COMPLEX DYNAMICS IN A SIMPLE STOCK MARKET MODEL

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A deterministic model is introduced in terms of conservation principles to describe the (qualitative) price dynamics of a stock market. It is shown how the fundamentalist and chartist patterns of the trader behavior affect the price dynamics. The model can display complex oscillatory behavior with transient erratic oscillations, which resembles the behavior found in actual price dynamics. By means of numerical simulations, it is shown that the model can produce price time-series with statistics similar to that found in real financial data.

Keywords: Stock market; trading; price dynamics; oscillations.

1. Introduction

Several models have been introduced in the financial community which attempt to capture the complex behavior of stock market prices and of market participants. Based on the competition between supply and demand, the effort is to model the main observed statistical facts. The underlying idea behind statistical (i.e. stochastic) approaches is that future asset prices tend to be largely random [Batchelier, 1900; Working, 1934; Samuelson, 1965]. In fact, to interpret the price time-series from given information and to predict future values is a very hard task, because the universally accepted approach is currently of probabilistic/statistical nature [Samuelson, 1965]. In the physics literature, the statistical approach gave rise to an avalanche of publications demonstrating the application of mechanical statistical tools to explain regularities in for example, price volatility and fundamentals [Mantegna & Stanley, 2000]. An example of application of stochastic analysis with a physics viewpoint is the description of price changes in open markets [Takayasu & Takayasu, 1999; Ausloos, 2000; Bonano et al., 2000].

A drawback of the statistical approach is that understanding the underlying dynamics governing price evolution is not easy. For instance, it is not clear how the heterogeneity and composition of agents in a stock market affect the price dynamics [Malliaris & Stein, 1999]. This situation
has led some researchers to take alternative approaches. Basically, the following question has been posed: Is there a nonlinear methodology (e.g., deterministic models) as an alternative to the stochastic approach, which generates a time-series sequence of price changes that appear random when in fact such sequence is nonrandom? The issue of deterministic chaos to explain price volatility has been documented (see for instance [Peters, 1990] and [De Grauwe et al., 1993]). The empirical analysis on price time-series presented in De Grauwe et al.’s book was not conclusive of chaos in asset markets. Malliaris and Stein [1999] have suggested that price volatility processes reflect the output of a higher-order dynamic system with an underlying stochastic foundation, and analyzed the economic scenarios which may generate seemingly chaotic processes that can be interpreted statistically. Recently, Schmidt [1999] proposed a deterministic model to describe the dynamics of high-frequency foreign exchange market. The model is constructed in terms of observable variables (price and excess demand assumed to be proportional to number of buyers). It is shown that the model is able to describe oscillatory behavior about the equilibrium price. In a second step, the deterministic model is equipped with an additive stochastic component in the price dynamics, which leads to a more realistic description of high-frequency price dynamics.

These results constitute interesting advances because they show how a deterministic approach can be the departing point to gain insight about the mechanisms governing the dynamics of actual stock markets. Besides, they show how a deterministic model can be endowed with a stochastic component to account for observed high-frequency modes in price dynamics.

In the spirit of Schmidt’s approach, a deterministic framework for modeling stock market dynamics is presented in this paper. The main objective of this paper is to describe the qualitative dynamics of the proposed model and to show that the resulting price time-series can display qualitative statistical behavior similar to that found in actual financial index [Mantegna & Stanley, 1995]. The underlying methodological idea is to undercover certain basic structures that can be analyzed in a mathematical dynamics framework. The model is based on assets conservation principles to reflect asset trading among different traders’ classes. The main goal of our modeling approach is to gain insight and understanding of the intrinsic mechanisms driving the price dynamics. To this end, a simple stock market dynamics is addressed; namely, one asset with a small number of investor types and simple trading rules. Since some universally accepted determinants of the dynamics of stock markets are psychological reactions of traders, our idea is to model some attitudes towards levels and changes of prices that make different types of investors, and to conform all together within an interaction model.

The model consists of a series of differential equations describing the dynamics of asset trading among different agent groups, and a functional equation describing trade conservation. In this way, the dynamics of the asset is determined by the trading dynamics. An equilibrium price is achieved when certain demand/supply equations are satisfied. Attention is devoted to a specific case for which the trading activity is based on an infinitely divisible asset and two trader groups (fundamentals and chartists) as these groups have been characterized by De Long et al. [1990]. It is shown that the resulting dynamic model is equivalent to a differential equation of neutral-type [Bellman & Cooke, 1963; Hale & Lunel, 1993]. Numerical simulations show a variety of dynamic behaviors, ranging from simple exponential stability/instability to complex oscillatory price behaviors, which resemble the random behavior of actual price dynamics.

2. Main Results

It should be stressed that this paper is not aimed at providing a model describing all the quantitative issues of stock markets. Since dynamics of stock markets is the result of complex interactions of human (psychological), cultural and economic factors, a comprehensive model seems to be a very hard task. From a scientific methodological point of view, we will focus on simple dynamical models to gain insights on some mechanisms inducing the stock market dynamic behavior.

We assume for simplicity that (a) only one holding company is listed at the stock exchange so that a stake in that company is the same as a stake in the stock market index, (b) the equity of the company consists of an infinitely divisible amount of one asset with unitary price \( p(t) \), and (c) the
asset is held by \( N \) investors, who are grouped into two classes.\(^1\)

### 2.1. Preliminaries

Let \( z_1 \) and \( z_2 \) be number of shares (also called as inventory) owned by the first and second investor classes at time \( t \geq 0 \). It is clear that

\[
  z_1(t) + z_2(t) = M, \quad \text{for all } t \geq 0 \quad (1)
\]

and

\[
  0 \leq z_i(t) \leq M, \quad i = 1, 2. \quad (2)
\]

That is, the shares owned by the \( i \)th class is non-negative and non-larger than the amount available in the stock market. Let

\[
  x_i(t) \triangleq z_i(t) - z_i(0), \quad i = 1, 2
\]

be the number of shares traded by the \( i \)th class at time \( t \geq 0 \). Then,

\[
  x_i(0) = 0, \quad i = 1, 2. \quad (3)
\]

From (1), we have that

\[
  x_1(t) + x_2(t) = 0, \quad \text{for all } t \geq 0 \quad (4)
\]

which states a type of traded shares conservation principle.

Let \( p(t) \) be the actual (objective) instantaneous asset price and let \( p_i^*(t) \) be the expected (subjective) instantaneous asset price of the \( i \)th class.\(^2\) The model construction is based on the economic fact that the (objective) price of a good is determined by the market dynamics. That is, the price formation process is given by means of trading, which can be seen as a kind of demand/supply dynamics. In this way, it seems natural to model the dynamics of the traded amount \( x_i(t) \), rather than the dynamics of the owned ones \( z_i(t) \) [Kandel & Pearson, 1995].

### 2.2. Trading dynamics

In the spirit of the recent model described by Schmidt [1999], our model framework discerns the “fundamentalists” that react to differences between the current and the fundamental prices, and the “chartists” that follows the price trends. Traders can switch from one type of trading behavior to another and the dynamics of these switches depend on the price values and changes. Price in turn is determined by the excess of demand.

In a continuous-time market framework, the trading dynamic behavior can be represented as a first-order relaxation processes:

\[
  \dot{x}_i(t) = \tau_i^{-1}[x_i^*(p(t), P(t), p_i^*(t)) - x_i(t)], \quad i = 1, 2 \quad (5)
\]

where \( \dot{x}_i = dx_i/dt, \ x_i^*(p(t), P(t), p_i^*(t)) \) is the trading task (also called expectation functions [Stein, 1994]) of the \( i \)th investor class, \( \tau_i > 0 \) is a time-constant, and \( P(t) \) is the price history. Notice that the trading task depends on the actual price \( p(t) \) (fundamental behavior) and the price history \( P(t) \) (chartist behavior). The parameter \( \tau_i^{-1} > 0 \) can be seen as the effort rate by the \( i \)th class to achieve the trading task \( x_i^*(P(r; t), p_i^*(t)) \).

The rationale behind formulation (5) is the following. If \( p(t), P(t) \) and \( p_i^*(t) \) are constant, then \( x_i^*(p(t), P(t), p_i^*(t)) \) is also a constant, say \( x_i^* \). Then, the tracking error \( e_i(t) = x_i^* - x_i(t) \) would decay exponentially to zero, with decaying rate equal to \( \tau_i^{-1} > 0 \) (i.e. \( e_i(t) = e_i(0) \exp(-t/\tau_i) \)). From the perspective of the \( i \)th class alone, the trading strategy (5) would stabilize its asset exchange.

The next step in modeling the trading dynamics is to incorporate suitable expressions for the trading task \( x_i^*(p(t), P(t), p_i^*(t)) \). To this end, we shall follow the ideas described by De Long et al. [1990] to model fundamentalist and chartist trading behaviors. For simplicity in the presentation, it is assumed that the underlying economy does not grow. Fundamental trading sticks to the following strategy:

\[
  N_i \alpha_i(p_i^* - p), \quad i = 1, 2 \quad (6)
\]

where \( \alpha_i > 0 \) is a trading coefficient representing the aggressiveness of rational speculators in betting on reversion to fundamentals. The parameter \( \alpha_i \) can be taken as \((\gamma \sigma^2_\eta / 2)^{-1}\), where \( \gamma > 0 \) is a risk aversion coefficient and \( \sigma^2_\eta > 0 \) is the mean-variance of the risky dividend paid by the stock. The demand \( N_i \alpha_i(p_i^* - p) \) of fundamental trading can be obtained by maximizing a mean-variance

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\(^1\) The model can be easily extended to the case of \( N \) investors grouped into \( n \) classes.

\(^2\) During certain time interval, the prices \( p_i^*(t) \) can be identified with the fundamental price \( p^* \) of the asset, given usually by macroeconomic considerations.
utility function with risk aversion coefficient $\gamma > 0$ [Korn, 1997]. On the other hand, chartist trading is also sensitive to the rate of change in prices. Basically, chartist trading places a market order today in response to a past price change. A simple linear model accounting for the rate of change in prices is given as follows:

$$\frac{N_i \beta_i (p - p_{h_i})}{h_i}, \quad i = 1, 2$$  (7)

where $p_{h_i} \equiv p(t - h_i)$ and $h_i > 0$ is a time-horizon. Note that $(p - p_{h_i})/h_i$ can be seen as a first-order approximation from the past (left) to the time-derivative $\dot{p}$. In fact, $\lim_{h_i \to 0} (p - p_{h_i})/h_i = \dot{p}$.

In a stock market, traders execute the customer orders as well as trade their company assets as follows:

$$x^*_i (p(t), \mathcal{P}(t), p^*_i (t))$$

$$= N_i \alpha_i (p^*_i - p) + \beta_i (p - p_{h_i})/h_i \quad i = 1, 2$$  (8)

In this way, the trading strategy $x^*_i (p(t), \mathcal{P}(t), p^*_i (t))$ depends on the price history $\mathcal{P}(t)$ via the past price $p_{h_i}$. When $(p - p_{h_i})/h_i \approx 0$, traders behave as pure fundamental traders. On the other hand, when $|(p - p_{h_i})/h_i|$ is large, traders behave as pure chartist traders (for an empirical discussion of this behavior [see De Long et al., 1990]).

Equations (5) and (8) together with the constraint (4) describe the dynamics of traded assets $x_i(t)$. It is noted that such dynamics depends on the price dynamics $p(t)$. To obtain an expression that governs the price dynamics, we proceed as follows. From (4) we know that $x_1(t) + x_2(t) = 0$, for all $t \geq 0$. Hence,

$$\dot{x}_1 + \dot{x}_2 = 0, \quad \text{for all } t \geq 0$$  (9)

We can use (5) to obtain

$$\tau^{-1}_i [x^*_i (p(t), \mathcal{P}(t), p^*_i (t)) - x_1(t)]$$

$$+ \tau^{-1}_i [x^*_2 (p(t), \mathcal{P}(t), p^*_2 (t)) - x_2(t)]$$

$$= 0, \quad \text{for all } t \geq 0$$  (10)

The expressions in (8) can be used in (10) to obtain the following one:

$$\tau^{-1}_1 [N_1 \alpha_1 (p^*_1 - p) + \beta_1 (p - p_{h_1})/h_1] - x_1(t)$$

$$+ \tau^{-1}_2 [N_2 \alpha_2 (p^*_2 - p) + \beta_2 (p - p_{h_2})/h_2] - x_2(t)]$$

$$= 0, \quad \text{for all } t \geq 0$$  (11)

Given the dynamics $x_1(t)$ and $x_2(t)$, the expected asset prices $p^*_1 (t)$ and $p^*_2 (t)$ (which are provided by each investor group), the instantaneous price $p(t)$ is computed as the solution of the functional equation (11). In this way, the dynamics of the asset price $p(t)$ are induced by the dynamics of the asset trading $x(t)$ via the trading conservation equation (4). To avoid unrealistic situations, the following additional (non-negativity) constraint is imposed:

$$p(t) \geq 0, \quad \text{for all } t \geq 0$$  (12)

Summarizing, the trading model is composed by the trading dynamics

$$\dot{x}_i (t) = \tau^{-1}_i [N_i \alpha_i (p^*_i - p) + \beta_i (p - p_{h_i})/h_i]$$

$$- x_i(t)], \quad i = 1, 2$$  (13)

the trading conservation constraint (4) (i.e. $x_1 + x_2 = 0$), the functional price equation (11), and the non-negativity constraint (12). At each time instant $t \geq 0$, the inventory owned by each group is computed as

$$z_i(t) = z_i(0) + x_i(t), \quad i = 1, 2$$  (14)

Besides, the constraint (2); namely, $0 \leq z_i(t) \leq M(t), i = 1, 2$, imposes the following constraint on the traded amounts:

$$-z_i(0) \leq x_i(t) \leq M(t) - z_i(0), \quad i = 1, 2$$  (15)

### 2.3. Equilibrium points

At an equilibrium point $(x_{1,eq}, x_{2,eq}, p_{eq})$, the individual supply/demand conditions are met, which defines the equilibrium asset price $p_{eq}$. Assume that $p^*_i (t) = \text{constant}$. Since $p_{eq} = p_{h_{i,eq}}$ for $i = 1, 2$, and $x_{2,eq} = -x_{1,eq}$, the equilibrium point $(x_{1,eq}, -x_{1,eq}, p_{eq})$ can be computed from the equations:

$$0 = N_1 \alpha_1 (p^*_1 - p_{eq}) - x_{1,eq}$$

$$0 = N_2 \alpha_2 (p^*_2 - p_{eq}) + x_{1,eq} \quad p_{eq} \geq 0$$  (16)

$$-z_1(0) \leq x_{1,eq} \leq M - z_1(0)$$

That is, an equilibrium point is reached as long as the demand/supply expectations of all the trader groups are satisfied. It will be said that $(x_{eq}, p_{eq})$ is a feasible equilibrium point if constraint (15) is satisfied. That is, $x_{eq} \in \Sigma$. Specifically, we can classify the equilibrium points as follows:
• If \( x_{eq} \in \text{interior}(\Sigma) \) (i.e. \( -z_i(0) < x_{i,eq} < M - z_i(0), \ i = 1, 2 \)), \( x_{eq} \) is called a cooperative equilibrium point since no investor class is eliminated off the stock market.

• If \( x_{eq} \in \text{boundary}(\Sigma) \), \( x_{eq} \) is called a noncooperative equilibrium point since one of the investor classes is eliminated off the stock market. This situation appears, for instance, when \( -z_1(0) \geq x_{1,eq}^{(16)} \) where \( x_{1,eq}^{(16)} \) is given by the first equality in (16). Hence, the actual equilibrium position is located at \( x_{1,eq} = -z_1(0) \) and so that \( z_{1,eq} = 0 \).

For simplicity in presentation, we will assume that \( -z_1(0) \leq x_{1,eq} \leq M - z_1(0) \) is satisfied. From the first two expressions in (16), we get an equation defining the equilibrium price:

\[
\text{Demand of the 1st trader class} \\
N_1 \alpha_1 (p_1^* - p_{eq})
\]

\[
\text{Supply of the 2nd trader class} = -N_2 \alpha_2 (p_2^* - p_{eq}) \tag{17}
\]

That is, an equilibrium price is achieved when the demand of the first trader group equals the supply of the second trader group. Equation (17) can be easily solved to give:

\[
p_{eq} = \frac{N_1 \alpha_1 p_1^* + N_2 \alpha_2 p_2^*}{N_1 \alpha_1 + N_2 \alpha_2} \tag{18}
\]

Since \( N_1 \alpha_1 > 0, N_2 \alpha_2 > 0, p_1^* \geq 0 \) and \( p_2^* \geq 0 \), we have that \( p_{eq} \geq 0 \). Notice that \( p_{eq} \) is given as the weighted mean, with weights \( N_1 \alpha_1 \) and \( N_2 \alpha_2 \), of the expected prices \( p_1^* \) and \( p_2^* \). Moreover, if \( p_1^* \geq p_2^* \) (resp. \( p_2^* \geq p_1^* \)), then \( p_1^* \geq p_{eq} \geq p_2^* \) (resp. \( p_2^* \geq p_{eq} \geq p_1^* \)).

### 2.4. Price equation

Since the underlying stock dynamics model is linear, an equation governing the price dynamics can be obtained explicitly. To this end, we make a translation to center the variables \( x_1, x_2 \) and \( p \) with respect to their equilibrium values \( x_{1,eq}, x_{2,eq} \) and \( p_{eq} \). Using Laplace transforms and recalling that \( p_{h_i}(s) = \exp(-h_is)p(s) \), we get

\[
G(s)p(s) = \delta \tag{19}
\]

where \( s \) is the Laplace variable,

\[
G(s) = a_1 s + a_2 \exp(-h_1 s)s + a_3 \exp(-h_2 s)s + a_4 + a_5 \exp(-h_1 s) + a_6 \exp(-h_2 s) \tag{20}
\]

and

\[
\delta = \tilde{\alpha}_1 p_1^* + \tilde{\alpha}_2 p_2^* \tag{21}
\]

with

\[
\begin{align*}
 a_1 &= \tau_1 (\tilde{\beta}_2 - \tilde{\alpha}_2) + \tau_2 (\tilde{\beta}_1 - \tilde{\alpha}_1) \\
 a_2 &= -\tau_2 \tilde{\beta}_1 \\
 a_3 &= -\tau_1 \tilde{\beta}_2 \\
 a_4 &= \tilde{\beta}_1 + \tilde{\beta}_2 - \tilde{\alpha}_1 - \tilde{\alpha}_2 \\
 a_5 &= -\tilde{\beta}_1 \\
 a_6 &= -\tilde{\beta}_2
\end{align*} \tag{22}
\]

\( \tilde{\alpha}_1 = N_1 \alpha_1, \tilde{\alpha}_2 = N_2 \alpha_2, \tilde{\beta}_1 = N_2 \beta_2/h_1 \) and \( \tilde{\beta}_2 = N_2 \beta_2/h_2 \). That is, the price dynamics are governed by the differential equation [Hale & Lunel, 1993]

\[
a_1 \dot{p} + a_2 \dot{p}_h + a_3 \dot{p}_{h_2} + a_4 p + a_5 p_{h_1} + a_6 p_{h_2} = \delta \tag{23}
\]

and the non-negativity constraint \( p(t) \geq 0 \). It is clear that, if \( a_5 + a_6 \neq 0 \), the linear equation has only one equilibrium point; namely, \( p_{eq} = \delta / (a_5 + a_6) \) [see Eq. (18)].

Equation (23) is a linear differential equation of the neutral-type [Bellman & Cooke, 1963; Hale & Lunel, 1993]. For this type of differential equation (23), the initial-value problem can be defined as follows [Bellman & Cooke, 1963; Hale & Lunel, 1993]. Suppose \( \phi \) is a given continuously differentiable function on \([-h^*, 0]\), where \( h^* = \max(h_1, h_2) \). A solution \( p(t) = \phi(\phi(t), t) \) of Eq. (23) through \( \phi \) is a continuously differentiable function \( p(t) \) defined on \([-h^*, \infty) \) with \( p(t) = \phi(t) \) for \( t \in [-h^*, 0] \); \( p(t) \) continuously differentiable except at the points \( kh_i, \ i = 1, 2 \) and \( k = 0, 1, 2, \ldots \) and \( p(t) \) satisfying Eq. (23) except at these points. In other words, the price dynamics \( p(t) \) depends on the past price dynamics \( \phi(t), t \in [-h^*, 0] \). It is shown that there always exists a unique solution of Eq. (23) through \( \phi \).

The stability of the equilibrium price \( p_{eq} \) [see Eq. (18)] is determined by the stability of the characteristic equation \( G(s) \). That is, \( p_{eq} \) is an asymptotically stable equilibrium price (i.e. \( p(t) \to p_{eq} \)) in the general \( n \)-dimensional case, \( p_{eq} \) is given as a linear weighted combination of the individual subjective prices.
as $t \to \infty$) for suitable initial conditions if all the roots of $G(s)$ lie in the open left-hand side $\mathbb{C}^-$ of the complex plane. We have the following asymptotic behavior. In the limit as $h_1, h_2 \to \infty$, we have that $\exp(-h_i, s) \to 0$ and $G(s) \to a_1 s + a_4$. That is, as $h_1, h_2 \to \infty$ the price dynamics $p(t)$ are governed by a first-order linear differential equation

$$a_1 \dot{p} + a_4 p = \delta$$

In such a case, $\beta_1, \beta_2 \to 0$ and $G(s)$ is a first-order polynomial with root located at

$$\rho_\infty = -\frac{a_4}{a_1}$$

$$= \frac{(\alpha_1 + \alpha_2)}{(\gamma_1 \alpha_2 + \gamma_2 \alpha_1)}$$

If $\gamma_1 = \gamma_2$, then $\rho_\infty = \tau^{-1}_1 > 0$. Hence, continuity arguments imply that $\rho_\infty > 0$ for $\tau_1 \approx \tau_2$, and the instability of the equilibrium price is guaranteed. From the stock market viewpoint, this means that, in the limit as $h \to \infty$, the price dynamics is very likely to be unstable. In this case, the equilibrium price corresponds to a boundary equilibrium (i.e. $x_{eq} \in \text{boundary}(\Sigma)$). From the stock market viewpoint the interpretation is straightforward: one of the trader class gets out of the market.

The most interesting cases arise for finite values of $h_1, h_2 > 0$. In such cases, we shall focus on the properties of characteristic equation $G(s)$ given by (20). Although some results on the stability of neutral differential equations are available in the literature, the geometry of the root-set of the characteristic equation $G(s)$ is a rather complex problem and has been studied only to some extent (see [Bellman & Cooke, 1963, Chap. 5], and [Hale & Lunel, 1993, Appendix]). The difficulty lies in the fact that $G(s) = 0$ has an infinite number of solutions. To gain some insight on the dynamic behavior of the price equation (23), the following results were borrowed from [Hale & Lunel, 1993] and [Kolmanovskii & Myshkis, 1992]:

(a) If $\alpha_0 = \sup \{\Re(s) : G(s) = 0\}$ and $p(\phi(t), t)$ is the solution of (23), then, for any $\alpha > \alpha_0$, there exists $K = K(\alpha)$ such that

$$|p(\phi(t), t)| \leq K \exp(\alpha t)|\phi|, \quad t \geq 0, \quad |\phi| = \sup_{-h \leq \theta \leq 0} |\phi(\theta)|$$

That is, stability of the price dynamics $p(\phi(t), t)$ are of the exponential type.

(b) There are two real numbers $\zeta_1$ and $\zeta_2$ such that all solutions of $G(s) = 0$ lie in a vertical strip $\{\zeta_1 < \Re(s) < \zeta_2 \}$ in the complex plane. This means that all roots of large modulus lie in a neutral strip of the complex plane.

(c) There are at least two fundamental frequencies. This means that, in sections where $p(\phi(t), t)$ is continuous, oscillations evolve with at least two fundamental frequencies, say $\omega_{f,1}, \omega_{f,2} > 0$.

(d) For $h_1$ and $h_2$ sufficiently small and $|(a_2 + a_3)/a_1| < 1$, all the roots of $G(s)$ satisfy $\Re(s) < -\varepsilon < 0$ for some $\varepsilon > 0$. That is, for $h_1$ and $h_2$ sufficiently small, the equilibrium price $p_{eq}$ is asymptotically stable.

(e) The condition $|\tau_1(\beta_2 - \alpha_2) + \tau_2(\beta_2 - \alpha_1)|^{-1}(\tau_1 \beta_2 + \tau_2 \beta_2) < 0$ is necessary to achieve (exponential) stability of the equilibrium price $p_{eq}$. This inequality can be satisfied if $\tau_1$ and $\tau_2$ are small enough. The interpretation from a stock market dynamics is as follows: quick reaction of fundamental traders is necessary to stabilize the stock market.

A more detailed analysis of the distribution diagram of the roots of $G(s)$ would help to describe the qualitative behavior of the price dynamics $p(\phi(t), t)$. However, this analysis requires more involved mathematics [Hale & Lunel, 1993] that are beyond the aim of this paper.

3. Numerical Simulations

We have carried out some numerical simulations to illustrate some dynamical features of the model described above. To this end, we have taken the following set of parameters: $\alpha_1 = 1.0, \alpha_2 = 1.0, \beta_1 = 0.3, \tau_1 = 4.0, \tau_2 = 2.0, p_1^* = 2.0, p_2^* = 4.0, M = 10.0$, and $z_1(0) = 5.0$. It should be stressed that these numerical values do not correspond to any real case, and they were chosen only for illustration purposes.

3.1. Dynamical behavior

In what follows, the dynamical behavior of price will be illustrated by taking different values of the model parameters. It should be stressed that time is in arbitrary scale. In fact, as mentioned in the introduction, the main purpose of the numerical simulations is to illustrate that our modeling framework can produce price time-series with qualitative behaviors.
similar to those found in actual price time-series. In this way, the parameters used in numerical simulations are not intended to calibrate time scales and price-magnitude of real price dynamics.

(a) Effects of chartist trading. Figure 1 presents the price dynamics $p(t)$, for $h_1 = 0.0$, $h_2 = 5.0$, $\beta_1 = 0$, $\phi(t) = 2.5$, $t \in [-h_2, 0]$, and four different values of the parameter $\beta_2$; namely, (a) 0.2, (b) 0.75, (c) 0.8 and (d) 0.9. Notice that the first trading group does not include chartist trading. We have selected $h_1 = 0$ to show more clearly the role of chartist trading in the price dynamics. For completeness, a detail (for $t > 300$) of the corresponding phase-portrait $(x_1, p)$ is presented in Fig. 2. For a better characterization of the price dynamics, the maximum Lyapunov exponent, denoted by $\lambda_{\text{max}}$, has been also computed from the time series. For $0 < \beta_2 \lesssim 0.72$, the stock dynamics is stable with $\lambda_{\text{max}} \approx -1.12$ and the price achieves exponentially its interior equilibrium value $p_{\text{eq}} = 3.0$. For $\beta_2 \gtrsim 0.72$, the equilibrium price $p_{\text{eq}}$ is unstable and the stock dynamics undergoes oscillatory behaviors. In this case, the second largest Lyapunov exponent has been also computed, giving $\lambda_{\text{max}} \approx 0.0$ and $\lambda_{2nd,\text{max}} \approx -0.012$. These values seem to state that the complex oscillatory behavior is of transient nature and that, for long times, the price dynamics will converge to a pure oscillatory behavior. For $\beta_2 = 0.75$ and $\beta_2 = 0.8$, the oscillatory behavior is quite complex, with discontinuities at the points $t = h_2k$, $k = 0, 1, 2, \ldots$ [Hale & Lunel, 1993]. Notice that, in sections where the oscillations are continuous, there is only one fundamental frequency since $\beta_1 = 0$. That is, discontinuity points located at $h_2k$, $k = 0, 1, 2, \ldots$, connect sections of oscillations with only one harmonic frequency. In this case, $\lambda_{\text{max}} \approx 0.0$ and $\lambda_{2nd,\text{max}} \approx 0.0$, although no stable convergence was obtained for $\lambda_{2nd,\text{max}}$. That is, $\lambda_{2nd,\text{max}}$ oscillates between positive and negative values of small amplitude. For $\beta_2 = 0.75$, after an initial transient behavior, the constraints (4) and (12) are not active. However, for $\beta_2 = 0.8$, such constraints become active, which lead to a billiard-type behavior with $\lambda_{\text{max}} = 0.0$. It is interesting to note that the time-series $p(t)$ display transient erratic
behavior, although these transient behaviors may last a long time, induced by displacements of regular trajectory sections between neighbor discontinuities $h_2 k$ and $h_2 (k+1)$. Such dynamic behavior induced by the neutral nature of the price equation (27) can explain qualitatively the erratic behavior observed in price time-series [Peters, 1990; De Grawe et al., 1993]. For $\beta_2 = 9$, such erratic behavior is not present. Instead, a pure high-amplitude periodic behavior is displayed, which reflects the high-gain nature of the parameter $\beta_2$ (i.e. “chartist” trading behavior is so aggressive that stock market dynamics becomes very sensitive to changes in prices).

(b) Effects of the time-horizon. Let us illustrate the effects of the time-horizon $h_2$ on the price dynamics. As above, Figs. 3 and 4 present the stock dynamics for $h_1 = 0.0$, $\beta_2 = 0.75$, $\phi(t) = 2.5$, $t \in [-h_2, 0]$, and three different values of the time-horizon $h_2$; namely, (a) 2.0, (b) 5.0 and (c) 10.0. In all cases, the stock dynamics displays an oscillatory behavior with discontinuities at $h_2 k$, $k = 0, 1, 2, \ldots$. Notice that the larger the time-horizon $h_2$, the smaller the oscillation frequency. This is due to the fact that past large amplitude price changes are brought to the present by the price prediction $(p - p_{h_2})/h_2$. For small values of $h_2$, only past neighbor prices are accounted for, which yield a noise-like behavior.

(c) Effects of initial conditions. In the simulations above, the uniform initial condition $\phi(t) = 2.5$, $t \in [-h_2, 0]$ was taken. As pointed out by Hale and Lunel [1993], the initial condition $\phi(t)$ has a large effect on the dynamics of neutral differential equations. To illustrate this, let us take the initial condition $\phi(t) = 2.5 (1.0 + 0.25 \sin(0.5t) + 0.75 \cos(0.25t))$, $t \in [-h_2, 0]$. Figure 5 presents the stock dynamics for $h_2 = 10.0$ and $\beta_2 = 0.75$. As compared with Figs. 3(c) and 4(c), the oscillatory behavior contained in $\phi(t)$ is inherited by the stock dynamics. That is, in addition to the intrinsic oscillatory behavior of the stock dynamics, additional oscillations induced by $\phi(t)$ are displayed.

(d) Effects of double time-horizon. In the simulations above, we have taken $h_1 = 0.0$. As mentioned in the previous section, the addi-
tion of another time-horizon introduces at least one more fundamental frequency. That is, if $h_1 > 0$, the induced price dynamics should be more complex. Figure 6 presents the dynamical behavior of the stock market model for the same parameters as in Fig. 5 and $h_1 = 3.25$. In this case, the dynamics $p(t)$ contains at least one more fundamental frequency induced by the chartist trading strategy of the first investors group. Notice that the oscillatory behavior of $p(t)$ is more complex than in the previous ones.

It should be stressed that, except the constraint (15), the proposed model is of linear nature. In this way, the complex oscillatory behavior presented in Figs. 1–6 is the result of neutral-type differential structure of the model and the propagation of discontinuities and initial conditions induced by delayed time-derivatives [Hale & Lunel, 1993; Kolmanovskii & Myshkis, 1992]

The economic interpretation of these numerical simulations is straightforward. An increase in $\beta_2$ entails that some fundamental traders turn into chartist traders, or that the existing chartist traders become more aggressive. This may have a tremendous effect on the stability of $p_{\text{eq}}$, as intuition would suggest: the stock market dynamics can be more complex with complex oscillatory behavior. On the other hand, it seems that changes in the time-horizon does not eliminate the oscillatory behavior of the stock market dynamics. However, traders with a better technology to process information and enhance forecasting would use smaller values of $h_i$. As observed in Fig. 3, smaller values of $h_i$ reduce the amplitude of oscillations, hence leading to more stable stock market dynamics.

The behavior of oscillations in Figs. 1–6 is quite interesting. In fact, stock markets often display a sort of fluctuating behavior. Moreover, even the circumstance of a diverging price is not entirely devoid of economic meaning. Stock markets are sometimes exposed to a rise in prices that seems endless. However, such a growth is sooner or later stopped by a sudden and severe fall of prices, and a (total or partial) elimination of chartist traders. In our modeling framework, this phenomenon can be accounted
for if variations in the number of chartist traders are allowed.

### 3.2. Basic statistics

The simulations above have shown that our model is able to display complex oscillatory behavior. It is apparent that such oscillatory behavior resembles the qualitative dynamics of prices in real stock markets. Let us show that the price dynamics $p(t)$ can display transient erratic behavior and statistics similar to that found in real price histories.

(a) **Power spectra.** Several authors (see e.g. [Peters, 1990] and [De Grauwe et al., 1993]) have looked for chaotic behaviors in real stock price dynamics. As mentioned in the introduction, the empirical analysis on price time-series was not conclusive of chaos in asset markets. For instance, short period (e.g. monthly or quarterly) rates of inflation cannot be predicted, but in the long run period the quantity theory of money is valid. This would imply that the price time-series contains very high-frequency modes, which makes difficult short period forecasting. Figure 7(a) shows the price differences $d(t; \tau_r) = p(t) - p(t - \tau_r)$, $\tau_r = 1.0$, for the price time-series of Fig. 6. The time-series $d(t; \tau_r)$ displays a noise-like behavior with intermittent high-amplitude incursions. The corresponding power spectra is presented in Fig. 8, which shows a large concentration of high-frequency modes. The presence of these high-frequency behavior induces a stochastic-type behavior that makes difficult the forecasting of $d(t; \tau_r)$ for $t < \tau_r$. On the other hand, Fig. 7(b) shows low-pass filtered version $d_f(t; \tau_r)$ of $d(t; \tau_r)$. Notice that $d_f(t; \tau_r)$ displays a regular behavior which, in agreement with empirical observations [Stein, 1994], makes possible long period forecasting.

(b) **Probability distribution.** Mantegna and Stanley [1995] explored the probability distribution of a particular economic index — the Standard & Poor's 500. They found that the probability distribution of the price differences $d(t; \tau_r)$
could not be fitted using Gaussian distribution

\[ P(d) = P_0 + A \exp\left(-\frac{(x - x_c)^2}{w^2}\right) \]

which means that simple random processes cannot describe the dynamics of financial systems (see also [Mandelbrot, 1999]). They also found that the power-law behavior \(1/d^\beta\), similar to a phase transition phenomenon, describe better the data. Figure 9 presents the probability distribution of the price differences \(d(t; \tau_r)\) of Fig. 7. Figure 9 presents our best Gaussian fit with zero bias (i.e. \(P_0 = 0\)) and nonzero bias (i.e. \(P_0 > 0\)). In both cases, \(\chi^2 < 10^{-4}\). The curve fitting was carried out with MATLAB and Microcal Origin 4.0 softwares, giving practically the same results. Notices that nonzero bias Gaussian fits better in the low return rates and zero bias Gaussian fits better for moderate return rates. It could seem as if the probability distribution of return rate is composed by piecewise Gaussian functions. Of course, the nonzero bias Gaussian function does not make sense from a probabilistic viewpoint since its integral (i.e. accumulative distribution) has no finite limit. Such nonzero bias Gaussian fit is used as an artifact to illustrate that the price difference distribution has only regional Gaussian structure. A better fit is obtained if a power-law \(\sim 1/d^\beta\) is used (it should be mentioned that the actual power-law expression is \(P(d) = a/(d^\beta + b)\), where \(a\) is a positive parameter and \(b > 0\) is such that \(a/b = P(0)\)). In this case, \(\sigma = 0.4861\) with \(\chi^2 = 2.35 \times 10^{-5}\) is obtained. The mechanism that generates the power-law distribution (large tails) is the propagation of price history, including initial
conditions, along the price dynamics. Recall that this dynamical phenomenon is caused by the neutral-type nature of the price dynamics [Hale & Lunel, 1993]. That is, initial conditions with large price differences induce distributions with large tails in price difference dynamics. Recently, Mandelbrot [1999] have used multifractal methodologies to explain the presence of large fluctuations in price dynamics. In this way, static self-affinity structure in price dynamics is explained as the propagation of one basic structure along the price time-series via the neutral-type nature of the stock dynamical equations.

The above set of simulations shows that the model described in this paper is able to produce price time-series with statistics similar to that found in real financial time-series. It should be mentioned that we have found analogous power-law distributions for wide variations of $r$, which demonstrate that the time-series $d(t; \tau_r)$ displays interesting scaling properties similar to those found in real financial indexes [Mantegna & Stanley, 1995].

4. Conclusions

We have introduced a model of a stock market in terms of the price and traded assets. It is shown that the price dynamics can be represented as the solution of a linear differential equation of the neutral-type. Neutral-type differential equations, an important kind of functional-differential equation, can display a rich variety of dynamic behaviors, ranging from exponential stability and instability to complex oscillatory behaviors. Numerical simulations were provided to illustrate the main characteristics of the price dynamics and to interpret them in terms of stock market dynamics. In principle, the addition of more investor groups with different expectations and estimation (chartist) time-horizons would yield price dynamics with many fundamental frequencies [Kolmanovskii & Myshkis, 1992]. It is apparent that the resulting price time-series would produce strange “attractors” with very complex geometries. Moreover, discontinuities in the dynamics of neutral differential equations and heterogeneity of investors classes would make very hard short-run price forecasting, which is in agreement with early observations [Malliaris & Stein, 1999].

The results in this paper are not intended to demonstrate that stochastic approaches are useless. On the contrary, along the same lines of Schmidt’s paper [1999], the deterministic modeling approach must be seen as a complement to stochastic methodologies to gain understanding of real financial markets.

References


