

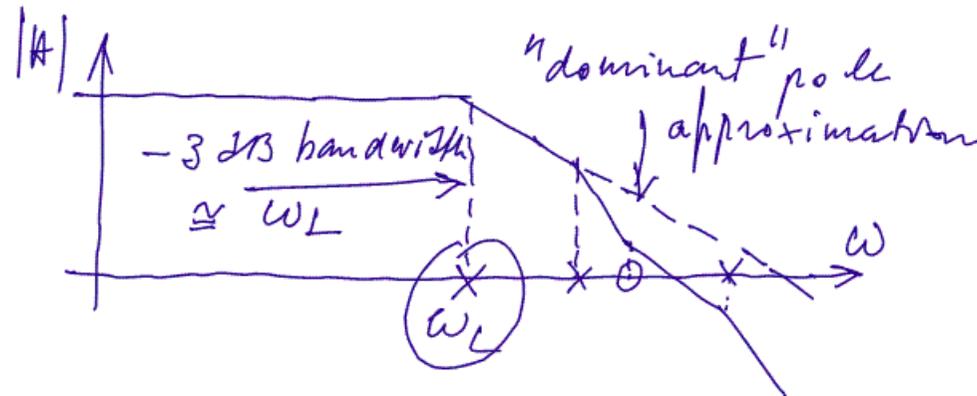
Approximate methods for poles/zeros computation

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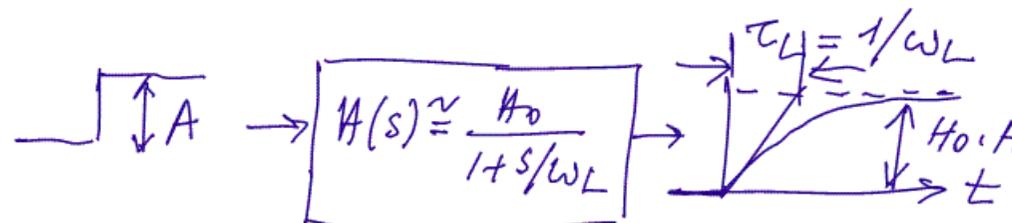
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Dominant pole approximation

- The response of a circuit to any excitation source can be computed from the knowledge of the transfer function $H(s)=V_{out}/V_{in}$ which is defined by its singularities (poles and zeros).
- Take the example of an amplifier with nearly constant amplification from DC to a maximum frequency:



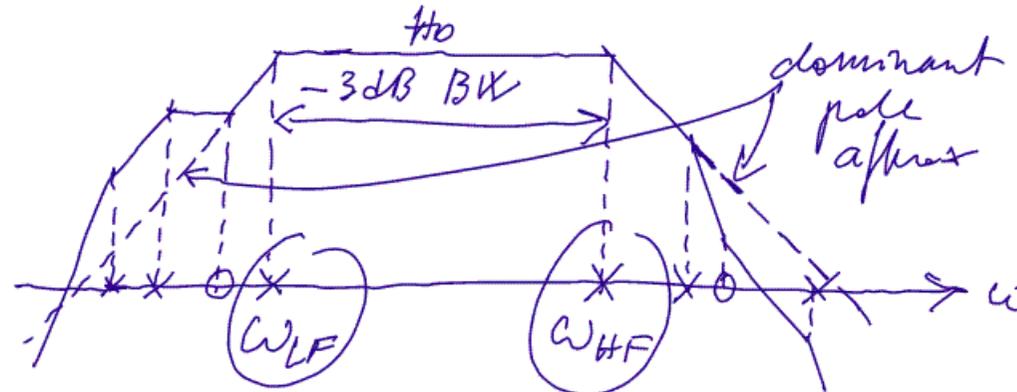
- Although all singularities contribute to the response of the system, it is clear that the pole at *lowest* frequency (ω_L) gives a first-order approximation of the -3dB bandwidth of the amplifier. In order to estimate the main properties of the amplifier we can compute the system response on the basis of the single-pole model shown in the figure with dashed line (*dominant pole approximation*). The farther (or more separated) are the rest of the singularities from the dominant pole, the better this approximation.



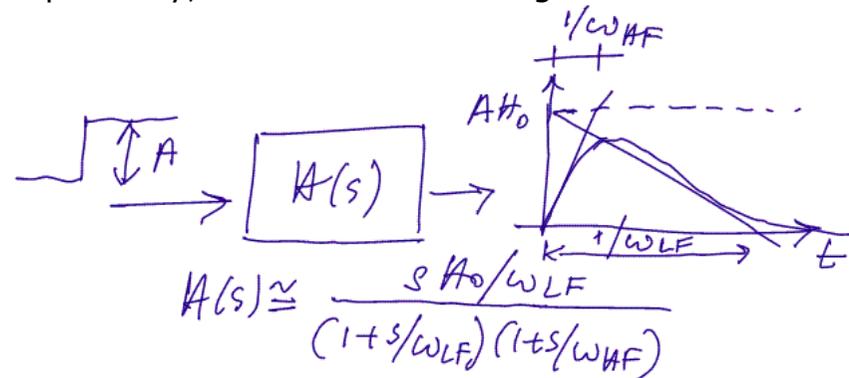
- The estimated response of this amplifier to an input step is the well-known exponential curve with time constant $\tau_L=1/\omega_L$. It must be reminded that in case of a "dominant" zero (i.e. occurring before ω_L , the -3dB bandwidth is no more given by ω_L .

Dominant pole approximation (2)

- This further example shows an amplifier with nearly constant amplification between two corner frequencies (e.g. an amplifier with AC-coupled input). The low-frequency (LF) corner is defined by the group of singularities at LF, while the high-frequency (HF) corner is defined by the group of singularities at HF.



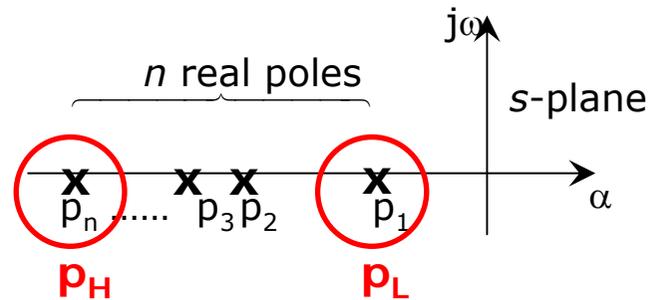
- It is often straightforward to identify the group of (larger-value) capacitances giving rise to the LF singularities and the (smaller-value) capacitances giving rise to the HF singularities. Therefore if we assume that the 2 groups of singularities are well separated in frequency (i.e. by several decades), then we can split the analysis of the circuit: for the analysis of the LF singularities we set the HF capacitances to open-circuits, while for the analysis of the HF singularities we set the LF capacitances to short-circuits. Among the two groups of singularities, ω_{LF} and ω_{HF} are the "dominant" poles, respectively, for the LF and HF regions.



- As before, we can estimate the response of this amplifier to an input step on the basis of a dominant pole approximation (dashed line).

Method of Time Constants (MTC)

- In general, the complete computation of the frequency response of a multi-stage amplifier with many capacitive elements may become difficult and the exact result can be of little use. Therefore approximate methods have been developed to aid the *design phase*, and computer simulation is used to verify the final design. A method that provides an estimation of the highest/lowest frequency poles in a group of singularities is given below.
- Let us consider a circuit with n capacitors, *independent* and *interacting*, giving rise to n poles.



We can estimate the pole at LF (p_1) and the pole at HF (p_n) with the following expressions:

$$p_L = -\frac{1}{\tau_L} = -\frac{1}{\sum_{i=1}^n \tau_i^0}$$

τ_i^0 is the time constant associated to the i -th capacitance, when all the other capacitances are assumed at zero frequency (i.e. open-circuits)

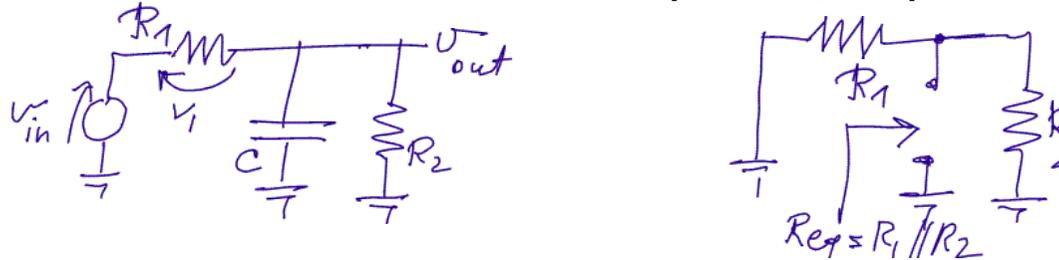
$$p_H = -\frac{1}{\tau_H} = -\sum_{i=1}^n \frac{1}{\tau_i^\infty}$$

τ_i^∞ is the time constant associated to the i -th capacitance, when all the other capacitances are assumed at *infinite* frequency (i.e. short-circuits)

- The level of approximation is better for separated poles $|p_1| \gg |p_2|, |p_3|, \dots \gg |p_n|$
- It should be reminded that the MTC does not give any information on the location of the other intermediate poles and on the presence of zeros (which must be found by direct inspection on the circuit).

Poles/zeros in basic circuits

- In general, the no. of poles of the circuit is equal to the no. of independent reactive elements (i.e. capacitors, inductors). If we restrict to circuits in which the reactive elements are only capacitors (the most frequent case) we can say that the **no. of poles of the circuit is equal to the no. of independent capacitors**. Any transfer function of that circuit will show the same poles. On the contrary, the **no. of zeros** depends on the considered transfer function of the circuit.
- Let us now review some basic case -- **Circuit with 1 capacitor (n=1 pole)**



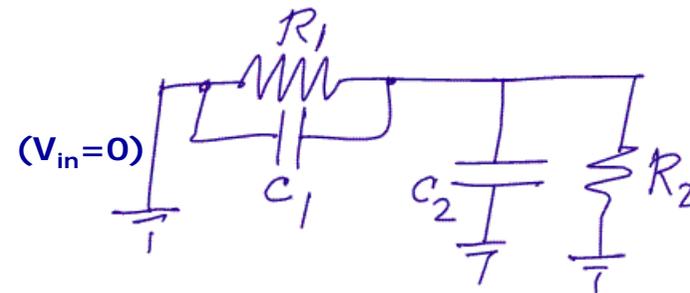
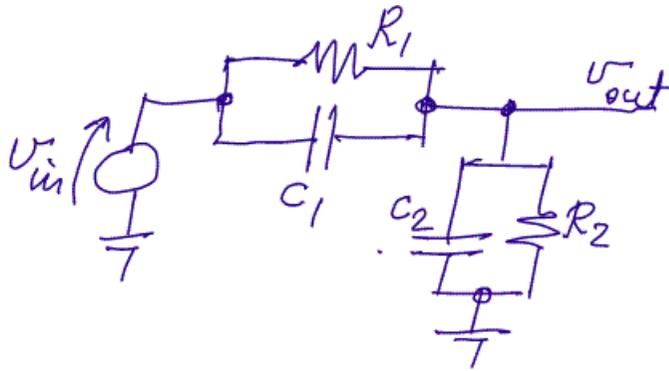
- There is only $n=1$ pole. The time constant associated with the pole is obtained by computing the equivalent resistance "seen" by the capacitance, setting the independent source to zero:
 $\tau_p = C(R_1 // R_2)$ ($\rightarrow p = -1/\tau_p$).
- From the definition, in order to find possible zeros in $H(s) = (V_{out}/V_{in})$ we seek a finite value of s that satisfies the condition $V_{out} = 0$. In the example this occurs only when the impedance of the capacitor is zero that is at infinite frequency. Therefore there is no zero in $H(s)$. Another practical way to seek possible zeros is to evaluate $H(j\omega)$ at $\omega=0$ (C open-circuit) and $\omega=\infty$ (C short-circuit) and deduce the number/location of zero(s).
- The found transfer function is:

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2} \frac{1}{1 + s\tau_p}$$
- Note-- If we take the voltage V_1 as output variable, the transfer function $K(s) = V_1/V_{in}$ will show the same pole as $H(s)$ (once V_{in} is set to zero we have the same circuit), but now we have also 1 zero at $s = -1/(R_2C)$ (in fact $V_1 = 0$ when $Z_2 = R_2 // (1/sC)$ goes to infinity, or $Y_2 = 1/Z_2 = 0$):*

$$K(s) = \frac{V_1}{V_{in}} = \frac{R_1}{R_1 + R_2} \frac{1 + sR_2C}{1 + s\tau_p}$$

Poles/zeros in basic circuits (2)

- Circuit with 2 capacitors, but not independent (n=1 pole)



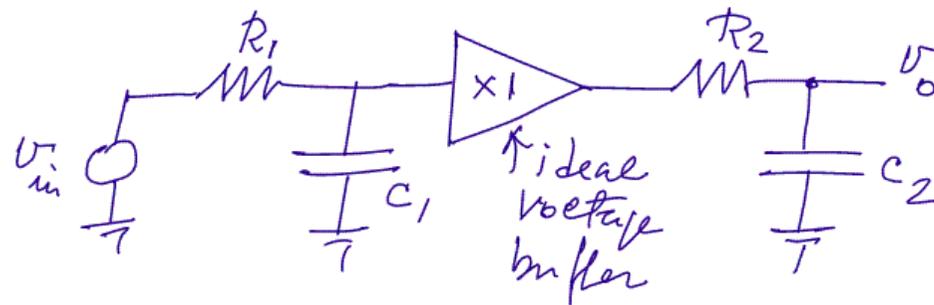
$$\tau_p = (C_1 + C_2)(R_1 // R_2)$$

- When you set the independent source to zero, you immediately realize that C_1 and C_2 are in parallel, therefore not independent(*) and they can be seen as one equivalent capacitor ($C_1 + C_2$). Therefore we have only one pole, with time constant equal to the equivalent capacitance times the equivalent resistance $\tau_p = (R_1 // R_2)(C_1 + C_2)$ ($\rightarrow p = -1/\tau_p$).
- The pair $R_1 - C_1$ gives rise to a zero in the $H(s) = V_{out}/V_{in}$, as $V_{out} = 0$ when $Z_1 = R_1 // (1/sC_1)$ tends to infinity, i.e. for $s = -1/(R_1 C_1)$. The pair $R_2 - C_2$ does not give a zero because $Z_2 = 0$ only for $\omega \rightarrow \infty$.

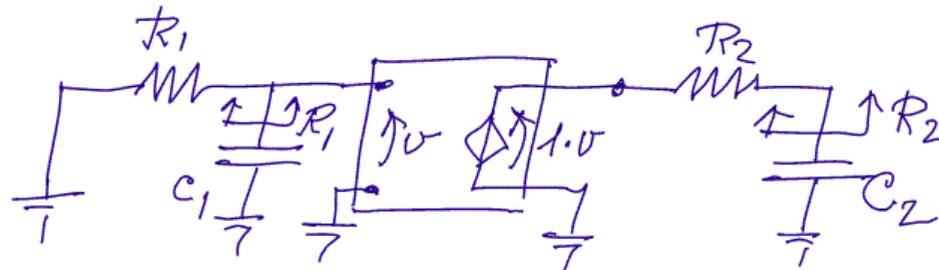
(*) C_1 and C_2 are independent if the voltage (or charge) on each capacitor can be assigned arbitrarily. In this case the KVL imposes $V_1 = V_2$ as C_1 and C_2 are in parallel. Typical cases when C_1 and C_2 are not independent is when they are in parallel or in series.

Poles/zeros in basic circuits (3)

- **Circuit with 2 independent capacitors (n=2)**
- When we have 2 independent capacitors we have n=2 poles.
- To compute the poles we have to check if they are *interacting* or *non-interacting*. Non-interacting means that the equivalent resistance associated to one capacitance is not affected by the other capacitance(s). Let us see it through the following examples.
- **Case of 2 RC's decoupled by an ideal buffer**

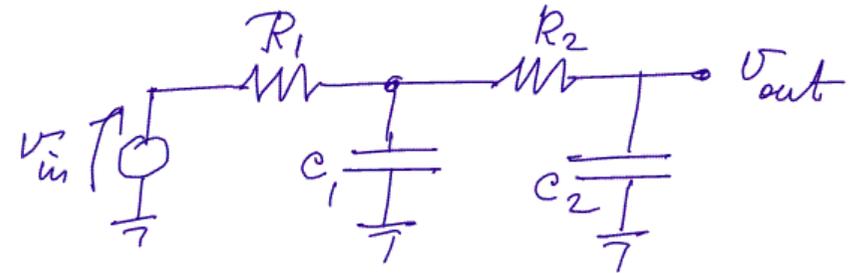


- The caps are clearly *independent* but also *non interacting* because of the ideal voltage buffer. We have two separated subcircuits, therefore each capacitance gives rise to a pole: $p_1 = -1/(R_1C_1)$, $p_2 = -1/(R_2C_2)$. It can be easily checked that the equivalent resistance seen by C_1 is R_1 , irrespective of the state of C_2 (for instance if C_2 is an open-circuit or a short-circuit).

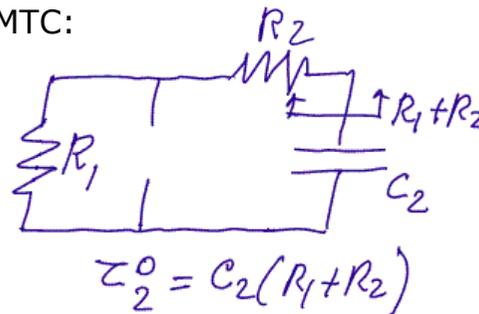
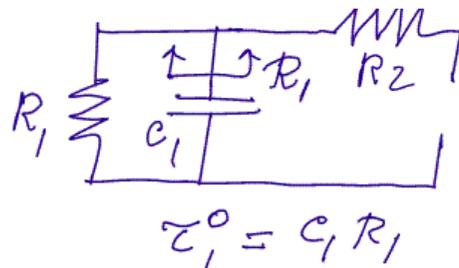


Poles/zeros in basic circuits (4)

- **Case of 2 coupled RC's**
- In this case, due to the presence of R_2 , we can define the voltages on C_1 , C_2 arbitrarily, so they are *independent* (once you define V_1 , the KVL at loop C_1 - R_2 - C_2 still allows any value for V_2). Moreover now the caps are also *interacting* as the equivalent resistance seen by one capacitance is influenced by the other cap. In this case p_1 and p_2 can be estimated with the MTC.

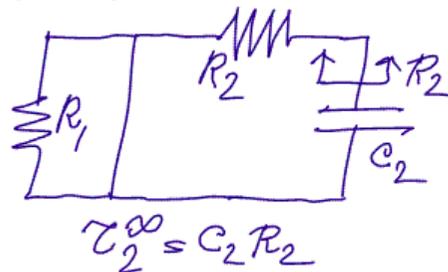
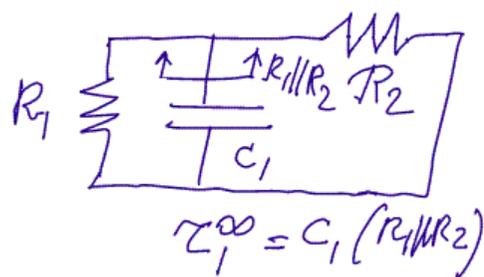


- Computation of $p_L = -(1/\tau_L)$ with MTC:



$$\tau_L = C_1 R_1 + C_2 (R_1 + R_2)$$

- Computation of $p_H = -(1/\tau_H)$ with MTC:



$$\frac{1}{\tau_H} = \frac{1}{C_1 (R_1 // R_2)} + \frac{1}{C_2 R_2}$$

- *Note: the equivalent resistances seen by C_1 at $\omega=0$ and at $\omega=\infty$ are different. The same occurs for C_2 . This is the confirmation of "interacting" capacitances.*
- Finally, we can easily check that there are no zeros in the $H(s) = V_{out}/V_{in}$.

Poles/zeros in basic circuits (5)

- In the case of 2 poles, a special situation that is often encountered is *when one of the time constants is dominant over the other*. For instance, while computing τ_L , you may find that the time constant associated with C2 is dominant:

$$\tau_L = C_1 R_1 + C_2 (R_1 + R_2) \stackrel{C_2(R_1 + R_2) \gg C_1 R_1}{\cong} C_2 (R_1 + R_2)$$

- This happens when $C_2(R_1 + R_2) \gg C_1 R_1$. By multiplying both sides by R_2 (and rearranging the terms) the previous condition becomes $C_2 R_2 \gg C_1 R_1 R_2 / (R_1 + R_2) = C_1 (R_1 // R_2)$ which highlights that also the time constants for the computation of τ_H are in the same ratio.
- Therefore τ_H can be approximated with:

$$\frac{1}{\tau_H} = \frac{1}{C_1 (R_1 // R_2)} + \frac{1}{C_2 R_2} \stackrel{C_2 R_2 \gg C_1 (R_1 // R_2)}{\cong} \frac{1}{C_1 (R_1 // R_2)}$$

- In other words, from the computation of τ_L we may realize if one time constant is dominant. If so, we can approximate τ_L with the dominant time constant (C2) while the other capacitance (C1) is considered still "open" at LF. For the computation of τ_H it will be now the smaller time constant that dominates, that is you can approximate τ_H as given by C1 only, assuming C2 already "closed" at HF.

Exact result for the double RC

- The exact $H(s) = V_{out}/V_{in}$ (from circuit theory) is:

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{1}{s^2 R_1 R_2 C_1 C_2 + s[R_1(C_1 + C_2) + R_2 C_2] + 1}$$

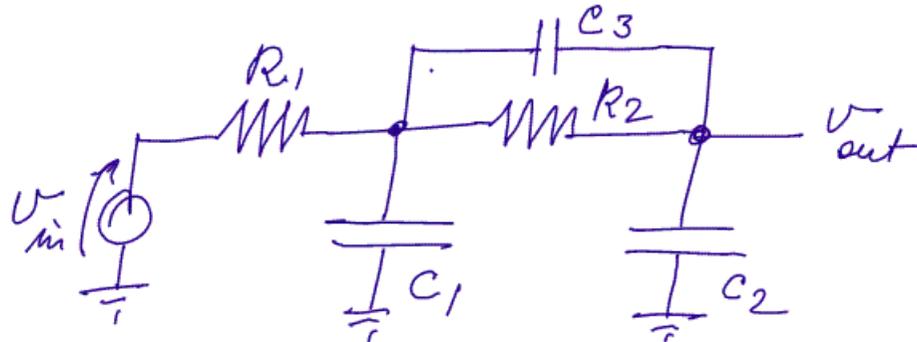
- The exact poles p_1, p_2 can be found with the known expressions for the roots of the quadratic form at the denominator.
- Comparison of the quadratic form with the estimates p_L, p_H - found with MTC previously - reveals the following interesting link:

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{1}{s^2 \frac{1}{p_L p_H} - s \frac{1}{p_L} + 1} = \frac{1}{s^2 \tau_L \tau_H + s \tau_L + 1}$$

- This shows how the MTC can be used not just for the direct estimation of the poles p_1, p_2 but also to compute the coefficients of the quadratic form, from which the exact roots can be computed.
- The possibility of computing the exact roots applies only to the case $n=2$, while for the general case of n poles we can obtain only the estimates p_L, p_H .

A frequent case: dependent capacitors

- Let us add now a 3rd capacitor C3 in parallel to R2 in the previous circuit. We obtain a circuit with 3 caps which are linearly dependent due to the KVL at the C1-C2-C3 loop. Therefore we have only n=2 poles. As the capacitors are dependent we cannot apply the MCT to find pL and pH. This is a frequent situation encountered when analyzing multi-transistor amplifying stages.



- However, if we compute τ_L , as in the MTC, we get:

$$\tau_L = \tau_1^0 + \tau_2^0 + \tau_3^0 = R_1 C_1 + (R_1 + R_2) C_1 + R_2 C_3$$

- The exact transfer function, computed from basic circuit theory, is:

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{1 + sR_2C_3}{s^2 R_1 R_2 [C_1(C_2 + C_3) + C_2 C_3] + s[R_1(C_1 + C_2) + R_2(C_2 + C_3)] + 1}$$

- In the denominator, the coefficient of s still corresponds exactly to τ_L , as in the previous case of 2 independent capacitors. **Therefore τ_L can be still used as an estimate of the low-frequency pole also in case of dependent capacitors.**
- The zero at $s = -1/(R_2 C_3)$ was expected, as the pair R2-C3 becomes an open-circuit (i.e. infinite impedance) at that complex frequency.
- We do not have any information on the HF pole if we do not solve the circuit from basic theory.