A) = P(K_1; F_2) = P(K_1)P(F_2|K_1) = \frac{4}{52}\,\frac{11}{51} = 0,0165\ldots$$

$$p(B) = \frac{12}{52} + \frac{12}{52} - \frac{12}{52}\,\frac{11}{51} = 0,411\ldots$$

Chapter 2

P.2.1 Given the function $f(x) = \dfrac{C}{\alpha^2 + x^2}$, determine the relationship between C e α in order to make $f(x)$ a pdf. (Cauchy). (3.1)

Solution. $\pi C = \alpha$

P.2.2 A point P uniformly chosen in a square of Side L centered at the origin and the x-axis. Find the pdf of RV X, coordinate of the orthogonal projection of P on the horizontal axis.

Solution.

$$f_X(x) = \frac{1}{L}, \qquad -L/2 \le x \le L/2$$

P.2.3 A point P uniformly chosen in a circle of radius R centered at the origin and the x-axis. Find the pdf of RV X, coordinate of the orthogonal projection of P on the horizontal axis.(3.2)

Solution.

$$f_X(x) = \frac{2}{\pi R}\sqrt{1 - (\frac{x}{R})^2}, \qquad -r/2 \le x \le R/2$$

P.2.4 Find the first order moment of pdf $f(x) = \lambda^2 x e^{-\lambda x}, x \ge 0$, and 0 elsewhere.

Solution. $2/\lambda$.

P.2.5 Find the first order moment of the integer distributions

1. $P(X = k) = (1-p)^{k-1}p, \quad k \ge 1;$
2. $P(X = k) = (1-p)^k p, \quad k \ge 0.$

Solution. $1/p$, and $(1-p)/p$.

P.2.6 2 points are chosen uniformly and independently in a segment of length L. Find the pdf of RV X distance to the origin of the point closest to the origin. Find the joint pdf of (X, Y) where Y is distance to the origin of the point farthest to the origin. Extend the result to the case of n points.(3.6)

Solution.
As usual

$$P(x < X \le x + \Delta x) = P(\text{one point in } \Delta x\text{; the other beyond x})$$

We get

$$f_X(x) = \frac{2}{L}\frac{L-x}{L}, \qquad 0 \le x \le L$$

For the joint pdf we have

$$P(x < X \le x + \Delta x;\ y < Y \le y + \Delta y) = 2\frac{\Delta x}{L}\frac{\Delta y}{L}, \qquad x < y$$

$$f_{XY}(x, y) = \frac{2}{L^2}, \qquad x < y$$

P.2.7 2 points are chosen uniformly and independently in a circle of radius R. Find the pdf of RV X distance to the center of the point closest to the center.(3.8)

Solution.

$$f_X(x) = 2n \, \frac{x}{R^2} \left[1 - (\frac{x}{R})^2\right]^{n-1} \qquad (0 \leq x \leq R)$$

P.2.8 Take a number X from one to six, throw three dices. You win C if X appears once, $2C$ if X appears twice, $3C$ if it appears three times, and you lose C if X does not appear. Check whether this is a fair game. (3.12)

Solution.
The number of possible outcomes is $6^3 = 216$. X appears once with probability $3 \times 25/216$, twice with probability $3 \times 5/216$, and thrice with probability $1/216$. The probability of winning is the sum, i.e, $91/216$; therefore your loss is on the average $(125/216)C$. On the other side, if your win is, on the average,

$$(75/216)C + (15/216)2C + (1/216)3C = (108/216)C$$

Then on the average you loose $(17/216)C$ at each bet.

P.2.9 Assume the RV X, lifespan of a component, is uniform in $[0; L]$. We know that the component age is z; find the pdf of its lifespan. Find the pdf of Y, remaining lifespan.

Solution.
We look for

$$f_X(x|\text{age=z}) = f_X(x|X > z) = \lim \frac{P(x < X \leq x + \Delta x; X > z)}{\Delta x P(X > z)} =$$

$$= \lim \frac{P(x < X \leq x + \Delta x)}{\Delta x P(X > z)}, \qquad x > z$$

Therefore

$$f_X(x|X > z) = \frac{f_X(x)}{P(X > z)} = \frac{1}{L - z}, \qquad z \leq x \leq L$$

Then

$$f_Y(y) = f_X(y + z|X > z) = \frac{1}{L - z}, \qquad 0 \leq x \leq L - z$$

P.2.10 Repeat the previous exercise assuming that the pdf of X is negative exponential. Find the fair amount a a customer of age z must pay to get a capital C if he dies before the year.

Solution.
As in the previous case we have

$$f_X(x|X > z) = \frac{f_X(x)}{P(X > z)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda z}}, \qquad x > z$$

$$f_Y(y) = f_X(y+z|X>z) = \frac{\lambda e^{-\lambda(y+z)}}{e^{-\lambda z}} = \lambda e^{-\lambda y}, \qquad x>0$$

Note that the remaining life span has still the same pdf as the original lifespan. This is because the negative exponential is "memoryless".

The probability of dying within a year is $P(X<1) = 1-e^{-\lambda}$. The fair amount a is such that

$$ae^{-\lambda} = C(1-e^{-\lambda})$$

P.2.11 Check whether functions of x and y below can represent joint pdfs' and if so check whether X and Y are statistically independent. (5.1)

1. $f(x,y) = 4xy \quad (0 \le x \le 1;\ 0 \le y \le 1)$,
2. $f(x,y) = 8xy \quad (0 \le x \le y;\ 0 \le y \le 1)$,
3. $f(x,y) = 4x^2y \quad (0 \le x \le 1;\ 0 \le y \le 1)$

P.2.12 A person in phone booth makes a phone call whose duration is represented by RV X, with negative exponential pdf with mean value $1/\mu$. A second person comes after a time Y. RV negative exponentially with average $1/\lambda$, independent of X. Find the pdf of RV W, the time the latter has to wait to the end of the call. (5.6)

Solution.
If $Y>X$ the second person arrives when the first has already finished his phone call and, therefore $W=0$. On the other side, we take the condition $Y=y, Y<X$. RV W is then "the remaining lifespan" of Problem 2.10. By this problem we have learned that, with the negative exponential pdf, the remaining lifespan has the same pdf. Therefore

$$f_W(w|Y=y;Y<X) = \mu e^{-\mu w}, \qquad y \ge 0$$

that doesn't depend on y. The, using the total probability Theorem

$$f_W(w|Y<X) = \mu e^{-\mu w}, \qquad y \ge 0$$

Finally

$$f_W(w) = \delta(w)P(Y>X) + \mu e^{-\mu w}P(Y<X), \qquad y \ge 0$$

See Problem 2.13 to see $P(Y>X)$.

P.2.13 Given two independent RVs' X, and Y, find the probability of the event $\{Y \le X\}$ when

1. f_X, f_Y are uniform within intervals respectively $[-1;3], [0;4]$;
2. f_X, f_Y with the same pdf (you do not need to know the pdf);
3. f_X, f_Y are negative exponentials with parameters λ and μ;

What about event $\{Y \leq X/2\}$?

Solution.
The first way finds event $\{Y \leq X\}$ in the plane x, y, and then integrates the joint pdf in such event. Case 1) is very simple since the joint distribution is uniform and the integration takes the volume of the regular solid where the base is the area of the event. The area of the vent is the portion of the rectangle of base $[-1; 3]$ and height $[0; 4]$ that lies beneath straight line $y = x$. The height of the pdf is 1/16 .

The second way uses conditioning.

$$P(Y \leq X) = \int P(Y \leq X|X = x) f_X(x) dx$$

In the first case we have

$$P(Y \leq X|X = x) = x/4, \qquad 0 \leq x \leq 3$$

$$P(Y \leq X|X = x) = 0, \qquad -1 \leq x \leq 0$$

$$P(Y \leq X) = \int_0^3 x/4 \; 1/4 dx = 9/32$$

In case 2, since both RVs' obeys the same law, outcomes $X > Y$ or $Y > X$ are equally probable.

In case 3 we use conditioning again.

$$P(Y \leq X) = \int P(Y \leq X|Y = y) f_Y(y) dy = \int_0^\infty P(X \geq y|Y = y) f_Y(y) dy =$$

$$= \int_0^\infty e^{-\mu y} \lambda e^{-\lambda y} dy = \frac{\lambda}{\lambda + \mu} \int_0^\infty (\lambda + \mu) e^{-(\lambda+\mu)} dy = \frac{\lambda}{\lambda + \mu}$$

P.2.14 Find the pdf of RV $Z = \min(X, Y)$, where X and Y are two independent negative exponential RVs' with parameters λ and μ respectively. (Hint: observe that $\min(X, Y) > z$ if $x > z$ and $Y > z$. Also, we may take the condition $Y = y$...)

Solution.
The suggestion says

$$P(Z > z) = P(\min(X, Y) > z) = P(X > z; Y > z) = P(X > z)P(Y > Z) = e^{-\lambda z} e^{-\mu z}$$

Therefore

$$P(Z > z) = e^{-(\lambda+\mu)}$$

or

$$f_Z(z) = (\lambda + \mu) e^{-(\lambda+\mu) z}, \qquad z \geq 0$$

Take notice: The minimum of two negative exponential RVs is again a negative exponential RV with parameters sum of parameters.

P.2.15 Take interval $[0, X]$, where X is a RV Erlang-2. Then take a point P uniformly within the preceding interval. Find the pdf of Y, length of $\overline{0P}$.

Solution.
We use the Total Probability Theorem

$$f_Y(y|X=x) = \frac{1}{x}, \qquad y \leq x$$

$$f_Y(y) = \int f_Y(y|X=x) f_X(x)dx = \int_y^\infty \frac{1}{x}\lambda^2 x e^{-\lambda x}dx = \lambda \int_y^\infty \lambda e^{-\lambda x}dx =$$

$$= \lambda(1 - F_X(y) = \lambda e^{-\lambda x}$$

The solution could be expected. Why?

P.2.16 n points are uniformly taken within $[0;T]$. Find the probability that k out of n point lie within an interval $[0;X]$ where RV X is uniform in $[0;T]$.

Solution.
The probability can be written as:

$$P(N(X) = k/N(T) = n) = \int_0^T P(N(X) = k/N(T) = n, X = x) f_X(x)dx =$$

$$= \int_0^T \binom{n}{k} \left(\frac{T-x}{T}\right)^{n-k} \left(\frac{x}{T}\right)^k \frac{1}{T}dx = \frac{1}{n+1}$$

It seems strange that the solution does not depend on k. This becomes apparent if we solve the problem in this other way. The extreme of interval X is itself a uniform point in $[0, T]$, exactly as the others n. Therefore, the sought probability is the probability that this boundary point lies the $k+1$-th position out of $n+1$. But all positions are equally probable and therefore the sought probability is $1/(n+1)$.

P.2.17 Two RVs' X and Y are independent and uniformly distributed in $[0;1]$. Find $f_X(x|X>Y)$, $f_{XY}(x,y|X>Y)$ and $P(X>2Y|X>Y)$.

Solution.

$$f_X(x|X>Y) = P(X>Y|X=x)\frac{f_X(x)}{P(X>Y)} = x\frac{1}{1/2} = 2x, \qquad 0 \leq x \leq 1$$

$$f_{XY}(x,y|X>Y) = P(X>Y|X=x, Y=y)\frac{f_{XY}(x,y)}{P(X>Y)} = 2 \qquad x > y$$

Since the above is uniform in $0 \leq y \leq x \leq 1$, $P(X > 2Y|X > Y)$ is simply the ratio of the areas of events $X > 2Y$ and $X > Y$, equal to $1/2$.

Chapter 3

P.3.1 f_X, f_Y are uniform within intervals respectively $[0;5]$ $[-3,-1]$. Find the pdf of RVs' (6.1)

1. $Z = X + Y$
2. $W = X - Y$

Solution.

$$f_z(z) = \begin{cases} (1/10)(z+3) & -3 \leq z \leq -1 \\ 1/5 & -1 \leq z \leq 2 \\ (1/10)(4-z) & 2 \leq z \leq 4 \end{cases}$$

$$f_W(w) = \begin{cases} (1/10)(z-1) & 1 \leq w \leq 3 \\ 1/5 & 3 \leq w \leq 6 \\ (1/10)(8-w) & 6 \leq w \leq 8 \end{cases}$$

P.3.2 Let X e Y be independent RVs' with negative exponential pdfs' and average value $\frac{1}{\lambda}$. Find the pdf of RVs (6.2)

1. $Z = X - Y$
2. $W = X + \dfrac{Y}{2}$

Solution.

$$a) \quad f_Z(z) = \frac{1}{2}\lambda e^{-\lambda|z|} \quad \text{(Laplace)}$$

$$b) \quad f_W(w) = 2\lambda(e^{-\lambda z} - e^{-2\lambda z}) \qquad (z > 0)$$

P.3.3 Find $P(Z = n)$ where $Z = X + Y$ is the sum of the numbers that appear in the rolling of two dices. (6.6) **Solution**.

$$P(Z = n) = \begin{cases} \dfrac{n-1}{36} & (2 \leq n \leq 7) \\[2ex] \dfrac{13-n}{36} & (7 \leq n \leq 12) \end{cases}$$