Stochastic characterization of the two band two player spectrum sharing game

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***Abstract*—We consider two pairs of communicating users sharing two bands of spectrum under a sum band power constraint. Our earlier work proposed a natural spectrum sharing game for this problem and characterized the Nash equilibria as a function of the signal and interference distances, when the positions of the four nodes were assumed fixed. In this work, we derive $i)$ the joint distribution of the interference distances conditioned on the transmitter separation distance, as well as $ii)$ the unconditioned interference distance distribution when we place one transmitter at the origin and the second uniformly at random over a disk. This allows us to compute the distribution of the random Nash equilibria and random prices of anarchy and stability as a function of the random interference distances. We leverage the analysis to give an asymptotic expression for the *coupling probability* in a game where the transmitter positions form a (low density) Poisson process, which may be interpreted as the fraction of players essentially playing a two player game.**

I. Introduction

In this paper, we analyze the dynamics among multiple pairs of users that want to communicate sharing the same portion of the spectrum. We model this interaction using game theory, since, in general, the quality perceived by a user strictly depends on the behavior of other entities. We consider a Gaussian Interference Game (GIG) [1], where two transmitter and receiver pairs have to decide how to split the total power budget P across two orthogonal channels of the spectrum with equal bandwidths B, maximizing the sum of the Shannon rate achieved on each band. Non–cooperative game theory is suitable in distributed networks, where control and management are decentralized. For this reason, we model the problem as a non–cooperative game, in which selfish users aim to maximize their achieved throughput.

In our previous work [2], the game is characterized analytically assuming that the positions of the two transmitter/receiver couples are known. In this scenario, we derive the quality/number of the Nash equilibria, highlighting the dependence on the parameters that define the topology (interference distances x_1, x_2 and signal distance d) and the propagation model (pathloss exponent α and noise power η). Since the wireless network performance is strongly influenced by the spatial distribution of the communicating/interfering nodes, a natural objective is to analyze the dependence of the equilibria distribution on the node positions. Therefore, we fix one transmitter at the origin and place the second transmitter uniformly at random in a disk of radius L. The two receivers are placed uniformly at random along the circles of radius d centered at the corresponding transmitter. Basic stochastic geometry is used to characterize the joint distribution of the interference distances x_1, x_2, and to compute the distribution of the random equilibria.

The stochastic characterization of the 2–player game can be leveraged as a starting point to tackle the more general N–player game. Namely, the analysis of large networks game can be simplified, identifying couples of pairs in the N–player game that play small 2–player sub–games. The key result of this paper is to provide, for the low–density regime, the fraction of nodes whose game can be characterized using the stochastic analysis of the 2–player game. The complete analysis of the N–player game is part of the ongoing work and the results of this paper are fundamental in order to assess the network performance of large networks with game theoretic dynamics.

The paper is organized as follows: §II reviews prior work on spectrum sharing games; §III sets the reference scenario and recalls previous results for the spectrum sharing game; §IV provides a stochastic analysis of the 2–player game; and §V shows how these results can be used to evaluate the performance in large N–player networks.

II. Related work

Spectrum sharing games have been widely studied in the literature. In [1], the authors propose a spectrum sharing game for multiple players that share the same portion of spectrum. In particular, they introduce self-enforcing rules that allow users to reach an efficient and fair solution. In [3], the authors consider a power control problem with SINR as objective function, in both the selfish and the cooperative scenario. In contrast, non–cooperative games among operators that share the spectrum are proposed in [4] and [5]. Both these papers characterize the Nash equilibria of the system. Games based on the water–filling algorithm are proposed in [6] and [7]. The authors consider a scenario composed by two contending communicating systems and study the existence and uniqueness of the Nash equilibrium. The water–filling algorithm is used also in [8] and [9]. A unified view of main results presented in the literature is proposed in [10]. This work shows how the different approaches proposed in the literature can be unified following a unique interpretation of the waterfilling solution. Furthermore, a unified set of sufficient conditions that guarantee the uniqueness of the equilibrium is derived.

Power control games in the context of cognitive radio networks are considered in [11] and [12]. Existence and properties of the Nash equilibria are investigated assuming the interference temperature model. Furthermore, the channel competition is studied in [13] and [14]. The authors characterize the equilbria of the game and assess their quality with respect to the optimal solution. In [15], the power allocation game is modeled as a potential game that is shown to possess a unique equilibrium. When the channels are assumed to fluctuate stochastically over time, stochastic approximation theory is used to show convergence to equilibrium.

There exist other few works that propose a stochastic analysis of games in the context of wireless networks. In [16], the authors propose a stochastic game that models the interactions among users that compete for the available spectrum opportunities. In particular, users make bids for the required resources. A best–response algorithm is proposed in order to improve the users' bidding policy. A random access game is proposed also in [17], where the authors characterize the Nash equilibria of the random access game and analyze the asymptotic properties of the system as the number of users goes to infinity. [18] provides a useful tutorial on different techniques based on stochastic geometry and the theory of random geometric graphs. Different from previous work, we introduce the use of stochastic geometry in order to characterize the equilibria of the game in terms of probability distribution over the set of possible equilibria.

III. Reference scenario

We consider the spectrum sharing game, where two transmitter and receiver pairs (the players) have to decide how to split the total power budgets over two available bands (the actions), maximizing the sum of the achievable Shannon rates over the two bands (the payoffs). Namely, the reference topology is reported in Fig. 1. We label the distance between the two transmitters t and assume that the corresponding receivers are placed uniformly around the transmitters at a fixed distance d, i.e., a and b are uniform angles in the range $[0, 2\pi]$. (Note that in this section we assume a and b to be fixed, whereas we will treat them as random variables in the next section.) Let x_1 and x_2 be the distance between receiver 1 and transmitter 2 and the distance between receiver 2 and transmitter 1, respectively. The strategy space of the generic transmitter i is $P_i \in [0, 1]$. Namely, P_i is the fraction of P that pair i uses in the left band and $\bar{P}_i = 1 - P_i$ is the fraction of power in the right band. The utility function of each player is defined as the sum achievable Shannon rate over the two bands when the interference from the other player is treated as noise. We assume a channel model with pure pathloss attenuation, where $\alpha > 2$ is the pathloss exponent. Assuming a noise power of η on each band, the utility function of player 1 (and symmetrically of player 2) for transmission powers $P \equiv (P_1, P_2) \in [0, 1]^2$ is:

$$U_1(P) = \log_2\left(1 + \frac{P_1 d^{-\alpha}}{\eta + P_2 x_1^{-\alpha}}\right) + \log_2\left(1 + \frac{\bar{P}_1 d^{-\alpha}}{\eta + \bar{P}_2 x_1^{-\alpha}}\right). \quad (1)$$

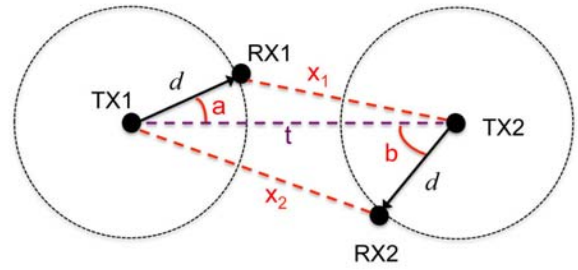

Fig. 1. 2–player topology.

The characterization of pure Nash equilibria derived in [2] is reported in Fig. 2, where $\beta_i^{\pm} = 0.5\left(1 \pm x_i^{-\alpha}/d^{-\alpha}\right)$. Note that there exist four regions. When $x_1 x_2 < d^2$, the game admits three equilibria and the optimum is in $(0, 1)$ and $(1, 0)$. In particular, when both x_1 and x_2 are smaller than d, the PoS is one, since the best equilibrium and the optimum coincide. In contrast, when x_1 (or x_2) is greater than d, the best equilibrium is worse than the optimum, then the PoS is greater than one. The PoA is in both the two cases greater than one. When $x_1 x_2 > d^2$ and $x_1^{-\alpha} x_2^{-\alpha} + \eta x_1^{-\alpha} + \eta x_2^{-\alpha} - \eta d^{-\alpha} < 0$ the game admits a unique equilibrium, that does not coincide with the optimum, then PoS and PoA coincide and are greater than one. In contrast, when

$$x_1^{-\alpha} x_2^{-\alpha} + \eta x_1^{-\alpha} + \eta x_2^{-\alpha} - \eta d^{-\alpha} > 0 \quad (2)$$

the optimal solution and the unique equilibrium coincide. Finally, along the curve $x_1 x_2 = d^2$ the two best responses coincide and there exists an infinite number of equilibria.

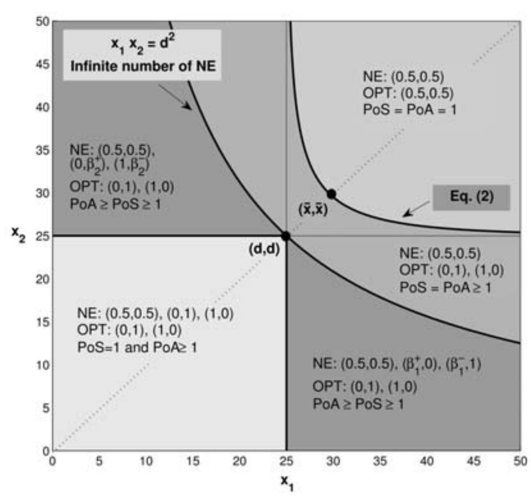

Fig. 2. Nash equilibria and optimal allocations as a function of x_1 and x_2 (for $d = 25$).

From this analysis, we can conclude that the equilibrium chosen by the two players (as well as the corresponding PoS/PoA) depends upon the two interference distances (x_1, x_2). When node positions are random, it follows that (x_1, x_2) are random, and thus the equilibrium is random. In short, we compute probabilities of being in the various regions in Fig. 2.

IV. Stochastic Analysis of the 2–player Game

We use standard probabilistic notation: sans-serif letters (e.g., x) denote random variables (RV) and italic characters denote their corresponding value (e.g., x). The cumulative distribution function (cdf), its complement (ccdf), and probability density function (pdf) are denoted $F_{\mathsf{x}}(x), \bar{F}_{\mathsf{x}}(x) = 1 - F_{\mathsf{x}}(x), f_{\mathsf{x}}(x)$, respectively, with the natural notational extensions for conditional and joint distributions.

A. Conditional joint pdf of the interference distances

We suppose the two transmitters are placed at random and we condition on the distance separating them t. We further suppose each receiver is placed uniformly at random on the circle of radius d centered at the corresponding transmitter. It follows that the interference distances $\mathsf{x}_1, \mathsf{x}_2$ are conditionally independent given t:

$$F_{\mathsf{x}_1,\mathsf{x}_2|\mathsf{t}}(x_1, x_2|t) \equiv \mathbb{P}(\mathsf{x}_1 \leq x_1, \mathsf{x}_2 \leq x_2|t) = = \mathbb{P}(\mathsf{x}_1 \leq x_1|t)\mathbb{P}(\mathsf{x}_2 \leq x_2|t). \quad (3)$$

Using the law of cosines $\mathsf{x}_1^2 = d^2 + t^2 - 2dt\cos\mathsf{a}$:

$$\mathbb{P}(\mathsf{x}_1 \leq x_1|t) = \mathbb{P}(d^2 + t^2 - 2dt\cos\mathsf{a} \leq x_1^2|t) = = \mathbb{P}\left(\cos\mathsf{a} \geq \frac{(d^2+t^2) - x_1^2}{2dt}\middle| t\right). \quad (4)$$

The following proposition is elementary.

Proposition IV.1. *The random variable* w*, cosine of a uniformly distributed angle, has the following distribution:*

$$f_{\mathsf{w}}(w) = \frac{1}{\pi\sqrt{1-w^2}},\ F_{\mathsf{w}}(w) = \frac{1}{2} + \frac{1}{\pi}\sin^{-1}w,\ w \in [-1, 1].$$

Using this result, we can now state the following theorem.

Theorem IV.2. *The joint cdf and pdf of* $\mathsf{x}_1, \mathsf{x}_2$ *conditioned on* t *are given by Eq.* (5) *and Eq.* (6) *(reported on page 4), respectively, where* $F_{\mathsf{w}}(w)$ *is given in Prop. IV.1.*

Proof: For the sake of brevity, we only report some mathematical steps. From Eq. (4) and Prop. IV.1:

$$\mathbb{P}\left(\cos\mathsf{a} \geq \frac{(d^2+t^2) - x_1^2}{2dt}\middle| t\right) = \bar{F}_{\mathsf{w}}\left(\frac{(d^2+t^2) - x_1^2}{2dt}\middle| t\right). \quad (7)$$

From Eq. (3):

$$F_{\mathsf{x}_1,\mathsf{x}_2|\mathsf{t}}(x_1, x_2|t) = \bar{F}_{\mathsf{w}}\left(\frac{(d^2+t^2) - x_1^2}{2dt}\right)\bar{F}_{\mathsf{w}}\left(\frac{(d^2+t^2) - x_2^2}{2dt}\right). \quad (8)$$

Eq. (5) follows by requiring the argument of the cdf $F_{\mathsf{w}}(w)$ to be between -1 and 1. Eq. (6) follows directly from taking the double partial derivative with respect to x_1 and x_2:

$$f_{\mathsf{x}_1,\mathsf{x}_2|\mathsf{t}}(x_1, x_2|t) = \frac{\partial^2}{\partial x_1 \partial x_2}F_{\mathsf{x}_1,\mathsf{x}_2|\mathsf{t}}(x_1, x_2|t). \quad (9)$$

■

Fig. 3 shows the conditional joint pdf and cdf when $d = 1$ and $t = 3$. Note that we obtain a function different from zero only in the region $\{(x_1, x_2) : |d - t| \leq x_1 \leq d + t, |d - t| \leq x_2 \leq d + t\}$, consistent with the topology constraints mentioned at the beginning of this section.

B. Joint pdf of the mutual interference distances

For any distribution on the random distance t separating the transmitters, we obtain the joint (unconditioned) distribution for $(\mathsf{x}_1, \mathsf{x}_2)$ by the total probability theorem:

$$f_{\mathsf{x}_1,\mathsf{x}_2}(x_1, x_2) = \int_0^\infty f_{\mathsf{x}_1,\mathsf{x}_2|\mathsf{t}}(x_1, x_2|t) f_{\mathsf{t}}(t)\mathrm{d}t. \quad (10)$$

We will henceforth assume the random transmitter separation distance t is determined by placing one of the transmitters at the origin of a disk of radius L, and placing the second transmitter uniformly at random in the disk. We emphasize, however, that Eq. (10) holds for any distribution on t, and our assumption is merely for the purpose of concreteness. Under this assumption, the RV t has cdf and pdf:

$$F_{\mathsf{t}}(t) = \left(\frac{t}{L}\right)^2,\ f_{\mathsf{t}}(t) = \frac{2t}{L^2},\ 0 \leq t \leq L. \quad (11)$$

The RVs $\mathsf{x}_1, \mathsf{x}_2$ are not (unconditionally) independent. Due to the geometry of the problem (the sum of any two sides of a triangle must be greater than the third side), the following inequalities must hold:

$$\begin{aligned} |d-t| \leq &\quad x_1 \quad \leq d+t \\ |d-t| \leq &\quad x_2 \quad \leq d+t \\ \max\{|d-x_1|, |d-x_2|\} \leq &\quad t \quad \leq \min\{d+x_1, d+x_2\} \end{aligned} \quad (12)$$

We can now state the following theorem.

Theorem IV.3. *Suppose the distance* t *between the two transmitters has the distribution in Eq.* (11)*. Then the joint pdf of* $\mathsf{x}_1, \mathsf{x}_2$ *is given by Eq.* (13) *(reported on page 5) and the support of* $(\mathsf{x}_1, \mathsf{x}_2)$ *is given by Eq.* (12).

Proof: Substitute Eq. (11) for $f_{\mathsf{t}}(t)$ and Eq. (6) for $f_{\mathsf{x}_1,\mathsf{x}_2|\mathsf{t}}(x_1, x_2|t)$ in Eq. (10), using the constraints in Eq. (12). ■

Note that, in general, it is not possible to provide close form expression of the integral in Eq. (13). Numerical evaluation, however, is straightforward.

C. Distribution on the Nash equilibria

We can now use the joint distribution in Thm. IV.3 to compute the distribution of the equilibria in Fig. 2. Each equilibria's probability can be evaluated via:

$$\int_{\mathcal{P}} f_{\mathsf{x}_1,\mathsf{x}_2}(x_1, x_2)dx_1dx_2 \quad (14)$$

where $\mathcal{P}$ is the set of points that defines that region. Table I summaries the different cases, which are illustrated in Fig. 4. Namely, *unique* refers to the probability that the equilibrium is unique. When the equilibrium is not unique, we consider two cases. The case in which the three equilibria are $(0.5, 0.5)$, $(0, 1)$ and $(1, 0)$, i.e., *(0.5,0.5)-(0,1)-(1,0)*, and the case in which the equilibria depend on the parameter β, (*mixed*). For completeness, we also provide the probability that there is an infinite number of equilibria (*infinite*). Finally, we evaluate the probability that the unique equilibrium and the optimal solution coincide (*coincide w/opt*).

$$F_{\mathsf{x}_1,\mathsf{x}_2|\mathsf{t}}(x_1,x_2|t) =$$

$$= \begin{cases} 0 & x_1 < |d-t| & \text{or} & x_2 < |d-t| \\ \left[1 - F_{\mathsf{w}}\left(\frac{(d^2+t^2)-x_1^2}{2dt}\right)\right]\left[1 - F_{\mathsf{w}}\left(\frac{(d^2+t^2)-x_2^2}{2dt}\right)\right] & |d-t| \le x_1 \le d+t & \text{and} & |d-t| \le x_2 \le d+t \\ \left[1 - F_{\mathsf{w}}\left(\frac{(d^2+t^2)-x_1^2}{2dt}\right)\right] & |d-t| \le x_1 \le d+t & \text{and} & x_2 > d+t \\ \left[1 - F_{\mathsf{w}}\left(\frac{(d^2+t^2)-x_2^2}{2dt}\right)\right] & x_1 > d+t & \text{and} & |d-t| \le x_2 \le d+t \\ 1 & x_1 > d+t & \text{and} & x_2 > d+t \end{cases} \tag{5}$$

$$f_{\mathsf{x}_1,\mathsf{x}_2|\mathsf{t}}(x_1,x_2|t) =$$

$$= \begin{cases} \frac{x_1 x_2}{\pi^2 d^2 t^2 \sqrt{\left[1-\left(\frac{d^2+t^2-x_1^2}{2dt}\right)^2\right]\left[1-\left(\frac{d^2+t^2-x_2^2}{2dt}\right)^2\right]}} & |d-t| \le x_1 \le d+t \text{ and } |d-t| \le x_2 \le d+t \\ 0 & \text{else} \end{cases} \tag{6}$$

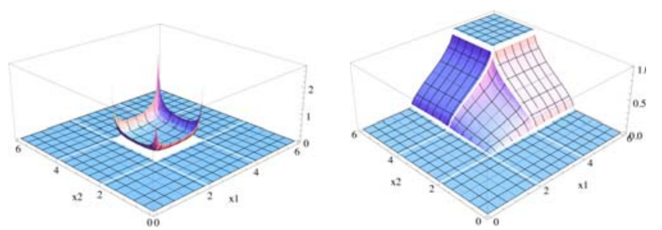

Fig. 3. Conditional joint pdf (left) and cdf (right) of x_1 and x_2 when $d = 1$ and $t = 3$.

TABLE I
DIFFERENT EVENTS THAT ARE CONSIDERED IN THE STOCHASTIC ANALYSIS OF THE GAME.

"event"	$\mathcal{P}$
unique	$x_1 x_2 > d^2$
(0.5,0.5)-(0,1)-(1,0)	$x_1 < d \wedge x_2 < d$
mixed	$x_1 x_2 < d^2 \wedge (x_1 > d \vee x_2 > d)$
infinite	$x_1 x_2 = d^2$
coincide w/opt	$(x_1 x_2)^{-\alpha} + \eta(x_1^{-\alpha} + x_2^{-\alpha} - d^{-\alpha}) > 0$

Numerical results for all these probabilities are reported in Fig. 5. We consider four different values of d, with L varying from 0.1 to 50. We assume $\eta = 10^{-3}$ and $\alpha = 4$. Clearly, the probability of having a unique equilibrium, that corresponds to the case in which players decide to use the whole spectrum, increases with L. This is reasonable since increasing L, also the average distance between the interfering pairs increases (x_1 and x_2), then, for a fixed d, the probability of $x_1 x_2 > d^2$ increases. This corresponds to the case in which users share the same spectrum since the interfere is small. In particular, we can see that when $L < d$ that probability is around 50% and then starts increasing, approaching 1 when $L \simeq 5d$. Note that the probability of having an infinite number of Nash equilibria is very close to zero, as expected (sometimes it is slightly different from zero for numerical issues in the evaluation of the integral). Furthermore, the probability that the optimum and the unique equilibrium coincide has a similar behavior of the probability of uniqueness, but it is lower. This is obvious, since when the equilibrium and the optimum coincide, the equilibrium is also unique, but the opposite is not always true.

Moreover, when $L = 0$, the two transmitters coincide, then $x_1 = x_2 = d$, that corresponds to the case in which there exists an infinite number of Nash equilibria. In contrast, when $L = \epsilon$, for any small $\epsilon > 0$, each x can be a little bit greater or a little bit smaller than d (remember that $x^2 = d^2 + \epsilon^2 - 2d\epsilon \cos a$). We can see this in Fig. 6 (left).

In particular, the zoom reported in Fig. 6 (right) explains why when L approaches zero, the probability of unique equilibrium is equal to 0.5 (given by the probability that both the xs are greater than d). Therefore, also the probability of having three equilibria is 0.5. In particular, this is half split between having $\{(0,1),(1,0),(0.5,0.5)\}$ (both x's are less than d and this happens with probability 0.25) and the *mixed* case (when only one x is greater than d, but both below the curve $x_1 x_2 = d^2$).

Finally, note that when L goes to zero the optimum never coincides with the equilibrium in $(0.5, 0.5)$. In fact, the optimal solution always corresponds to split and use different portions of the spectrum. The reason why, when $L = \epsilon$, with ϵ small, this probability approaches zero is clear also from Fig. 6 (right). In fact, when $L = \epsilon$, the two x's are both close to d and they do not satisfy the condition given by Eq. (2).

V. COUPLING PROBABILITY

In the previous section we provided a stochastic characterization of the 2–player game. We now illustrate the value of this analysis in the corresponding N-player game. Consider the Binomial Point Process (BPP) $\Pi_N = \{\mathsf{u}_1, \ldots, \mathsf{u}_N\}$ where N transmitters are positioned independently and uniformly at random in a compact set $\mathcal{A} \subset \mathbb{R}^2$ with $N/|\mathcal{A}| = \lambda$; we refer to λ as the density. The N receivers $\{\mathsf{v}_1, \ldots, \mathsf{v}_N\}$ are each positioned independently and uniformly at random on the circle of radius d centered at the corresponding transmitter. The sum and maximum (shot noise) interference processes seen at receiver v_j under BPP Π_N are:

$$\Sigma_{\Pi_N}(\mathsf{v}_j) \equiv \sum_{\mathsf{u}_i \in \Pi_N \setminus \{\mathsf{u}_j\}} \|\mathsf{u}_i - \mathsf{v}_j\|^{-\alpha}, \; \mathsf{M}_{\Pi_N}(\mathsf{v}_j) \equiv \max_{\mathsf{u}_i \in \Pi_N \setminus \{\mathsf{u}_j\}} \|\mathsf{u}_i - \mathsf{v}_j\|^{-\alpha}. \tag{15}$$

$$f_{\mathsf{x}_1,\mathsf{x}_2}(x_1,x_2) = \frac{2x_1x_2}{\pi^2 d^2 L^2}\int_{\max\{|d-x_1|,|d-x_2|\}}^{\min\{d+x_1,d+x_2\}} \frac{1}{t\sqrt{\left[1-\left(\frac{d^2+t^2-x_1^2}{2dt}\right)^2\right]\left[1-\left(\frac{d^2+t^2-x_2^2}{2dt}\right)^2\right]}}\mathrm{d}t. \quad (13)$$

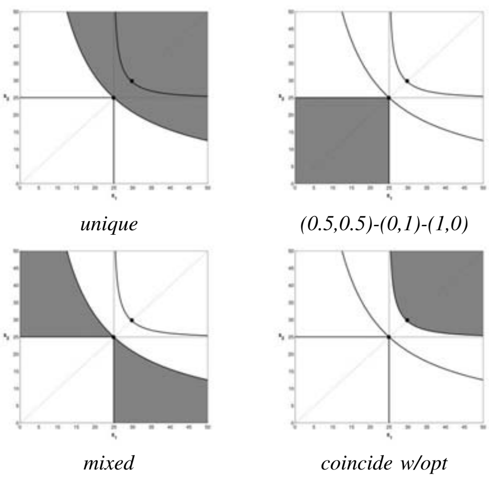

Fig. 4. Regions of Fig. 2 that we are considering in Table I .

As discussed in [19] (§2.5), the sum interference RV Σ is subexponential, meaning

$$\lim_{z\to\infty}\frac{\mathbb{P}(\Sigma_{\Pi_N}(\mathsf{v}_j) > z)}{\mathbb{P}(\mathsf{M}_{\Pi_N}(\mathsf{v}_j) > z)} = 1. \quad (16)$$

That is, the sum interference is large when the maximum interference term is large, i.e., intuitively approximating the sum interference by the maximum is a valid approximation.

The key insight is that under the approximation $\Sigma \approx \mathsf{M}$ the N-player game will decouple into subgames, described in what follows. We introduce the following definition:

Definition V.1 (Coupled users)**.** *Users* $(\mathsf{u}_i,\mathsf{v}_i)$ *and* $(\mathsf{u}_j,\mathsf{v}_j)$ *are* coupled *if the nearest interfering transmitter of receiver* v_j *is* u_i *and the nearest interfering transmitter of receiver* v_i *is* u_j*.*

Under the approximation $\Sigma \approx \mathsf{M}$, a pair of coupled users in the N–player game are playing a 2–player game. We define the probability of a typical user being coupled:

Definition V.2 (Coupling probability)**.** *The* coupling probability $P_c(\lambda)$ *is the probability that a typical user, say* $(\mathsf{u}_i,\mathsf{v}_i)$*, is coupled with the user whose transmitter, say* u_j*, is the closest interferer to* v_i*, i.e., the probability that* u_i *is in fact also the closest interferer to* v_j*:*

$$P_c(\lambda) = \mathbb{P}(i = \arg\min_{k\neq j}\|\mathsf{u}_k - \mathsf{v}_j\| \mid j = \arg\min_{k\neq i}\|\mathsf{u}_k - \mathsf{v}_i\|). \quad (17)$$

Refer to Fig. 1. Assume that TX_2 is the nearest interferer for RX_1. Note that this means that there is no other transmitter in the circle (say, $\mathcal{C}_1$) of radius x_1 with center RX_1, i.e., $\Pi_N(\mathcal{C}_1) = 0$. The probability that TX_1 is the nearest interferer for RX_2 is $P_c(\lambda) = \mathbb{P}(\Pi_N(\mathcal{C}_2) = 0|\Pi_N(\mathcal{C}_1) = 0)$, where $\mathcal{C}_2$ is the circle of radius x_2 with center RX_2. Although occupancy counts of disjoint regions are dependent in a BPP, the dependence vanishes in the limit as the BPP becomes a Poisson Point Process (PPP), i.e., as $N, |\mathcal{A}| \to \infty$ with $N/|\mathcal{A}| \to \lambda$. Under this approximation we have

$$P_c(\lambda) \approx \mathbb{P}(\Pi_\lambda(\Delta) = 0) = \mathrm{e}^{-\lambda|\Delta|}, \quad (18)$$

i.e., the coupling probability is approximately the void probability for a PPP Π_λ of intensity λ on the lune $\Delta \equiv \mathcal{C}_2 \setminus \mathcal{C}_1$.

Theorem V.3. *The value for the coupling probability when* λ *goes to* 0 *is the following constant:*

$$\lim_{\lambda\to 0} P_c(\lambda) = C_{cp} \triangleq \frac{6\pi}{3\sqrt{3}+8\pi} \approx 0.6215.$$

Proof: In the low density (small λ) regime the BPP behaves like the PPP since $|\mathcal{A}|$ must be large. Recall that the average distance to a nearest neighbor in a PPP with density λ is $1/\sqrt{\lambda}$, and thus for small λ the average distance is large. To obtain the value of the coupling probability in the low density regime, we can assume that $x_1 \gg d$ and $x_2 \gg d$. Thus as $\lambda \to 0$, d can be neglected and we can assume that $x_1 = x_2$. In this scenario, reported in Fig. 7, we have two circles with the same radius, whose centers are separated by a distance equal to the radius itself. Therefore, the area of the lune depends only on the parameter x_1. In particular, using the formula of the area of the lune, we obtain that in this case the area of the lune is the following:

$$\Delta = \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)x_1^2. \quad (19)$$

Applying the total probability theorem:

$$P_c(\lambda) = \int_0^\infty P_{c|\mathsf{x}_1}(\lambda, x_1) f_{\mathsf{x}_1}(x_1)\mathrm{d}x_1. \quad (20)$$

The distribution of the nearest interferer is:

$$f_{\mathsf{x}_1}(x_1) = 2\pi\lambda x_1 \mathrm{e}^{-\pi\lambda x_1^2} \qquad x_1 \geq 0. \quad (21)$$

Substituting the expression for $f_{\mathsf{x}_1}(x_1)$ in Eq. (21) and the void probability over the area in Eq. (19), we obtain:

$$P_c(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)x_1^2} 2\pi\lambda x_1 \mathrm{e}^{-\pi\lambda x_1^2}\mathrm{d}x_1 = C_{cp}. \quad (22)$$

■

Fig. 8 reports different curves for the coupling probability $P_c(\lambda)$ vs. λ. The black point is C_{cp}. Thus, in the low density regime, under approximation that the sum interference equals the max interference, 62% of the nodes play a 2–player game, for which the analysis in §IV applies. Note the sum/max

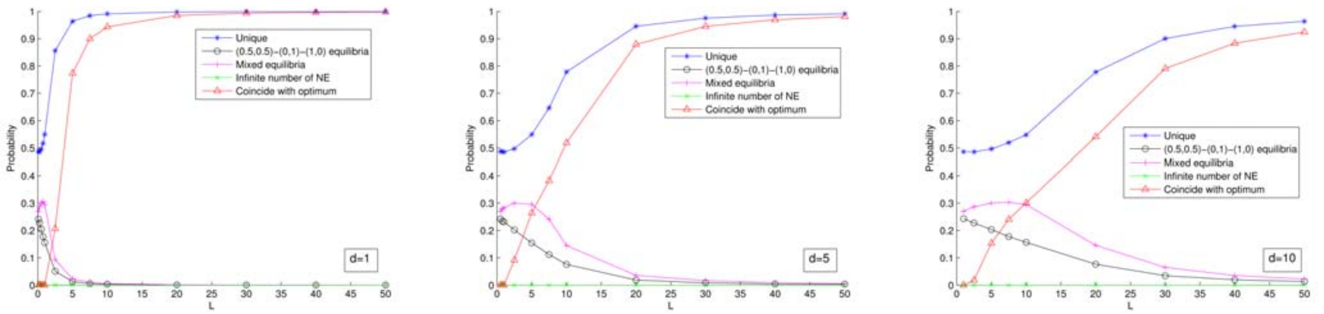

Fig. 5. Probability of having different Nash equilibria for the 2–player game, increasing L, for different values of d, assuming $\eta = 10^{-3}$ and $\alpha = 4$.

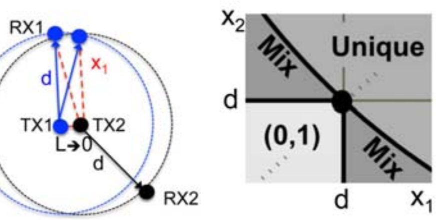

Fig. 6. Scenario that explains the limit as L that approaches zero.

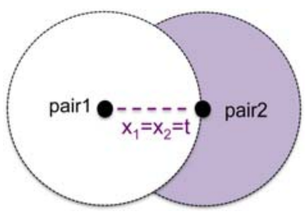

Fig. 7. Reference scenario when λ goes to zero.

approximation is expected to hold in the low density regime. Furthermore, the behavior of the nodes that are not *coupled* can be analyzed as a reaction to the small coupled sub–games, where non-coupled nodes are not playing a game, but instead employ best response dynamics to their nearest interferers. This is ongoing work to characterize the N–player game and predict the performance in large networks.

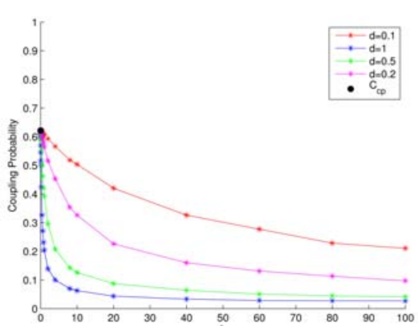

Fig. 8. Simulated coupling probability for different values of d.

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