Ex 1: Two runners

\[ \dot{x}_1 = p_1 (1 - p_2^2) \quad \dot{p}_1 = p_2 (1 - p_2^2) \quad x_1 = p_1 \cos \vartheta_1 \quad x_2 = p_1 \sin \vartheta_1 \]

\[ \dot{\vartheta}_1 = \omega_1 \quad \dot{\vartheta}_2 = \omega_2 \]

\[ x_3 = x_1 - w_1 x_2 - (x_1^2 + x_2^2) x_1 \]

\[ x_3 = w_4 x_4 + x_2 - (x_4^2 + x_2^2) x_2 \]

\[ x_1 = x_3 - w_3 x_4 - (x_3^2 + x_4^2) x_3 \]

\[ x_4 = w_2 x_2 + x_4 - (x_2^2 + x_4^2) x_4 \]

\[ Y = A x_2 + B x_4 \]

Notes:

- The invariant torus is the 2-dimensional manifold \( \{ x_1^2 + x_2^2 = 1, x_3^2 + x_4^2 = 1 \} \)

- If \( w_1/w_2 \) rational \( \Rightarrow \) there are infinitely many stable (but not asymptotically stable) cycles on the torus. Fixing \( e.g. \) \( x_1(0) = 1, x_2(0) = 0 \), two different initial conditions with \( x_3^2(0) + x_4^2(0) = 1 \) belong to two different cycles.

- If \( w_1/w_2 \) irrational \( \Rightarrow \) the dynamics on the torus is quasiperiodic. Each trajectory on the torus is aperiodic and densely fill the torus. Trajectories starting from close initial conditions remain close for all \( t > 0 \).

- The torus is \( k \)-dimensional, \( k > 2 \), if there are \( k \) incommensurable frequencies \( (\omega_i/\omega_j \text{ irrational for any } i \neq j, i, j = 1, \ldots, k) \) in the system.

Ex 2: Two friends running

\[ \dot{x}_1 = p_1 \cos \vartheta_1 - p_1 \sin \vartheta_1 \dot{\vartheta}_1 = p_1 (1 - p_2^2) \cos \vartheta_1 - p_1 \sin \vartheta_1 \left( \omega_1 + k \sin (\omega_2 - \omega_1) \right) \]

\[ = x_1 - w_4 x_2 - (x_1^2 + x_2^2) x_1 - \frac{k x_4 x_2}{\sqrt{x_2^2 + x_4^2}} \left( x_4 x_1 - x_2 x_3 \right) \]

Notes:

- Similarly for \( \dot{x}_2, \dot{x}_3, \dot{x}_4 \)

- If \( w_4 \approx k_1 + k_2 \) \( \Rightarrow \) there are two cycles on the torus (one asymptotically stable, the other unstable). The phase difference on the stable cycle is constant (phase locking).
Quasi-periodicity in discrete time

\[ y(t) = A \sin (\alpha t) \]

\[ \alpha \frac{\pi}{\Pi} = \begin{cases} \text{rational} & \text{(non-generic)} \\ \text{irrational} & \text{(generic)} \end{cases} \]

\[ \alpha \frac{\pi}{\Pi} = \frac{p}{q} \Rightarrow 2q \alpha = p \cdot 2\pi \Rightarrow y \text{ is periodic with period } T = 2q \]

\[ \alpha \frac{\pi}{\Pi} \text{ irrational } \Rightarrow y \text{ is quasi-periodic} \]

Ex: Rotation

\[ x(t+1) = R x(t) \quad \text{(linear system)} \]

\[ + \text{ h.o.t. to make the unit circle an attractive invariant curve} \]

\[ R = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \quad \text{(rotation matrix)} \]

\[ x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad y(t) = A x_2(t) \]

Notes:

- The invariant "torus" is the unit circle \( x_1^2 + x_2^2 = 1 \). It is called "torus" by analogy with the continuous-time case. It is an invariant curve.

- \( \alpha \frac{\pi}{\Pi} \) rational \( \Rightarrow \) there are infinitely many stable (not asymptotically stable) cycles. E.g., two different initial conditions on the unit circle belong to two different cycles.

- \( \alpha \frac{\pi}{\Pi} \) irrational \( \Rightarrow \) the dynamics on the torus is quasi-periodic (same comments as in continuous time).

- The torus is \( K \)-dimensional, \( K \geq 1 \), if there are \( K \) incommensurable rotations angles \( \alpha_i/\alpha_j \) irrational for any \( i \neq j, 1, 2, \ldots, K \) in the system.

* For example:

\[ p(t+1) = \bar{p}(t) + \frac{1 - p(t)}{2} = \frac{1 + \bar{p}(t)}{2} \Rightarrow \bar{p} = 1 \text{ is globally stable} \]

\[ \bar{\theta}(t+1) = \bar{\theta}(t) + \alpha \]

\[ x_1(t+1) = p(t+1) \cos \theta(t+1) = \frac{1}{2} (1 + \bar{p}(t)) (\cos \theta(t) \cos \alpha - \sin \theta(t) \sin \alpha) \]

\[ = \frac{1}{2} (x_1(t) \cos \alpha - x_2(t) \sin \alpha) + \frac{1}{2 \sqrt{x_1^2(t) + x_2^2(t)}} \left( x_1(t) \cos \alpha - x_2(t) \sin \alpha \right) \]

\[ x_2(t+1) = -\frac{1}{2} (x_1(t) \sin \alpha + x_2(t) \cos \alpha) \left( 1 + \frac{1}{2 \sqrt{x_1^2(t) + x_2^2(t)}} \right) \]