Optimizing WLAN Radio Coverage

E. Amaldi, A. Capone, M. Cesana, F. Malucelli
Politecnico di Milano - DEI
Address : Piazza L. da Vinci 32, 20133, Milano,Italy
Phone: +39.02.2399.3460, fax:+39.02.2399.3412
Email: {amaldi, capone, cesana, malucell}@elet.polimi.it

Abstract— **Wireless Local Area Networks (WLANs) are spreading all over the planet with impressive speed and market penetration. They will replace traditional indoor wired local networks and allow flexible access outdoor, eventually competing with classical cellular systems (GSM, $GPRS$, $UMTS$, etc.) in the provision of wireless services. Although the small systems currently installed are planned using rules of thumb, their rapid spread and size increase requires quantitative methods to determine proper Access Points (AP) positioning. Previously proposed approaches to the coverage planning neglect the effect of the IEEE802.11 access mechanism, which limits system capacity when Access Points coverage areas overlap. Here we propose a new modelling approach that directly accounts system capacity and show that the resulting optimization problems of WLAN coverage planning can be seen as extensions of the classical set covering or maximum coverage problems. We present and discuss different formulations based on quadratic and hyperbolic objective functions and report some preliminary results on synthetic instances we generated.**

Keywords: WLAN, radio planning, coverage problem, set covering problem, 802.11, wireless hot spots.

I. Introduction

Wireless Local Access Network (WLAN) is the emerging technology used to build up local wireless networks of computers providing connectivity and cable replacement in different scenarios such as hospitals, airports, firms and bars [1]. A WLAN is basically constituted by one or more wireless Access Points (APs) connected to the backbone network which provide wireless connectivity to the covered area. The number of WLAN "hot spots" is amazingly growing all over the planet due to the simplicity in the deployment process and the effectiveness of the offered services [2].

Most of today's WLAN systems run the IEEE802.11b protocol [3] which defines physical and MAC layer features. The IEEE802.11b was originally devised explicitly for a single Access Point scenario, replicating the Local Area Network structure in a wireless environment. In these conditions of negligible interference the protocol is shown to be highly effective in providing bandwidth to the mobile users. On the contrary, many problems arise when 802.11b is used in a scenario where interference is no longer negligible, such as an infrastructured cellular-like scenario with different APs cooperating to provide connectivity [4].

In many situations, the deployment of a single AP is not enough to provide the required connectivity. As an example, large facilities, such as an office complex, apartment buildings, hospitals, university campuses, or warehouses generally require many cooperating APs in order to provide the required services to the end users.

Obviously, the correct placement of the APs is crucial in order to cover the service area and to obtain high efficiency. Although the small systems currently installed are planned using rules of thumb, their rapid spread and size increase calls for quantitative methods to determine cost efficient solutions with service quality guarantees.

In order to access the network, a user terminal needs to receive the radio transmission of an AP at an adequate level of power. A simple way to provide the radio coverage is to consider possible positions of user terminals (Test Points, TP) in the service area and AP candidate sites (CS). The APs are then installed in a subset of candidate sites so that the signal level is high enough in all the considered TPs. The problem of selecting the minimum cardinality subset of candidate AP positions able to cover all points is a combinatorial optimization problem, namely the well known minimum cardinality set covering problem [5]. This problem is NP-hard and heuristics are usually adopted to obtain sub-optimal solutions. This is the basic approach that has been adopted in most of the previous papers appeared in the literature on wireless networks design [6].

However, not all feasible solutions able to cover all points provide the same system capacity and level of service. Due to the radio access mechanism, if a user terminal is covered by more than one AP and is transmitting/receiving to/from one of them, the other APs are prevented to transmit/reveice to/from other users (*exposed terminal problem*). Therefore, the coverage overlaps of different APs should be taken into account during the radio planning phase.

In this paper we address the problem of coverage planning for the emerging wireless LANs. We present novel mathematical programming models which take into account the the coverage overlap between APs and its impact on the system capacity. Some algorithmic approaches and preliminary numerical results obtained on synthetic instances are finally reported.

The paper is organized as follows. In Section II we discuss the issues related to the radio planning of WLAN and we briefly review the main features of the access mechanism which plays an important role in radio coverage. Section III presents the optimization models we have developed to describe the planning problem, while in Section IV some results are reported. Finally, Section V contains our concluding

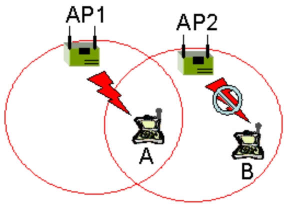

Fig. 1. *Spatial Reuse Impairment in WLAN.*

remarks.

II. WLAN DESIGN

WLAN systems follow the 802.11 standard which is a wireless extension of the well known ethernet. In particular, almost all the wireless devices composing today's wireless networks run the 802.11b standard which uses the unlicensed 2.4 GHz band and allows data rate up to 11 Mbps. The standardization bodies have recently approved new versions which allow to increase the maximum speed to 54 Mbps both in the 2.4 and 5 GHz bands.

As already pointed out in the previous section, the 802.11b access mechanism, based on *Carrier Sensing Multiple Access with Collision Avoidance* (CSMA/CA) is proven to be very effective if the interference is negligible. The carrier sense mechanism has been devised explicitly to get rid of any interference during ongoing transmission. For this reason no spatial reuse in classical 802.11b based networks is allowed. Figure 1 reports one possible pitfall of 802.11b based cellular networks. In the figure the mobile node A is covered by both the APs AP_1 and AP_2. If user A is engaged in a communication with AP_1, every other mobile user in the union of the coverage ranges of AP_1 and AP_2 is forbidden to start any new communication, and no spatial reuse is allowed.

As a consequence, the peculiar characteristics of 802.11b access mechanism affect the coverage planning process and the planning procedure should take into account the incidence of overlapping regions, beside all the other optimization parameters.

One possible solution of the planning problem would be installing disjoint access points, that is to say separate WLANs with disjoint coverage areas. This solution is optimal from the capacity point of view, but, on the other hand, could bring to coverage "holes", i.e. zones not reached by the wireless service. The opposite strategy would place the access points with large overlapping areas in the coverage ranges of one another in order to achieve higher connectivity. Although the full coverage is achieved, this strategy suffers from the throughput point of view, that is to say introduces potential interference which can impair the effectiveness of the access mechanism and consequently the overall capacity [7].

As in classical planning problems, the optimum solution comes from a trade off choice between the installation cost and the required quality. On the other hand, it's quite clear that the problem of WLAN planning is slightly different from the traditional approach of cellular networks. Firstly, as shown above WLAN can't cope with interference while cellular systems are based on it. Second, the installation costs of WLAN APs is definitely lower than the installation cost of cellular networks base stations. Finally, as already pointed out, WLAN weren't thought to provide cellular coverage, and this should be taken into account in the planning phase.

For the reason above, up to now the development of coverage planning tools for WLANs has been looked over as too expensive with respect to the price of access points and has been run following empirical ways rather than quantitative approaches. Nowadays the continuous increase of WLAN systems and of the services they can provide drives the focus on finding effective methods to determine high capacity and cost efficient solutions to the coverage planning problem.

To this end, very few work has appeared on the issue, and most of the papers appeared in the literature are based on the simple minimum cardinality set covering approach, which is the same adopted also for base station location problem arising in 2G cellular systems (see e.g. [8]). Classical methods for coverage planning based on random search heuristics can be applied to the problem [9]. In [10] the authors propose a formulation driven by the maximization of the signal quality in the service area and the contemporary minimization of the areas with a poor signal quality. The objective function comes from a combination of the above objectives. Some heuristics based on pruning, local search and simulated annealing are proposed. Rodrigues et al [11] propose a ILP formulation of the problem based on the maximization of the signal quality in the Test Points. Each Test Point is assigned a priority parameter based on the percentage of use and on the physical dimension of the Test Point area. This parameter is introduced in the objective function. The proposed formulation, which does not consider the full coverage constraint, is solved using CPLEX.

In general, all the works cited above focus on the problem of achieving high coverage level in terms of received signal quality. Very few of them consider network capacity as an optimization parameter. In [12] authors assign a traffic intensity to each TP and propose a formulation based on the maximization of the channel utilization of each access point. The proposed formulation is a particular case of the capacitated facility location problem.

Since the planning strategy is affected by many factors and the formulation of the problem often depends on a trade-off choice, our purpose here is to provide a novel mathematical framework to the coverage planning problem of WLANs, presenting different formulations according to the parameter(s) to be stressed in the planning process.

In the next section we introduce the notation and define the mathematical models for WLAN planning.

III. MATHEMATICAL MODELS

The North-American and European bandwidth regulation institutions have allowed the use of three separated frequency channels when deploying a WLAN, while only one channel is available for use in Japan.

Although using three channels has obvious advantages in the planning process, on the other hand it can create some problems in the network management. As a matter of fact, handover between access points on different frequency channels can be tricky and affect the service offered to the user. Furthermore, many of the 802.11b Network Interface Connectors (NIC) available on the market can't handle fast frequency channel switching, i.e., when switching to one frequency to another the connection must be torn down and it is re-established with high consequent delays. Therefore, it is reasonable to think that the first multi-access point WLANs will use one single frequency channel shared among all the APs.

For these reasons, at this first stage of analysis we assume that only one frequency channel is available within the same WLAN, i.e., no frequency assignment must be performed within the network.

Let's now formalize the planning problem. Let $J = \{1,\ldots,n\}$ denote the set of candidate sites to host access points and $I = \{1,\ldots m\}$ be the set of users. For each $j \in J$ a subset of users $I_j \subseteq I$ is given. This subset represents the users which can use the access point j. A second family of subsets I'_j, $j \in J$ representing the users affected by the interference of access point j, can be also defined. Clearly $I_j \subseteq I'_j$.

As mentioned above, we assume that each user is connected to a single AP whose capacity is shared by all users within its coverage range. Without loss of generality we assume that the overall capacity of an AP is equal to 1. However, due to the multiple access mechanism a user that is in the interfering range of a set of AP blocks the transmissions to/from all APs in this set. Therefore, we can assume that the capacity offered by the network to each user is equal to the reciprocal of the number of users in the interference range of the set of APs the considered user can interfere with.

Assuming uniform traffic for all users, the network capacity can be measured by the following expression:

$$c(S) = \sum_{i \in I(S)} \frac{1}{\cup_{j \in S: i \in I_j} |I'_j|}. \tag{1}$$

where $S \subseteq J$ is the subset of activated access points and $I(S)$ the subset of users covered that is included in some $I_j, j \in S$.

Alternatively, the *fairness* of the network, that is, the minimum capacity among all users accessing the network, can be considered:

$$f(S) = \min_{i \in I(S)} \frac{1}{\cup_{j \in S: i \in I_j} |I'_j|}. \tag{2}$$

The maximum capacity WLAN planning problem without covering constraints is then formalized as follows:

$$P : max\{c(S) : S \subseteq J\}, \tag{3}$$

while the maximum capacity WLAN planning problem with covering constraints is:

$$PC : max\{c(S) : S \subseteq J, \forall i \in I \exists j \in S \text{s.t.} i \in I_j\}. \tag{4}$$

Similarly, we may also define the maximum fairness WLAN planning problem:

$$PF : max\{f(S) : S \subseteq J\}. \tag{5}$$

All the three problems proposed above can be proved to be NP-Hard [13].

Without loss of generality, in the sequel we will assume that $I_j = I'_j$ for each $j \in J$. All the results that we describe can be easily extended to the more general case.

A. 0-1 hyperbolic formulations

The intrinsic difficulty of formulating problems P, PC, and PF resides in the particular objective function which is fractional. The fundamental decision variables are, as in any covering problem, those selecting which subsets are part of the solution:

$$x_j = \begin{cases} 1 & \text{if an access point is installed in } j \\ 0 & \text{otherwise} \end{cases}$$

Moreover we need to measure the cardinality of the union of subsets containing each element. To this end we introduce variables, which actually depend on variables x:

$$y_{ih} = \begin{cases} 1 & \text{if elements } i \text{ and } h \text{ appear together in some selected subset} \\ 0 & \text{otherwise} \end{cases}$$

Consider the usual elements-subsets incidence matrix, that is $a_{ij} = 1$ if element i belongs to subset j, and 0 otherwise, for each $i \in I$ and $j \in J$, the formulation of problem PC is:

$$PCH : \max \quad \sum_{i \in I} \frac{1}{\sum_{h \in I} y_{ih}} \tag{6}$$

$$subject\ \ to:$$

$$\sum_j a_{ij} x_j \geq 1 \qquad i \in I \tag{7}$$

$$a_{ij} a_{hj} x_j \leq y_{ih} \qquad j \in J, i, h \in I \tag{8}$$

$$x_j \in \{0,1\} \qquad j \in J \tag{9}$$

$$y_{ih} \geq 0 \qquad i, h \in I \tag{10}$$

where 18 imposes the complete coverage and 19 defines the variables y_{ih}.

Consider now problem P. The formulation that we gave for problem PC cannot be adapted by simply deleting the covering constraints because of the fractional objective function. Without covering constraints we cannot guarantee that all terms of the objective function have significant values. Indeed for the uncovered elements the sum at the denominator are equal to zero. Therefore we introduce explicit variables z_i which are equal to one if and only if element i is covered, and 0 otherwise. The formulation is the following:

$$PH : \max \quad \sum_{i \in I} \frac{z_i}{\sum_{h \in I} y_{ih}} \tag{11}$$

$$subject\ \ to:$$

$$\sum_j a_{ij} x_j \geq z_i \qquad i \in I \tag{12}$$

$$a_{ij} a_{hj} x_j \leq y_{ih} \qquad j \in J, i, h \in I \tag{13}$$

$$x_j \in \{0,1\} \qquad j \in J \tag{14}$$

$$0 \leq z_i \leq 1 \qquad i, h \in I \tag{15}$$

$$y_{ih} \geq 0 \qquad i, h \in I \tag{16}$$

where the constraints are similar to PCH's ones with the exception of 12 which introduces the case of uncovered users.

As far as problem PF is concerned, recalling that solutions covering all elements dominate the others, we can formulate the problem by explicitly imposing the covering constraints thus obtaining the following bottleneck formulation:

$$PFH: \max \quad \min_{i \in I} \frac{1}{\sum_{h \in I} y_{ih}} \tag{17}$$

$subject\ \ to:$

$$\sum_j a_{ij} x_j \geq 1 \qquad i \in I \tag{18}$$

$$a_{ij} a_{hj} x_j \leq y_{ih} \qquad j \in J, i, h \in I \tag{19}$$

$$x_j \in \{0,1\} \qquad j \in J \tag{20}$$

$$y_{ih} \geq 0 \qquad i, h \in I \tag{21}$$

Unfortunately hyperbolic 0-1 problems appear to be quite intractable [14], [15]. However, due to their particular structure, these problems can be solved using easy heuristics [13].

B. Quadratic formulations

Let us analyze alternative formulations of problems P and PC, which in some cases are equivalent to the original ones. When a subset I_j is added to the solution, if no other subset is present or if it does not intersect any other subset already in the solution, its contribution to the network capacity is $1/|I_j|$ for each element of I_j, which clearly gives globally 1. Thus, let us associate to each subset I_j a coefficient $c_j = 1$. Consider now two subsets I_j and I_ℓ, if their intersection is empty their simultaneous presence in the solution does not imply any decrement in the network capacity. On the contrary, if their intersection is not empty, the network capacity must be decreased proportionally with the cardinality of the intersection. In particular, the contribution of the intersection must be subtracted from the capacity of the two subsets and a new evaluation of the contribution of the intersection considering the union of the two subsets must be added to the capacity measure. In order to account for this capacity decrement let us define coefficients

$$q_{j\ell} = -\frac{|I_j \cap I_\ell|}{|I_j|} - \frac{|I_j \cap I_\ell|}{|I_\ell|} + \frac{|I_j \cap I_\ell|}{|I_j \cup I_\ell|}. \tag{22}$$

Note that these coefficients assume values between -1 (when $I_j \subseteq I_\ell$ or $I_\ell \subseteq I_j$) and 0 (when the two subsets are disjoint). Moreover they are symmetric, that is $q_{j\ell} = q_{\ell j}$.

Consider the following formulation:

$$QP: \max \quad \frac{1}{2} \sum_{j \in J} \sum_{\ell \in J, \ell \neq j} q_{j\ell} x_j x_\ell + \sum_{j \in J} c_j x_j \tag{23}$$

$subject\ \ to:$

$$x_j \in \{0,1\} \qquad j \in J \tag{24}$$

The above simple unconstrained quadratic 0-1 formulation is equivalent to problem P if each element of I belongs to at most two subsets. In the other cases, according to the definition of coefficients $q_{j\ell}$, the objective function (23) gives an underestimation of the actual network capacity. Unfortunately also this formulation, beside being an approximation, is difficult to tackle since it is equivalent to a max cut problem, due to the coefficients signs. However many studies on pseudo boolean functions can be exploited to obtain exact or approximate solutions.

In a similar way, we can give an alternative formulation of problem PC:

$$QPC: \max \quad \frac{1}{2} \sum_{j \in J} \sum_{\ell \in J, \ell \neq j} q_{j\ell} x_j x_\ell + \sum_{j \in J} c_j x_j \tag{25}$$

$subject\ \ to:$

$$\sum_j a_{ij} x_j \geq 1 \qquad i \in I \tag{26}$$

$$x_j \in \{0,1\} \qquad j \in J \tag{27}$$

which turns out to be an interesting (but difficult to solve) Quadratic Set Covering Problem. It should be noted that both formulations contain a substantially lower number of variables than the hyperbolic ones.

IV. Solving Some Problems

In the case of PFH we can exploit the structure of the hyperbolic 0-1 formulation and obtain the optimal solution by solving a sequence of mixed integer linear problems. Problem PFH can be rewritten in the following equivalent way:

$$\max \beta \tag{28}$$

$subject\ \ to:$

$$1 \geq \beta \left(\sum_{h \in I} y_{ih} \right) \tag{29}$$

$$\sum_j a_{ij} x_j \geq 1 \qquad i \in I \tag{30}$$

$$a_{ij} a_{hj} x_j \leq y_{ih} \qquad j \in J, i, h \in I \tag{31}$$

$$x_j \in \{0,1\} \qquad j \in J \tag{32}$$

$$y_{ih} \geq 0 \qquad i, h \in I \tag{33}$$

Variable β and constraints (29) play the role of the bottleneck function of problem PFH. It should be observed that β, being the value of the fairness, assumes values between 0 and 1.

For a given value of $\beta = \beta^*$ determining the existence of a feasible solution is equivalent to solving a linear inequalities system with continuous and 0-1 variables (let us call it $LS(\beta)$), therefore can be solved by means of any Mixed Integer Linear Programming (MILP) software. This implies that the optimal solution of PF can be determined by making a binary search on the value of β, each time solving a MILP problem. The search iterates the following steps until a stopping condition on the precision of β is reached:

- Let $[\underline{\beta}, \overline{\beta}]$ be the search interval (initially $[0,1]$).
- Set $\beta^* = (\underline{\beta} + \overline{\beta})/2$
- If LS(β^*) is feasible then set $\underline{\beta}$ to β^*, otherwise set $\overline{\beta}$ to β^*.

In order to test the effectiveness of the above algorithm we have considered a square simulation area with edge length equal to 1Km. Within this area, we have randomly generated positions for users and candidate sites. Namely, each planning instance is characterized by a tern of parameters:

- the number of Candidate Sites (CSNumber), i.e. the positions where a AP can be installed
- the number of Test Points (TPNumber), i.e. the end users
- the value of each AP's coverage range, expressed in meters (r).

Running these experiments we have supposed coverage areas modelled as circles of radius r.

For each instance, the optimal solution of PCH and QPC is obtained by enumeration, while the optimal solution of PFH is obtained with the algorithm proposed above. The optimal solution of one formulation is used to calculate the objective functions of all the other formulations. Since we need to compute exactly the solutions for the PCH and QPC cases we limited our analysis here to relatively small instances with $CSNumber = 10$ and $TPNUmber = 100$. Tables I, II and III report the obtained results. In particular results in table I are obtained solving PCH and using the solution found to calculate PFH and QPC, table II reports the case of the exact solution of QPC and finally table III the exact solution of PFH.

	SC	Exact PHC	QPC	PFH
r=50m	9.9	9.3340	9.3229	0.0509
r=100m	9.4	7.4243	7.2618	0.0358
r=200m	7.0	4.3440	4.0155	0.0192

TABLE I

CSNumber=10, TPNumber=100, randomly generated position within 1Km edge square area, exact solution of PCH.

	SC	PHC	Exact QPC	PFH
r=50m	9.9	9.3340	9.3229	0.0509
r=100m	9.4	7.4243	7.2618	0.0358
r=200m	7.0	4.3381	4.0605	0.0191

TABLE II

CSNumber=10, TPNumber=100, randomly generated position within 1Km edge square area, exact solution of QPC.

	SC	PHC	QPC	Exact PFH
r=50m	9.9	8.9035	8.8035	0.0509
r=100m	9.4	7.4243	7.2618	0.0358
r=200m	7.1	4.2905	3.9688	0.0193

TABLE III

CSNumber=10, TPNumber=100, randomly generated position within 1Km edge square area, exact solution of PFH.

As expected, QPC objective function provides a lower bound for the actual capacity predicted by PHC objective function. This bound is definitely good for all the tested instances, even if it obviously decreases when the overlaps between subsets increase, that is, when the coverage radius r increases. As an example, with low values of r ($r = 50m$) the values of the two objective functions almost coincide in all the three tables. The gap between the two values increases consistent for $r = 100, 200m$, when the overlap among different subsets augments.

As far as fairness is concerned, it obviously decreases whenever the number of user's interferers increases, that is, when the coverage radius r increases. The optimal solutions with respect to the fairness present slightly lower values of network capacity with respect to the exact solution of PCH.

Note that, even though for these small instances there are only slight differences among the results of different formulations, we do expect that the gap will increase if considering more complex overlap patterns. To this end, we plan to develop a more realistic instance generator, which considers more precise path-loss models specific for WLANs' environments.

V. Conclusions

In this work we have addressed the problem of coverage planning in WLANs, which is starting to attract the attention both of the industry and of the research community. In particular we have proposed different mathematical formulations based on hyperbolic and quadratic objective functions. Most of the proposed formulations focuses on capacity, that is, they aim at maximizing the capacity of either the overall network or the single end user.

Since the proposed formulations are difficult to solve to optimality (the problems are NP-Hard), efficient heuristics providing good solutions in a reasonable amount of computing time are needed. Anyways, a possible solution algorithm for the formulation which aims at maximizing the fairness is presented and some preliminary results are discussed. In particular, we have shown that the quadratic formulation QPC provides a good quality lower bound to the capacity of the overall network on some small synthetic instances.

Work is underway to develop effective heuristics able to provide nearly optimal solutions for larger instances in reasonable amount of computing time [13].

References

[1] http://www.eweek.com/article2/0,3959,485099,00.asp
[2] http://www.pcworld.com/news/article/0,aid,111418,00.asp
[3] http://grouper.ieee.org/groups/802/11/
[4] Haitao Wu, Shiduan Cheng, Yong Peng, Keping Long, Jian Ma, *Does the IEEE 802.11 MAC protocol work well in multihop wireless ad hoc networks?*, IEEE Communication Magazine, June 2001.
[5] S. Ceria, P. Nobili, A. Sassano , *Set Covering Problem*, in Annotated bibliographies in Combinatorial Optimization, Dell'Amico M., F. Maffioli, S. Martello eds, John Wiley and Sons - Chichester.
[6] A. Hills, *Large-Scale Wireless LAN Design*, IEEE Communications Magazine, vol. 39, no. 11, Nov. 2001, pp. 98-107.
[7] N.R. Prasad, *IEEE 802.11 system design*, IEEE International Conference on Personal Wireless Communications, 2000, Page(s): 490 494.
[8] K. Tutschku, *Demand-based radio network planning of cellular mobile communication systems*, IEEE INFOCOM 1998, pp. 1054–1061.
[9] S. Hurley, *Planning Effective Cellular Mobile Radio Networks*, IEEE Transactions on Vehicular Technology, vol. 51, no. 2, March 2002, pp. 243-253.
[10] M. Kamenetsky, M. Unbehaun, *Coverage planning for outdoor wireless LAN systems*, 2002 International Zurich Seminar on Broadband Communications Access, Transmission, Networking, 2002, Page(s): 491 -496.
[11] R.C. Rodrigues, G.R. Mateus, A.A.F. Loureiro, *On the design and capacity planning of a wireless local area network*, Network Operations and Management Symposium, 2000. NOMS 2000, Page(s): 335348.
[12] Y. Lee, K. Kim, Y. Choi, *Optimization of AP placement and Channel Assignment in Wireless LANs*, IEEE Conference on Local Computer Networks, 2002. LCS 2000.
[13] E, Amaldi, A. Capone, M. Cesana, F. Malucelli, *Maximum Capacity Coverage Problems in planning Wireless Local Area Networks*, manuscript, August 2003.
[14] P. Hansen, M. V. Poggi de Aragão, *Boolean query optimization and the 0-1 hyperbolic sum problem*, Annals of Mathematics and Artificial Intelligence (1990), 1, 97-109.
[15] P. Hansen, M. V. Poggi de Aragão, *Hyperbolic 0-1 programming and query optimization in information retrieval*, Mathematical Programming 52 (1991), 255-263.