Reduced order models for the prediction of the time of occurrence of extreme episodes

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Abstract

We describe in this paper a very special class of complex systems, namely those which admit reduced order models for the prediction of the peaks of a relevant variable (the output variable of the system). In particular, we show that if the attractor of the system is a torus or a low-dimensional strange attractor, such reduced order models exist and allow one to forecast the time of occurrence of the next peak, given the times of occurrence of the last three peaks. This property is particularly useful for forecasting recurrent extreme episodes in ecosystems. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Many ecosystems are characterized by recurrent extreme episodes related with sudden collapses and/or regenerations of a number of populations. Very often these episodes are associated with high social–environmental costs or risks. Budworm outbreaks in fir forests of North America, fires in Mediterranean forests, and algae blooms in shallow lakes are typical examples.

Recurrent extreme episodes can be, in general, associated with the maximum (or minimum) of a specific variable \( y(t) \) so that the \( k \)th episode is identified by its time of occurrence \( t_k \) and its intensity \( y_k = y(t_k) \) from now on called peak. Very often the dynamics are complex, in the sense that the return time \( \tau_{k+1} = t_{k+1} - t_k \), namely the time separating two successive peaks of the output variable, varies in a random fashion. The same holds for the peaks \( y_k \). In some cases (e.g., the lynx population in Canada [1]) the peaks vary much more than the return times, while in other cases (e.g., pest outbreaks in forests, see below) the opposite is true. Of course, there are also cases (e.g., fires in Mediterranean forests [2]) in which both the intensity and the return time have remarkable and irregular variations.

The real time forecast of extreme episodes is a difficult problem. In principle, one can approach the problem by using a detailed simulation model of the ecosystem of the form

\[
\dot{x}(t) = f(x(t)),
\]

where \( t \geq 0 \) is time, \( x \in \mathbb{R}^n \) is the \( n \)-dimensional state vector. But, in practice, this is very difficult because a paramount number of measurements is needed to tune models of this kind which have, very often, 3–10 state variables and 10–50 parameters. Thus, a reduced order model is needed.

The simplest model one can imagine for predicting extreme episodes is composed of a pair of one-dimensional maps of the form

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\[ y_{k+1} = F(y_k), \quad \tau_{k+1} = G(y_k). \] (2)

In fact, reduced order models of this kind would allow one to predict the intensity and the time of occurrence of the next episode through the equations
\[ \hat{y}_{k+1} = F(\hat{y}_k), \quad \hat{\tau}_{k+1} = \hat{\tau}_k + G(\hat{y}_k), \]
where \( \hat{\tau} \) and \( \hat{y} \) are measurements, \( \hat{y} \) and \( \hat{\tau} \) are forecasts.

Unfortunately, this forecasting scheme can be rarely adopted because the intensity of the episode is often difficult to measure. By contrast, the time of occurrence of the episode is very often known with relatively high precision. Thus, the following model would be more useful
\[ \tau_{k+1} = F(\tau_k) \] (3)
because it would allow the prediction of the time of occurrence of the next episode through the equation
\[ \hat{\tau}_{k+1} = \hat{\tau}_k + F(\hat{\tau}_k - \hat{\tau}_{k-1}). \] (4)

We will show in Section 2 that if the fractal dimension of the strange attractor of system (1) is close to 2, there are chances that a simple reduced order model (2) exists, while in the most complex case the low dimensionality of the attractor implies that
\[ \tau_{k+1} \in \mathcal{F}(\tau_k), \] (5)
where \( \mathcal{F}(\tau_k) \) is a finite set. This means that only a set of potential times of occurrence of the next extreme episode can be forecast. But, we will also indicate how the ambiguity of the forecast can be resolved by using some extra information. In particular, we will show that, in the most complex case, the reduced order model can be written as follows:
\[ \tau_{k+1} = F(\tau_k, \tau_{k-1}) \] (6)
so that the simple forecasting scheme (4) is substituted by a slightly more complex forecasting scheme, namely
\[ \hat{\tau}_{k+1} = \hat{\tau}_k + F(\hat{\tau}_k - \hat{\tau}_{k-1}, \hat{\tau}_{k-1} - \hat{\tau}_{k-2}). \]

In Sections 3–5, we will discuss three different chaotic ecosystem models: a sixth-order forest–pest model for which a simple reduced order model (3) allows the prediction of the occurrence of the pest-outbreaks; a fourth-order forest fire model for which a complex reduced order model (5) exists, and a tritrophic food chain model that shows that a reduced order model (3) can be determined also for predicting the times at which the food chain is at high risk of extinction.

As far as we know, reduced order models of the form (3), (5) and (6) have never been discussed in the ecological context. By contrast, the fact that, in some systems, return times can be recursively computed through suitable maps was discovered long ago through a series of experiments on dripping faucets [3].

2. Reduced order models of return times

Consider a continuous-time dynamical system described by Eq. (1), and suppose that the system is observed through a single-output variable \( y \in \mathbb{R} \) given by
\[ y(t) = g(x(t)). \] (7)
Assume that system (1) behaves on a strange attractor \( X \) and that the corresponding output variable is recorded for a long time interval. Thus, a return time plot (RTP), namely the set of all pairs \( (\tau_k, \tau_{k+1}) \) can be associated to the output record. In general, the RTP is a rather dispersed cloud of points, but sometimes it is filiform, i.e., its points roughly align along one or more curves.

Fig. 1 reports, for example, two RTP’s, concerning, respectively, the maxima and the minima of the consumer population density of a tritrophic food chain composed of resource \( (x_1) \), consumer \( (x_2) \) and
close to 1 (as in Fig. 2), i.e., $P$ shown in Fig. 1(b).

The Poincaré section. Of course, the Poincaré section is two states $x$ with the parameters fixed at the values indicated in the caption.

A return time of occurrence of the $k$th state, $S$ is associated to each point of the Poincaré section $S$. Thus, the intersection of the strange attractor with $S$ indicates the next output maximum after $t_k$ units of time. Such a manifold is generically transversal to the trajectory and is, therefore, a particular Poincaré section [5]. Figs. 2 shows, for example, the strange attractor of system (8a)–(8c) and the Poincaré section $S$ for $y = x_2$, which is, therefore, identified by (see Eq. (8b) with $x_2 = 0$ and $x_2 \neq 0$)

$$a_1 x_1 \frac{x_1}{1 + b_1 x_1} - a_2 x_3 \frac{x_3}{1 + b_2 x_2} - d_2 = 0.$$  

The trajectories cross the Poincaré section $S$ in two opposite directions when $y$ has a minimum or a maximum (see Fig. 2(b)). Thus, the intersection of the strange attractor $X$ with $S$ can be partitioned into two subsets: one, indicated by $P^+$, corresponds to the maxima of the output variable, while the other, indicated by $P^-$, corresponds to the minima of $y(t)$.

If the strange attractor $X$ has a fractal dimension close to 2, then the fractal dimensions of $P^+$ and $P^-$ are close to 1 (as in Fig. 2), i.e., $P^+$ and $P^-$ can be fairly well approximated on $S$ by a few curve segments. The curves approximating $P^+$ and $P^-$ are called skeletons, and indicated by $S^+$ and $S^-$. A return time $\tau$, as well as a first return point, is associated to each point of the Poincaré section. Thus, one can consider the lines $\tau = \text{const.}$ on $S$ and look for the intersections of these lines with $S^+$ and $S^-$. Figs. 2(c) and (d) show the lines $\tau = \text{const.}$ in a neighborhood of $S^+$ and $S^-$, respectively. Each line $\tau = \text{const.}$ has at most one intersection with $S^+$. This means that a particular value $\tau_k = t_k - t_{k-1}$ identifies only one point on $S^+$, namely the state $x(k-1) \in S^+$ (corresponding to a maximum of $y$) that gives rise to the next output maximum after $\tau_k$ units of time. Of course, the Poincaré map defined on $S$ determines the next state $x(k) \in S^+$ and, hence, the return time $\tau_{k+1}$. In conclusion, Eq. (3) holds for the return times concerning the maxima, as already shown in Fig. 1(a). By contrast, some lines $\tau = \text{const.}$ intersect the set $S^-$ twice. Thus, for some return times $\tau_k$, there are two states $x(k-1) \in S^-$ and, hence, two next states $x(k) \in S^-$ and two corresponding return times $\tau_{k+1}$, as shown in Fig. 1(b).
More in general, a finite cardinality $K$ of next return times $s_k^\ddagger$ is associated to a return time $s_k$, i.e., Eq. (5) holds with $F_s^\ddagger$ of cardinality $K$. In order to resolve the ambiguity among the $K$ possible forecasts, we can proceed as follows. First we partition the RTP, namely the set of all pairs $s_k; s_k^\ddagger$ into $K$ disjoint subsets with skeletons $R_1; R_2; \ldots; R_K$ in such a way that each point $s_k; F_s^\ddagger$ belongs to one and only one $R_i$ (see Fig. 3(a) where this is done for the RTP of Fig. 1(b), which has $K \approx 2$). This means that each skeleton $R_i$ is described by a one-dimensional map $s_k^\ddagger = F_i(s_k)$. Then, we determine from the time series the skeletons $R_1^-, R_2^-, \ldots, R_K^-$, called predecessors, with the following rule (see Fig. 3(b))

$$(\tau_k, \tau_{k+1}) \in R_i^- \quad \text{if} \quad (\tau_{k+1}, \tau_{k+2}) \in R_i.$$

Thus, by construction, the next return time can be forecast as follows (see Fig. 3(c)):

$$\tau_{k+1} = F_i(\tau_k) \quad \text{if} \quad (\tau_{k-1}, \tau_k) \in R_i^-.$$

This relationship shows that the knowledge of two consecutive return times $(\tau_k, \tau_{k+1})$ allows one to determine the next return time $\tau_{k+1}$, i.e., Eq. (5) holds independently upon the value of $K$.

3. Pest outbreaks

Through the analysis of a model, we show in this section that insect pest outbreaks can be chaotic and that the time separating successive outbreaks can obey a simple law of the form (3).

The model, we use, has been recently proposed by Gragnani et al. [6]. It is a sixth-order model with five state variables concerning the vegetation and one state variable indicating the biomass of the defoliator insects.
Hence, the first five equations $\dot{x}_i = f_i$ represent the vegetation submodel: they describe the mechanisms of carbon and nitrogen exchange between soil and vegetation, biomass decomposition and microbial mineralization, and defoliation by pest grazers. The sixth differential equation describes the dynamics of defoliators.

Realistic ranges for the parameters are indicated in [6], where the model is analyzed for different parameter values and five modes of behavior are uncovered. Although not shown explicitly in [6], the model can also have chaotic behavior, i.e., the pest outbreaks can be periodic. One of the most famous examples of this kind of regime is that of the spruce budworm, *Choristoneura fumiferana*, for which several models have been suggested (see, for example, [7–9]). Nevertheless, the models proposed until now were not capable to explain the randomness of the pest outbreaks.

Fig. 4(a) shows the series of pest outbreaks obtained by simulation for realistic values of the parameters. The return times are quite variable (from 40 to 100 yr), while the peaks of defoliator biomass are less variable. The RTP obtained through simulation is filiform (as shown in Fig. 4(b) and can be approximated by a model of the form (3), which says, in particular, that if the time interval separating two subsequent pest outbreaks is large, then the next one is going to be short.

4. Forest fires

A minimal model describing forest fire interactions in various forests has been recently proposed by Casagrandi and Rinaldi [10]. The model is composed of four state variables: two for the upper layer (green and burning trees) and two for the lower layer (green and burning bush).
The model can be interpreted as the coupling of two prey–predator sub-models, one for each layer (the green biomass being the prey and the burning biomass being the predator). The model has limit cycles for parameter values corresponding to savannas and boreal forests and strange attractors for parameter values corresponding to Mediterranean forests.

Fig. 5(a) shows the simulated pattern of the green biomass of the upper layer of a Mediterranean forest. Each maximum of the green biomass corresponds to a fire and the figure clearly shows that the fire return time is chaotic and quite variable.

The RTP obtained from this simulation is reported in Fig. 5(b). The plot is filiform but complex, in the sense that it gives a model of the form (5). Thus, the procedure described in Section 2 (Fig. 3) is applied. The RTP is partitioned into three skeletons $R_1$, $R_2$ and $R_3$ as indicated in Fig. 5(b) and the predecessors $R^{-1}_1$, $R^{-1}_2$ and $R^{-1}_3$ are determined through simulation (see Fig. 5(c)). Thus, the next return time $\tau_{k+1}$ can be forecast from the two previous return times $(\tau_{k-1}, \tau_k)$ as indicated in Fig. 5(d).

5. Resource–consumer bursts

Models (8a)–(8c) shows that for suitable parameter values chaotic tritrophic food chains can be characterized by recurrent bursts of resource–consumer oscillations. This is a consequence of the tea-cup geometry of the strange attractor [4]. Fig. 6(a) shows the typical time pattern of the consumer: after a period of very low density the consumer population quickly regenerates and reaches a maximum followed by a few lower maxima, before declining again. It might therefore be of interest to predict the next burst from informations concerning the last one(s). In particular, it might be interesting to check if the time $T_{k+1}^+$ separating two successive bursts (defined as the time separating the first maximum of a burst form the first
maximum of the next burst) can be directly computed from $T_k^+$. This is actually what happens, as shown by the RTP of Fig. 6(b) which is filiform and indicates that a model of the kind

$$T_{k+1}^+ = F(T_k^+)$$

can be used for forecasting the time of occurrence of the next burst.

One could also define the time separating two successive bursts as the time $T_k^-$ separating the minimum preceding one regeneration of the consumer from the minimum preceding the next regeneration. Fig. 6(c) shows the corresponding RTP which indicates that a simple map

$$T_{k+1}^- = F(T_k^-)$$

can be used to forecast the occurrence of the bursts. The use of this map would be particularly justified from a biological point of view, because it would allow one to forecast when the consumer population will reach particularly low densities, thus being at particular high risk of extinction.

6. Concluding remarks

We have shown in this paper that chaotic systems with strange attractors of dimension close to 2 enjoy a remarkable property called peak-to-peak dynamics: the time of occurrence of the peak of any relevant variable can be predicted from the time of occurrence of the last three peaks. This means that a complex model composed of $n$ nonlinear differential equations can be substituted by a simple recursive equation, if attention is restricted to the time of occurrence of the peaks.

A great number of chaotic ecological models, among which various food chain and food web models have been recently analyzed [11] are shown to admit peak-to-peak dynamics. Moreover, some laboratory experiments and the analysis of a few ecological time series, among which the famous canadian lynx time series [12], confirm that peak-to-peak dynamics might be relatively common also in the field. Thus, the
property pointed out in this paper is of great potential interest in natural resource management, where the prediction of extreme episodes (usually associated to the peak of a specific population) is a problem of major concern. The cases of insect pest outbreaks and forest fires analyzed in this paper are typical examples, but many others remain to be explored.

Since the randomness of the return times of the above-mentioned extreme episodes poses some problems, it might be of interest to conceive management policies aimed at regularizing the chaotic dynamics of the peaks. In principle, this can be done by formulating suitable optimal control problems involving the reduced order model. For example, one might try to maximize the minimum return time of the episodes. But also the amplitude of the peak, which is also important in determining the damage associated to the extreme episode, can be included in the formulation of the problem, as shown in [13]. In conclusion, peak-to-peak dynamics seem to be a very special but also very promising property of ecosystems, that might help managers of renewable resources to improve their forecasting techniques and their operating rules.

Acknowledgements

Part of this work has been carried out at the International Institute for Applied Systems Analysis (IIASA), Laxenburg, Austria by the first author, and at the Department of Ecology and Evolutionary Biology, Princeton University, NJ, USA, where the second author was visiting thanks to Simon Levin and to a prize from Italgas. Financial support came from Consiglio Nazionale delle Ricerche, Project ST/74 “Mathematical Methods and Models for the Study of Biological Phenomena”.

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