Non-linear Dynamics in Adaptive Control: Chaotic and Periodic Stabilization

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A specific example illustrates that adaptive control systems are essentially non-linear, which in particular is reflected in a positive way in their robustness properties with respect to undermodelling.

Key Words—Adaptive systems; non-linear control systems; (chaotic dynamics); robust control; stability.

Abstract—In this paper the non-linear features of adaptive control algorithms are highlighted. Using a simple "linear" discrete time example, based on classical design, it is demonstrated that in the presence of undermodelling errors, non-linear phenomena in the feedback gain such as limit cycles and even chaos arise. Despite these complicated dynamics, robust stabilization of the plant can still occur. For the class of undermodelling errors considered, the set of plants stabilizable by this adaptive controller is completely characterized.

1. INTRODUCTION

ADAPTIVE CONTROL is concerned with the automatic adjustment of feedback laws to control unknown fixed or slowly varying plants which are elements of some class of dynamical systems. The standard approach to adaptive control is to identify a parametrized model of the plant based on current input/output data, and then to use this parameter estimate on-line to compute a feedback control signal on the basis of the estimate being precisely the correct value. This certainty equivalence notion of using the parameter estimate underlies much of adaptive control, and stability studies typically rely on establishing convergence of the parameter value either to a point or to a set (Goodwin and Sin, 1984). When such adaptive controllers are applied to linear plants, or more correctly to linear plant models, the control signal is derived from a linear feedback control strategy (such as minimum variance, pole-positioning, LQG etc.) and the aim is that convergence of the parameter value results in asymptotic satisfaction of a linear design criterion. The same remarks hold for the Nussbaum high gain adaptive stabilization methods (Nussbaum, 1983; Willems and Byrnes, 1984). In spite of this linear formulation, however, the adaptive control methods yield fundamentally non-linear controllers. This follows because both the parameters of the feedback controller and the signals they multiply are functions of the measured plant output. Subject to parameter convergence, there may be asymptotic linearity and, subject to parameters lying in certain regions of the parameter space, there may be local linearizability, but prior to being able to establish these conditions one must face the fact that adaptive control is inherently a non-linear control strategy.

Given that adaptive control is non-linear, it is reasonable to ask whether such schemes exhibit characteristically non-linear dynamics and, more particularly, to ask whether the non-linear dynamics may even be favourable for the purposes of adaptive control and stabilization. It would be especially desirable to have access to possible improved robustness features of certain non-linear controllers since the class of plants to be controlled typically is only approximately described by the particular class of finite dimensional, linear, parametrized models and the unmodelled dynamics need to be accommodated.

A very simple adaptive control scheme will be presented which demonstrates vividly a wealth of non-linear dynamical behaviour with remarkably beneficial effects in certain circumstances. The problem under study is a first order adaptive controller used to regulate a linear second order plant. An adaptive controller is constructed using the certainty equivalence ideas above with a very fast (dead-beat) adaptive (scalar) parameter estimator coupled with a dead-beat feedback control law. Although simple in form, the dynamics of the closed loop system are highly complicated. For a range of plant systems the adaptive gain becomes periodic such that the plant output is regulated to zero. Still more interestingly, under different conditions the adaptive controller gain behaves chaotically but
still regulates the plant output. The class of second order plants can be partitioned completely into those stabilized by the first order adaptive controller and those not stabilized and hence the robustness with respect to this particular form of undermodeling can be completely characterized.

Recently there has been a considerable effort devoted to the local stability of adaptive control systems and the related robustness issues (Anderson et al., 1984; Åström, 1984; Riedle and Kokotovic, 1984; Riedle et al., 1984). The methods used in this area rely heavily on linearization of the adaptive control problem and so are capable of producing local results, typically requiring the parameter estimates always to lie within certain bounded sets or neighbourhoods—including the initial parameter values. When the adaptation is slow, averaging theory or other techniques may then be applied to the linearized system to derive conditions for robust stability of the adaptive control scheme. When the initial conditions are large or when the local neighbourhood is not stable the methods break down and say little or nothing about global response.

The algorithm treated here is not linearizable about a stabilizing trajectory because of the use of a very fast, i.e. dead-beat, identifier. Thus the closed loop dynamics are dictated entirely by non-linear effects. This formulation allows this behaviour to be made explicit and, as such, the example serves the primary purpose of displaying non-linear dynamical phenomena in adaptive control. However, the implications are more far-reaching in that the particular dynamics associated with this non-linearizable set-up describe the transient behaviour of very standard linearizable adaptive control systems with large initial conditions. The rudiments of a global theory are therefore constructed for the robust adaptive regulation of a limited class of linear systems with unmodelled dynamics.

Rubio et al. (1985) and Salam and Bai (1986) also report instances of chaotic dynamics in continuous time adaptive control schemes. In Rubio et al. the chaotic dynamics are due to the non-linear plant to be controlled, whilst in Salam and Bai the chaos is generated via a sinusoidal disturbance. Their analysis is based on a Melnikov technique. In this discussion however the chaos is due to the adaptive algorithm itself. A more complete analysis is also presented (Mareels and Bitmead, 1986) with less pessimistic conclusions. The aim in presenting this work is not to claim the complete solution of any particular adaptive control problem but rather to highlight the inherent non-linear dynamics of adaptive control systems and to illustrate the occurrence of unexpectedly complex behaviour, such as periodic or chaotic feedback gains, describing the evolution of the closed loop system. In spite of presenting formidable difficulties for analysis, the effect of these dynamics on the adaptive system performance need not be deleterious and, indeed, may be the basis of considerable robustness and stability.

The paper is organised as follows. Section 2 is devoted to the explicit formulation of the adaptive control problem studied, where the class of plants considered and parametrized models used, the identifier structure and the linear certainty equivalence control strategy are stated. In this section an explicitly non-linear difference equation, which describes the complete closed loop adaptive system, is derived. The dynamics of this difference equation fall into three distinct sets depending upon a single parameter b, measuring the unmodelled dynamics and the next three sections concentrate on describing the closed loop behaviour for each of these classes. Section 3 briefly describes the properties of the adaptive scheme when no modelling errors are present, i.e. \( b = 0 \). Section 4 considers negative values for the parameter b and proves that the feedback gain is asymptotically periodic. For a range of values of negative b this periodic gain stabilizes the plant. In Section 5 positive b is considered and the feedback gain is shown to be chaotic, again stabilizing the plant for a range of values of b. Section 6 deals more fully with the effects of the closed loop stability of the plant due to these different feedback gains and considers questions of performance and robustness of the adaptive control scheme. Section 7 concludes and draws together the threads of the previous sections to discuss the implications of the results of the paper.

2. THE ADAPTIVE SCHEME

In this section an example of an adaptive controller is presented, designed on the assumption that the plant to be controlled is a first order, linear, stationary, discrete time system. The control objective is to regulate the plant output to zero. The emphasis is on the robustness properties with respect to undermodelling by analyzing the behaviour of the controller–plant closed loop under the assumptions that there is no noise, and that the plant is a second order, time invariant (discrete time) system.

Suppose that the system to be controlled is given as follows.

\[
y_k = a y_{k-1} + b y_{k-2} + u_{k-1}, \quad k \in \mathbb{N}.
\]  

(2.1)

Here \( a, b \) are the unknown parameters. The
designer, however, believes that the system can be represented as a first order model.

The designer’s model:

\[ y_k = \Delta y_{k-1} + u_{k-1}, \quad k \in \mathbb{N}. \] (2.2)

Because the environment is noise free the adaptive controller can be based on the dead-beat identification rule.

The parameter estimator:

\[ \hat{\alpha}_k = \hat{\alpha}_{k-1} + \frac{1}{y_{k-1}} (y_k - \hat{\alpha}_{k-1} y_{k-1} - u_{k-1}) \]

\[ y_{k-1} \neq 0, \quad k \in \mathbb{N}. \] (2.3)

Here \( \hat{\alpha}_k \) is the parameter estimate for \( \alpha \) at time \( k \), and \( e_k \),

\[ e_k = y_k - \hat{\alpha}_{k-1} y_{k-1} - u_{k-1} \]

\[ = y_k - \hat{y}_k \] (2.4)

is the prediction error. The parameter estimator (2.3) can be seen as the limit of a normalized least mean square algorithm:

\[ \hat{\alpha}_k = \hat{\alpha}_{k-1} + \frac{y_{k-1}}{c + y_{k-1}^2} e_k, \quad c > 0, \quad k \in \mathbb{N}. \] (2.5)

The parameter estimator (2.3) follows from (2.5) by letting the step-size parameter \( 1/c \) tend to infinity, or \( c \) to zero. A dead-beat control law is used. Under the given structural assumptions this guarantees excellent performance when there is no model error (Goodwin and Sin, 1984). This is discussed further in Section 3.

The control law:

\[ u_{k-1} = -\hat{\alpha}_{k-1} y_{k-1}. \] (2.6)

Replacing \( u_{k-1} \) in (2.1) by the expression in (2.6) and using

\[ \hat{\alpha}_k = a - \hat{\alpha}_k \]

yields

\[ y_k = \hat{\alpha}_{k-1} y_{k-1} + b y_{k-2}, \quad k \in \mathbb{N}. \] (2.7)

Similarly substituting \( y_k \) in (2.3) using (2.1), leads to

\[ \hat{\alpha}_k = \hat{\alpha}_{k-1} + (\hat{\alpha}_{k-1} y_{k-1} + b y_{k-2})/y_{k-1} \]

hence

\[ a - \hat{\alpha}_k = a - \hat{\alpha}_{k-1} - \hat{\alpha}_{k-1} - b y_{k-2}/y_{k-1} \]

or

\[ \hat{\alpha}_k = -b y_{k-2}/y_{k-1}, \quad y_{k-1} \neq 0, \quad k \in \mathbb{N}. \] (2.8)

Obviously the parameter error \( \hat{\alpha}_k \) can be eliminated from (2.7–2.8) to obtain a single non-linear autonomous difference equation, which completely describes the dynamics of the adaptively controlled closed loop.

The closed loop:

\[ y_k = -b y_{k-3}/y_{k-2} + b y_{k-2}/y_{k-1}, \quad k \in \mathbb{N}. \] (2.9)

\[ y_{k-2} \neq 0. \]

This equation can be rewritten as,

\[ \frac{y_k}{y_{k-1}} = -b \frac{y_{k-3}}{y_{k-2}} + b \frac{y_{k-2}}{y_{k-1}}, \quad k \in \mathbb{N}. \] (2.10)

Introducing the ratio of successive outputs as a new variable finally gives the following.

Alternative description of closed loop:

\[ r_k = b \left( \frac{1}{r_{k-2}} - \frac{1}{r_{k-1}} \right), \quad k \in \mathbb{N}, \] (2.11)

where

\[ r_k = \frac{y_k}{y_{k-1}} \]

\[ y_k = \prod_{i=0}^{k} r_i \cdot y_{i-1}. \] (2.12)

For the parameter estimate,

\[ \hat{\alpha}_k = a + b \frac{1}{r_{k-1}}, \quad k \in \mathbb{N}. \] (2.14)

From (2.6), this relation also describes the controller gain.

Remarks.

(2.1) Notice that this design followed strictly the classical route of adaptive control scheme design based on the certainty equivalence principle. Note also that the implemented control law is the much celebrated minimum variance controller (Goodwin and Sin, 1984).

(2.2) Note the non-linear nature of the closed loop, expressed explicitly in (2.9). Further notice that it is impossible to linearize about \( y_k = 0 \),
which is the desired trajectory! Note that (2.9) or equivalently (2.11–2.13) gives a complete description of the non-linear closed loop adaptive scheme in terms of the output $y_k$ of the controlled plant. This non-linear equation describes the evolution of the closed loop consisting of the linear plant with the non-linear adaptive feedback controller.

(2.3) The introduction of the ratio $r_k$ of successive outputs of the plant admits a simplified analysis. Compare (2.11) with (2.9)—the state of (2.11) lies in $\mathbb{R}^2$, whilst the state of (2.9) is in $\mathbb{R}^3$. This reduction in dimension facilitates the analysis considerably.

(2.4) As it stands, the dead-beat identifier (defined in (2.4)) cannot produce a meaningful estimate when $y_k = 0$. This singularity is reflected in the closed loop equations (2.9)–(2.11). It transpires that this problem of a possible division by zero causes no real difficulties. However, this problem will be treated in detail in subsequent sections.

(2.5) Note that the parameter $a$, the sum of the values of the open loop poles of the plant, is irrelevant to the dynamics of the closed loop (e.g. 2.9). This is due to the combination of a dead-beat identifier and a dead-beat control law. In terms of robustness this feature looks most promising, i.e. if the adaptive scheme works, it works for both stable and unstable plants depending only on the $b$ parameter. Although the closed loop dynamics are not influenced by the parameter $a$, the parameter estimate $\hat{a}_k$ and hence the control signal $-\hat{a}_k y_k$ do depend on it. Both estimate and control action adapt to different values for different plants, i.e. the control scheme is really an adaptive control scheme and not merely a robust non-linear controller. Also, it does not belong to the class of high gain controllers as discussed, for example, in Nussbaum (1983) and Willems and Byrnes (1984).

This section ends with a preview of where the analysis will lead. First of all note that depending on the parameter $b$, three types of topologically different dynamical behaviour can be distinguished. For $b < 0$, $r_k$ becomes asymptotically periodic, for $b = 0$, a "trivial" dead-beat response results and for $b > 0$, $r_k$ behaves chaotically. That one can distinguish at most three types of different topological behaviour follows from the observation that one can rescale (2.11) to obtain

$$v_k = r_k/\sqrt{|b|}, \quad b \neq 0$$

$$v_k = (\text{sgn}(b)) \left\{ \frac{1}{r_{k-1}} - \frac{1}{r_{k-2}} \right\}, \quad k \in \mathbb{N}.$$  

One can immediately verify that (2.16) has two periodic trajectories with period two, $\{\sqrt{2}, -\sqrt{2}, \sqrt{2}, -\sqrt{2}, \ldots\}$ and $\{-\sqrt{2}, \sqrt{2}, -\sqrt{2}, \ldots\}$ if $b < 0$, and admits no such solution for $b > 0$. The trivial situation $b = 0$ corresponds to dead-beat control in the absence of model errors.

Section 3 contains comments on the case $b = 0$, in Section 4 $b < 0$ is treated and in Section 5 the dynamics for $b > 0$ are discussed. The analysis concentrates on (2.11–2.16).

3. THE DYNAMICS OF THE CLOSED LOOP I: $b = 0$

The combination of a dead-beat identifier and a dead-beat control law guarantees excellent performance with great "adaptation" power, when no "model errors" ($b = 0$) are present.

When $b = 0$, the closed loop reduces to:

**Plant:** $y_k = ay_k + u_{k-1} \quad k \in \mathbb{N}$.  

**Identifier:** $\hat{a}_k = a \quad \forall k \geq 1$.  

**Control law:** $u_{k-1} = ay_{k-1} \quad \forall k \geq 2$.  

**Hence:** $y_k = 0 \quad \forall k \geq 2$.

Note also that in the event $y_{-1} = 0$, the identifier does not identify anything useful, but there is also no need to identify anything, as in this case, $y_k \equiv 0$, $k \in \mathbb{N}$, without control effort.

The adaptive scheme is extremely robust to other types of model errors, such as non-stationarity of $a$, or additive and/or multiplicative noise (see Section 7).

4. THE DYNAMICS OF THE CLOSED LOOP II: $b < 0$

The relevant equation is:

$$v_k = -\left( \frac{1}{v_{k-1}} - \frac{1}{v_{k-2}} \right), \quad k \in \mathbb{N},$$  

where $v_k$ is the normalized (with respect to $b$) ratio of the outputs of the plant. The relations between $r_k$ and the output $y_k$ and the controller gain/parameter estimate are:

$$y_k = \sqrt{|b|} v_k y_{k-1}, \quad k \in \mathbb{N}$$

$$\hat{a}_k = a - \sqrt{|b|} \frac{1}{v_{k-1}} \quad k \in \mathbb{N}.$$  

First, a trivial time-dependent transformation is introduced which fixes the periodic orbits $\{\sqrt{2}, -\sqrt{2}, \sqrt{2}, -\sqrt{2}, \ldots\}$ and $\{-\sqrt{2}, \sqrt{2}, -\sqrt{2}, \ldots\}$ into two fixed point orbits $\{1, 1, \ldots\}$ and $\{-1, -1, \ldots\}$.

$$w_k = (-1)^k r_k/\sqrt{2}, \quad k \in \mathbb{N}.$$  

The correspondence between the plant output and
the $w_k$s is given by:

$$w_k = \frac{(-1)^k y_k}{\sqrt{2|b|} y_{k-1}}, \quad k \in \mathbb{N}$$

(4.5)

and

$$y_k = \sqrt{2|b|} ((-1)^{k+1} + 1) \prod_{i=1}^{k} w_i y_0.$$  

(4.6)

The correspondence between the parameter estimate/controller gain and $v_k$ is

$$\hat{a}_k = a + (-1)^k \frac{b}{2} \frac{1}{w_{k-1}}.$$  

(4.7)

The $w_k$s satisfy the recurrence relation:

$$w_k = \frac{1}{2} \left( \frac{1}{w_{k-1}} + \frac{1}{w_{k-2}} \right), \quad k \in \mathbb{N}.$$  

(4.8)

**Theorem 4.1.** All initial conditions $(w_0, w_{-1})$, except for a set of Lebesgue measure zero, yield solutions \{$w_q(w_0, w_{-1}); k \in \mathbb{N}$\} of (4.5) which are unique and well defined in the future: $0 < |w_q(w_0, w_{-1})| < \infty$, $k \in \mathbb{N}$. All trajectories \{$(w_{k-1}, w_k), k \in \mathbb{N}$\}, defined in the future, converge exponentially to either $(1, 1)$ or $(-1, -1)$.

The detailed proof of this result, together with some additional information about the dynamics of (4.5) is reported in Mareels and Bitmead (submitted for publication).

**Remarks.**

(4.1) Locally the “exponential” stability follows from the linearized difference equation

$$(w_k - w^*) + \frac{1}{2}(w_{k-1} - w^*) + \frac{1}{2}(w_{k-2} - w^*) = 0$$

(4.9)

which is valid in a neighbourhood of $w^* = \pm 1$. Outside this neighbourhood the convergence is actually faster.

(4.2) The phase plane $\mathbb{R}^2$ for the difference equation (4.8) is partitioned into regions of initial conditions which converge either to $(1, 1)$ or to $(-1, -1)$. The boundaries of these regions consist of those initial conditions which cannot be iterated indefinitely in time, these correspond to the “division by zero events”. Interpreting the result of Theorem 4.1 in terms of the (normalized) ratio of the output of the plant $(y_k)$ to $r_k$, the theorem says that the 2-periodic orbit is globally attractive in $\mathbb{R}^2$, except for a set of Lebesgue measure zero, i.e. the union of the above mentioned boundaries.

Hence, the theorem captures the generic dynamics of the closed loop adaptive scheme. 

Via (4.7) and (4.5) the properties of $w_k$ can be related to the controller gain and the plant output, respectively. Clearly, because $|w_k|$ converges exponentially to 1, the controller gain exponentially becomes periodic with period 2:

$$\hat{a}_k \rightarrow a + (-1)^k \left( \frac{|b|}{2} \right)^k.$$  

Hence, asymptotically the controller gain/parameter estimate has the correct averaged value $a$:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} \hat{a}_k = a$$

around which it oscillates with period 2 and an amplitude depending solely on $b$. Also, because $|w_k|$ is bounded and converges exponentially to 1, the plant output $y_k$, which from (4.6) satisfies

$$y_k = \sqrt{2|b|} ((-1)^k w_k) y_{k-1}, \quad k \in \mathbb{N}$$

converges exponentially to zero if $|b| < \frac{1}{2}$, and diverges exponentially if $|b| > \frac{1}{2}$.

These results, and their counterparts of the next section, which treats the case $b > 0$, are summarized in the main Theorems 6.1 and 6.2. Preferring to comment on the results of this section together with those of the next section, the discussion is postponed until Section 6, when the picture of the closed loop dynamics will be more complete.

5. THE DYNAMICS OF THE CLOSED LOOP III; $b > 0$

In this section the dynamical behaviour of the closed loop adaptive system, described by (2.11), is analyzed in the case $b > 0$. The difference equation is

$$v_k = \left( \frac{1}{v_{k-1}} - \frac{1}{v_{k-2}} \right), \quad k \in \mathbb{N}.$$  

(5.1)

The plant output $y_k$ is related to $v_k$ by

$$y_k = \sqrt{b} v_k y_{k-1}.$$  

(5.2)

The controller gain/parameter estimate is given by

$$\hat{a}_k = a + \sqrt{b} \frac{v_{k-1}}{v_k}.$$  

(5.3)

The state representation


\[ x_k = \begin{pmatrix} t_k \\ r_k \end{pmatrix} \]

is associated with (5.1), with the state transition map

\[ x_{k+1} = G(x_k), \]

where \( G \) is defined by

\[ G: \mathbb{R}^2 \rightarrow \mathbb{R}^2; \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rightarrow \begin{pmatrix} y_2 \\ \frac{1}{y_2 - \frac{1}{y_1}} \end{pmatrix}, \]

where \( X = \{(x, y) \in \mathbb{R}^2, x < 0 \} \).

In Mareels and Bitmead (1986) it was demonstrated that the difference equation (5.1) or equivalently (5.5) displays chaos. These complicated dynamics are exhibited in Figs 1 and 2. Figure 1 gives a time portrait of the \( v_k \)-sequence, whilst Fig. 2 contains the corresponding state space portrait. (Figure 1 contains only the first 500 samples, whilst Fig. 2 contains 40,000 samples.) For convenience of graphical representation the plane is mapped onto the square \((-1,1) \times (-1,1)\) by the way of the homeomorphism

\[ H: \mathbb{R}^2 \rightarrow (-1,1) \times (-1,1); (x, y) \rightarrow (x/(1 + |x|), y/(1 + |y|)). \]

This does not affect any topological properties a certain picture in \( \mathbb{R}^2 \) could have.

Chaos may be defined as follows.

**Definition 5.1.** (Guckenheimer and Holmes, 1983; Kloeden, 1981). The difference equation (5.5) is chaotic if there exists an invariant set \( S \subset \mathbb{R}^2 \) (\( G(S) \subset S \)) containing sets \( P, A_1, A_2 \) with the properties:

(i) (5.5) has a countably infinite number of periodic solutions with all periods above a certain integer, \( P \) is the collection of all the points visited by these trajectories;

(ii) (5.5) has an uncountably infinite number of aperiodic solutions which never become asymptotically periodic, \( A_1 \) is the collection of all the points by these trajectories

\[ \forall u_0 \in P, \; \forall y_0 \in A_1: \limsup_{k \rightarrow \infty} \|G^k(u_0) - G^k(y_0)\| > 0; \]

(iii) \( \exists \varepsilon > 0 \forall u_0, y_0 \in A_1: u_0 \neq y_0: \limsup_{k \rightarrow \infty} \|G^k(u_0) - G^k(y_0)\| > \varepsilon \)

(all aperiodic orbits separate);

(iv) \( A_1 \) contains an uncountable subset \( A_2 \) such that

\[ \forall u_0, y_0 \in A_2: \liminf_{k \rightarrow \infty} \|G^k(u_0) - G^k(y_0)\| = 0 \]

Properties (iii) and (iv) express an extreme sensitivity of the trajectories of (5.1) to changes in initial conditions. Trajectories (belonging to \( A_2 \)) merge and separate consecutively in time; coming closer to each other but then being forced to separate to at least a distance \( \varepsilon \) from each other. The phenomenon is easily understood when the periodic orbits are of saddle type, i.e. have one-dimensional...
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FIG. 2. Feedback parameter (normalized $X/(1 + |X|)$) phase plane. “The Gumleaf attractor.”

stable and unstable manifolds, and when their manifolds form a dense web. Close to stable manifolds trajectories are attracted to each other, whilst along the unstable manifolds they are separated. The observation is really the key to an intuitive understanding of the complete dynamical behaviour of the present closed loop adaptive system. This idea will be expanded, but first the presence of chaos is demonstrated.

Lemma 5.1. Some iteration of the map $G$ is chaotic in the sense of Definition 5.1.

Proof. A detailed proof is presented in Mareels and Bitmead (1986). Basically the existence of a fixed point $p$ for $G^4$ ($G^4(p) = p$) is demonstrated, which is a saddle and for which the unstable manifold intersects the stable manifold transversally. The Smale–Birkhoff Homoclinic Theorem (Guckenheimer and Holmes, 1983) then asserts the existence of a horseshoe-type of chaos for an iteration of $G^4$.

The points where the stable manifold intersects the unstable manifold transversally are called homoclinic points. These points converge to the periodic orbit both under the forward and backward mapping and signify the strange dynamical behaviour displayed by (5.1)–(5.5).

Remark 5.1. The invariant set established by the Smale–Birkhoff Homoclinic Theorem is a Cantor set $\times$ Cantor set. Because this set need not be attractive—its domain of attraction is typically a Cantor set $\times$ curve, this type of chaos is called transient. Although erratic behaviour can persist over a long period of time in many orbits not belonging to the invariant set, it eventually dies out. Of course, in order to address the stability question of the output $y$, it is precisely the asymptotic behaviour (of a generic orbit) which is of interest.

Theorem 5.1. The feedback gain/parameter estimate $\hat{\theta}$ behaves chaotically (in the sense of Definition 5.1).

Proof. This is obtained, directly from Lemma 5.1 and (5.3).

Remark 5.2. Note that the Definition 5.1 is rather academic. Any finite wordlength implementation of a difference equation—even if this difference equation is chaotic—has a finite state space and hence cannot be chaotic in the sense of Definition 5.1. The practical importance however is that whenever an algorithm has chaos according to Definition 5.1, its output from any reasonable implementation may closely resemble a random process. On the other hand, even if the implementation seems to behave erratically, it is not necessarily due to a chaotic difference equation. It could be that the difference equation had only strictly stable periodic orbits, but with such a small region of attraction that a numerical simulation never could reveal their existence!

Despite the previous comments, which force one to be cautious, the authors believe that the chaos is fundamentally present in the dynamics of the equation, in the sense that almost all trajectories exhibit a periodic behaviour and never settle down to a more regular periodic behaviour, i.e. the chaos
is not transient. This observation, not captured by Lemma 5.1, is formalized in the following conjecture, the technical details of which will be discussed subsequently.

**Conjecture 5.1.** The chaos displayed by the feedback gain is generically present and is ergodic, governed by a strange attractor, consisting of the union of all the unstable manifolds of the periodic orbits.

Demonstration of the existence of periodic orbits of all periods above period 3 and proof that they are all saddle type would prove this conjecture. Unfortunately, as yet this cannot be done, but all analytical and numerical evidence points this way.

The periodic orbits of period $n$, $V_{k+1} = V_k$, $l = 1, \ldots, n$, $k \in \mathbb{N}$; for (5.1) can be found by solving the set of (algebraic) equations:

\[
\begin{align*}
x_1 &= \frac{1}{x_2} - \frac{1}{x_{n-1}} \\
x_2 &= \frac{1}{x_1} - \frac{1}{x_n} \\
x_k &= \frac{1}{x_{k-1}} - \frac{1}{x_{k-2}}, \quad k = 3, \ldots, n.
\end{align*}
\]

This system of equations has no real solutions for $n < 4$. It has a unique solution for $n = 4$; unique after identifying those solutions which produce the same phase portrait in $\mathbb{R}^2$, which corresponds to identifying those solutions which differ only by a cyclic permutation of the indices. For each $n > 4$, there are at least two different solutions.

As Figs 1 and 2 suggest, the simulations indicate that these periodic orbits are not attractive. Indeed, by investigating their local stability properties—through linearization—one finds that they are saddle type. (Numerically verified up to period 55.)

The union of all the unstable manifolds is necessarily an invariant and attracting set (if all the periodic orbits are of saddle type, it is globally attracting). It is a strange attractor because it contains horseshoes—connected to the existence of the homoclinic points. (Figure 2 gives an idea of the shape of this attractor, it is the “Gumleaf attractor”.) It is then possible to think of the complicated dynamics in terms of “cycle slipping”. Generically, an orbit passing close to the stable manifold of a periodic orbit of period $p$, homes in on the periodic orbit, but because it did not belong to the stable manifold, to be captured by a stable and unstable manifold belonging to another periodic orbit, since the stable and unstable manifolds form a very dense web in $\mathbb{R}^2$.

The notion of ergodic chaos can be defined (Bowen and Ruelle, 1975; Ruelle, 1982; Guckenheimer and Holmes, 1983) and implies that an invariant measure $\mu$—invariant with respect to the state transition map $G$ for the difference equation (5.5)—can be defined on the invariant set, hence the strange attractor. (This corresponds to sets $A_1$ and $A_2$ in Definition 5.1 being equal and this invariant set being indecomposable. See the Appendix for a definition of the notion “indecomposable”.) Then the state transition map, restricted to this invariant set, represents an invertible measure-preserving transformation, so defining an ergodic map. In more intuitive terms this implies the convergence of Cesàro means of summable functions to constant values for $\mu$-almost all initial conditions, i.e. sample means of functions along trajectories converge to constants independent of the initial conditions. This constant value is actually the $\mu$-mean of the function over the invariant set, the ensemble average. This ergodicity property allows the stability questions for $y_k$, the output of the plant, to be answered. Indeed, it makes it possible to investigate the Cesàro mean of the logarithm of the normalized ratios $v_k$.

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \log |v_k|.
\]

because, if this limit exists along a typical orbit, it is independent of the particular orbit chosen—its value is typical for the generic dynamics of (5.1). Evaluating it through simulation, gives

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \log |v_k| \approx \frac{1}{2} \log 2.
\]

The relation (5.2) then implies that

\[
\lim_{n \to \infty} \frac{1}{n} \log |y_k| = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \log |b v_k| \approx \frac{1}{2} \log 2 b.
\]

Hence, $y_k$ converges “exponentially” to zero if $0 \leq b < \frac{1}{2}$; exponentially, in the sense that there exists a positive real constant $a$, larger than 1, such that

\[a^k y_k \to 0 \quad \text{as } k \to \infty.\]

This constant can be chosen as

\[1 < \infty < \frac{1}{\sqrt{2b}}.\]

The controller gain/parameter estimate $\hat{u}_k$ behaves
chaotically, because of \((5.3)\), and

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \log \left( \frac{\hat{a}_k - a}{\sqrt{2b}} \right) = 0
\]

for almost all initial conditions. Analogously,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \hat{a}_k \approx a.
\]

Hence, the estimate \(\hat{a}_k\) has the correct averaged value around which it oscillates erratically—almost as a random process.

6. STABILITY OF THE ADAPTIVE LOOP

In this section the adaptive closed loop described in (2.6–2.8) is reconsidered and the main robustness results are stated, summarizing the previous Sections 4 and 5. The obtained results are discussed and interpreted in the light of the available theories for establishing robustness of adaptive schemes. In particular it is argued that chaotic parameter estimates are not necessarily a bad thing to have.

The first result concerns the behaviour of the parameter estimate.

**Theorem 6.1.** For all initial conditions, except possibly for a set of Lebesgue measure zero, the parameter estimate \(\hat{a}_k\), \(k \in \mathbb{N}\), defined in (2.6–2.8) has the properties:

(i) if \(b < 0\), the parameter estimate is bounded, and exponentially becomes periodic with period 2,

\[
\hat{a}_k \to a + (\pm)(-1)^k \left( \frac{|b|}{2} \right)^{\sqrt{2b}}, \quad k \to \infty;
\]

(ii) if \(b = 0\), then \(\hat{a}_k \equiv a \forall k \geq 2\);

(iii) (subject to the veracity of Conjecture 5.1), if \(b > 0\), \(\hat{a}_k\) behaves chaotically.

Along almost all trajectories the cesaro mean of the logarithm of \(\hat{a}_k = a - \hat{a}_k\) is defined and is given by

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log |\hat{a}_k| \geq \frac{1}{2} \log \left( \frac{|b|}{2} \right), \quad b \neq 0.
\]

These results are valid independent of the size of \(b\). Note also that in the case that \(\hat{a}_k\) behaves chaotically, one does not have a boundedness result, on the contrary, almost certainly \(\hat{a}_k\) will exceed any given bound \((l)\), however, as will be argued, this is not dramatic, nor even bad, since the control signal and the plant output are bounded.

The robustness result, which gives a sharp stability–instability boundary in the \((a, b)\) parameter plane is as follows.

**Theorem 6.2.** For all initial conditions, except for a set of Lebesgue measure zero, and conditioned on the veracity of Conjecture 5.1, the plant output \(y_k\) defined in (2.6–2.8) is:

(i) bounded and regulated to zero if \(|b| < \frac{1}{2}\);

(ii) the rate of convergence is exponential, in the sense that

for \(b < 0\) \(|y_k| < C\left(\sqrt{2b}\right)^k\),

for \(b > 0\) \(y_k \leq C\left(\sqrt{2b}\right)^k\) as \(k \to \infty\),

for \(b = 0\) \(y_k \equiv 0 \forall k \geq 2\);

(iii) unbounded and diverges exponentially if \(|b| > \frac{1}{2}\).

**Remarks—discussions**

(6.1) Note first that this algorithm achieves exponentially fast regulation (in an appropriate sense) of the output in the presence of model-errors, without using—obviously—an external input. This demonstrates that there are alternative ways of obtaining robust adaptive schemes, other than forcing exponentially fast identification by the use of an external sufficiently exciting input.

(6.2) Here is another example of an adaptive scheme which regulates the plant output to zero, i.e. achieves its desired purpose, whilst the parameter error does not converge. Actually, because of the undermodelling it would be rather surprising if the parameter error did converge to zero. Note however that this "non-convergence" limits the robustness margin of the adaptive loop. Indeed, if it were possible to estimate \(a\) correctly, then the proposed controller would regulate the output to zero for all \(|b| < 1\).

(6.3) Notice that the proposed model by no means needs to be a good approximation for the real plant. Even in the situation \(b \neq 0\), \(a = 0\), where the first order model does not make any sense at all, good control action is obtained. This is clearly an advantage of the fast adaptive loop, which slow adaptation can never obtain.

(6.4) Averaging techniques in adaptive control are able to handle slow adaptation and fast parasitic plant states for model errors. Simplified, the theory states that as long as the adaptive algorithm together with the dominant slow part of the plant is slow compared to the parasitic states, even after closing the adaptive loop, all is well provided that the parameter
estimates in the adaptive loop are close to the real parameters. The theory is conditioned on the assumption that the controller can indeed stabilize the closed loop system (including the parasitic part) for parameters belonging to some set containing the real parameter. These ideas do not work in this environment—as the adaptive observer and control law have dead-beat response in the ideal circumstances. But something can still be learnt from averaging. Indeed, assuming that the time average of the parameter estimate $\hat{d}_k$ exists, i.e.

$$\langle \hat{d}_k \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N-1} \hat{d}_k = \bar{a}, \quad \text{uniformly in } l,$$

it is possible to obtain an "averaged" equation which has the same stability properties as the original system,

$$\bar{y}_k = \langle \hat{d}_k \rangle \bar{y}_{k-1} + b \bar{y}_{k-2}. \quad (6.3)$$

As indicated in Section 5, the assumption "the time average exists" is a non-trivial one. In order to be able to answer the stability question of the non-linear loop through analysis of (6.3) it is necessary to require that $b$ is small. It is possible to obtain estimates for how small $b$ has to be, typically $|b| \ll 1$, which has to be compared with the correct robustness margin of $|b| < \frac{1}{2}$. Note that the philosophy behind this kind of averaging is precisely the opposite of the one available in the literature; it is not the state of the fast part of the plant which is averaged out, but the fast adaptive estimate which is averaged by the plant! Why these results are not available in the literature is precisely due to the difficulties encountered in establishing the existence of time averages for the parameter estimate—caused by the complex dynamics of these estimates.

(6.5) Note that it is impossible by using a constant output or state feedback to stabilize the class of "uncertain" systems

$$y_k = ay_{k-1} + by_{k-2} + u_{k-1}$$

with constant parameters $a$ and $b$, which are unknown, but satisfy the bounds

$$|a - a_0| < a_0, \quad a_0 > \frac{1}{2}, \quad a_1 \text{ arbitrary}$$

$$|b| < \frac{1}{2}$$

($a_0$, $a_1$ are given constants). Notice that for a control input defined by

$$u_{k-1} = f_1 y_{k-1} + f_2 y_{k-2},$$

where $f_1$, $f_2$ are the gains of the controller, there is not a single gain setting which can stabilize all systems in the above class. This demonstrates that the adaptive controller is, in a sense, superior to a more complicated controller based on fixed gain design for uncertain systems.

(6.6) For $b < 0$, the periodic feedback gain situation, the stability is in the sense of Lyapunov, however for $b > 0$, chaotic feedback gain, this is not the case! The present technique (via ergodicity and césaro mean convergence) is closely related to techniques used to establish asymptotic properties of products of sequences of ergodic stochastic matrices (Bitmead and Boel, 1985; Oseledec, 1968). This is not surprising, as deterministic chaos seems to have a lot in common with random processes (Bowen and Ruelle, 1975; Ruelle, 1982).

(6.7) The problem of a possible division by zero in the parameter estimator does not cause any difficulty. The only trace of this problem in the theorem statements is the qualifier "except for a set of Lebesgue measure zero". Hence this analysis captures the generic properties of the adaptive scheme. Note that this is common practice in the discussion of the adaptive system in a stochastic context, where results are only almost surely valid (at their best). See for example Caines and Meyn (1985) and the references therein, where a division by zero is treated and disposed of by noting that it is an event of zero probability—in much the same style as this analysis.

Some examples

Figures 3 and 4 illustrate the adaptive loop's response. Figures 3.1 and 3.2 contain the trajectories of the plant output $\bar{y}_k$, the control action $u_k = \bar{d}_k \bar{y}_k$ and the parameter estimate/feedback gain $\hat{d}_k$, respectively, for the open loop unstable plant with parameters $a = 2$, $b = -0.3333$. The plant output is quickly regulated to zero, whilst the controller gain behaves asymptotically periodically. Figures 4.1 and 4.2 display, respectively, the same trajectories for the open loop unstable plant with parameters $a = 2$ and $b = +0.3333$. Again the output is regulated to zero, and also the control action disappears quickly, but the controller gain behaves quite erratically (see Fig. 1). (For both simulations the initial conditions were $y_{-2} = 1.5$, $y_{-1} = -0.5$, $\bar{d}_0 = 0$.)

7. COMPLEMENTS AND CONCLUSIONS

First, before presenting the final conclusions, some further features are briefly mentioned, without going into details, mainly referring to simulations experience.

The comments and claims of this section follow
exhaustive simulation experiments. Given the simplicity of the problem formulation (discrete time, low dimension, rational calculations) and the inherent scepticism of the readers of the adaptive control literature, these statements are more convincingly checked by the readers themselves. Because the modifications discussed in this section lead to much more complicated analyses without promising more insight, the authors feel justified in not pursuing their analyses in this early stage of the investigations.

In the simulations the influences of the various disturbances are easily identified in the "periodic case $b < 0$", whilst in the "chaotic case $b > 0$", the chaos tends to obscure any regularity. In this situation the effects are best identified by altering the initial conditions slightly and comparing the different trajectories.

**Higher order problem, time-varying parameters**

The observed dynamical features are not miraculously due to the combination of a second order
The adaptive control scheme, because of its ultra-fast identification law, promises good tracking properties in the case when the plant is time-varying. Suppose the plant can be represented as follows.

The plant:

\[ y_k = a_{k-1}y_{k-1} + b_{k-2} + c_{k-3} + u_{k-1}. \]  
(7.2)

Using the same adaptive scheme as outlined in

plant with a first order model, e.g. if the plant was third order:

\[ y_k = ay_{k-1} + b_{k-2} + cy_{k-3} + u_{k-1}, \]  
(7.1)

qualitatively, the same results would be obtained. Moreover, it appears as if the size of \( c \) is immaterial. (It is not difficult to demonstrate that for \( -\frac{1}{2} < b < 0 \), the periodic orbit of period two for the feedback gain indeed is independent of \( c \), and is asymptotically stabilizing.)
Section 2, gives the following.

The closed loop:

\[ r_k = b \left( \frac{1}{r_{k-2}} - \frac{1}{r_{k-1}} \right) + (a_{k-1} - a_{k-2}) \quad (7.3) \]

\[ y_k = r_k y_{k-1} \quad (7.4) \]

\[ \delta_k = a_{k-1} + b \frac{1}{r_{k-1}}. \quad (7.5) \]

From these equations and from the previous discussion of the homogeneous part of (7.3) it transpires that small time variations pose no threat to this controller. Simulations with \( ak \) being periodic, or stationary random or even being a random walk process or a ramp function can be conducted without experiencing difficulties.

**Effects of additive noise, rounding errors and clipping**

The proposed adaptive scheme is sensitive to measurement errors, though not as sensitive as one would suspect. Assume that the plant can be represented by:

\[ y_k = a y_{k-1} + b y_{k-2} + u_{k-1} + v_k, \quad (7.6) \]

where \( v_k \) is additive measurement noise. In this situation the parameter estimate is governed by:

\[ \delta_k = a + b \frac{y_{k-2}}{y_{k-1}} + \frac{v_k}{y_{k-1}}. \quad (7.7) \]

Of course, if the signal to noise ratio is negligible, \( \delta_k \) jumps crazily, driving \( y_k \) to large values, but therefore restoring the signal to noise ratio to acceptable levels, and hence bringing the adaptive scheme back to good behaviour and so on. In the case \( b > 0 \), the feedback parameter is chaotic and occasionally the feedback parameter can take on astronomical values, as can the control input, the more so as \( b \) is closer to \( \frac{1}{2} \), because then the stabilization requires more time and those rare events become more visible. Therefore it is natural to introduce clipping in the parameter estimator to limit its possible range. Done with care this does not alter the nature of the dynamics of the closed loop. More precisely, the true parameter should be within the allowed parameter range, and this range should be large enough to accommodate (most) of the periodic points of the adaptive algorithm. If these obvious criteria, which are easy to meet, are indeed met, clipping has a beneficial effect on the dynamics of the algorithm, certainly for values of \( |b| \) close to \( \frac{1}{2} \), without upsetting the earlier picture of the dynamics.

**Transient response of more standard algorithms**

The main importance of this analysis is that it captures the transient behaviour of the adaptive scheme with a finite step size estimator (\( c > 0 \) in 2.5):

\[ \delta_k = \delta_{k-1} + \frac{y_k - y_{k-1}}{c + y_{k-1}}(y_k - \delta_{k-1} y_{k-1} - u_{k-1}). \quad (7.8) \]

For \( c \) small compared to \( y_k^2 \), the analysis is valid. In this "transient region" of the state space, all the above discussed dynamical properties are present. In particular, for \( |b| < 1/2 \), this theory predicts convergence of \( y_k \) to zero, hence, after a transient period, \( y_k^2 \) becomes of the order of magnitude of \( c \) and then the dynamics become essentially linear, driving \( y_k \) further towards zero and \( \delta_k \) towards some constant, depending on \( b \). (Note that for the specified range of \( b \) the linearized equation predicts asymptotic stability.) On the other hand if the initial conditions (for \( y \)) are large (compared to \( c \)) and the parameter \( b \) is larger than \( 1/2 \) in modulus the algorithm behaves unstably, despite the fact that the linearized system could well be stable. Figure 5 illustrates this point. The step size was chosen as \( c = 0.0001 \), the other parameters were set as \( a = 2, b = -0.3333, y_0 = 10, y_{-1} = -0.34, \delta_0 = 0 \). One can clearly recognize the "transient periodicity" in the parameter estimate. As chaos obscures any regularity, no figures are presented for \( b > 0 \), but the readers may convince themselves that the transient behaviour does contain the features of "chaos"—which become especially clear when one alters the initial conditions and tries to compare trajectories!

The combination of a large-scale non-linear stability theory with a local linearized stability theory for more standard algorithms effectively completes a global stability analysis for the adaptive control of these particular undermodelled plants. Other combinations of large-scale unstable and small-scale stable results would yield local stability of the adaptively controlled loop while large-scale stability with local instability should give rise to limit-cycle types of closed loop behaviour.

**Conclusions**

The complexity of the non-linear dynamics of adaptive control schemes has been highlighted using a surprisingly standard and simple algorithm. These complex dynamics may be present in the majority of the available adaptive schemes (based on the certainty equivalence principle) and are explicitly present whenever situations of large initial conditions and/or fast adaptation are considered.

Although the authors do not yet promote, nor advocate the present ultra-fast adaptive controller,
note that the presented algorithm has very good robustness properties, both with respect to under-modelling and to time variations in the plant parameters, without losing its control objective. Hence, it deserves further analysis, not in the least in the directions of extending the present results to higher order models and controllers—which promises to be a particularly hard problem.

"It is hard to adapt to chaos, but it can be done. I am a living proof of that: It can be done."
Kurt Vonnegut Jr.

*Breakfast of Champions* (Chapter 19, p. 210)

**REFERENCES**


**APPENDIX**

**Symbols and definitions**

\( Z^+ \) the set of (positive) integers.

\( \mathbb{R} \) the set of real numbers.

A *homeomorphism* is a continuous function with a continuous inverse. Let \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a homeomorphism. Consider the difference equation \( x_{k+1} = f(x_k), \ x_0 \).

A *trajectory (orbit)* is \( \{ f(x_k), k \in \mathbb{N} \} \); a sequence of iterations starting from \( x_0 \).

A *fixed point* is a solution of \( x = f(x) \).

A *periodic orbit of period p* is an orbit \( \{ f^p(x), k \in \mathbb{N} \} \) such that
Nonlinear dynamics in adaptive control

\[ f^l(x) = x, \text{ and } f^l(x) \neq x \text{ for } l = 1, \ldots, p - 1. \]

The **stable manifold** of a fixed point \( x \) is the collection of all points converging to \( x \) under the forward iteration of \( f \).

The **unstable manifold** of a fixed point \( x \) is the collection of all points converging to \( x \) under the backward iteration of \( f^{-1} \).

An **invariant set** is \( S \subset \mathbb{R}^n : f(S) \subset S \).

An **invariant attractor** is an invariant set for which there exists a set \( D \), of positive Lebesgue measure such that \( \lim_{n \to \infty} f^n(D) \subset S \).

An **indecomposable** set is invariant set such that for any two points, \( x, y \) in this set and for any \( \varepsilon > 0 \), there exist \( n; x = x_0, x_1, \ldots, x_n = y \) and \( t_1, \ldots, t_n \) such that \( |f^{t_i}(x_{i-1}) - x_i| < \varepsilon \forall i = 1, \ldots, n \). (Intuitively, any two points can be linked arbitrarily closely by a certain trajectory, completely in the set.)

A **Cantor set** is a closed set, such that the largest connected subset is a point, and every point in the set is a limit point.