Quasi-periodicity

$y(t) = A \cos \omega_1 t + B \cos \omega_2 t$

$\frac{\omega_1}{\omega_2} = \begin{cases} \text{rational} & = \frac{p}{q} \quad (\text{probability} = 0) \\ \text{irrational} & (\text{probability} = 1) \end{cases}$

$\frac{\omega_1}{\omega_2} = \frac{p}{q} = \frac{T_2}{T_1} \Rightarrow \quad pT_1 = qT_2 \Rightarrow y(t) \text{ is periodic of period } T = pT_1 = qT_2$

$\omega_i = \frac{2\pi}{T_i}$

$\frac{\omega_1}{\omega_2} \text{ irrational } \Rightarrow \quad y(t) \text{ is quasi-periodic}$

Ex. 1. Two runners

$\dot{\theta}_1 = \omega_1, \quad \dot{\theta}_2 = \omega_2$

If $\frac{\omega_1}{\omega_2} = \frac{p}{q}$ runner 1 will complete p tours when runner 2 will complete q tours

Thus, after a period $T = p \frac{2\pi}{\omega_1} = q \frac{2\pi}{\omega_2}$ the two runners will be in the same conditions.
Visualization on torus

2-torus

\[
\frac{d\theta_2}{d\theta_1} = \frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{\omega_2}{\omega_1}
\]

If \( \frac{\omega_1}{\omega_2} = \frac{p}{q} \) we have a cycle on torus.

If \( \frac{\omega_1}{\omega_2} \) is irrational we have a trajectory on torus that never comes back to the initial condition but densely covers the entire torus.

**Ex. 2 Two friends running (phase locking)**

\[
\begin{align*}
\dot{\theta}_1 &= \omega_1 + K_1 \sin(\theta_2 - \theta_1) \\
\dot{\theta}_2 &= \omega_2 + K_2 \sin(\theta_1 - \theta_2) \\
\dot{\varphi} &= \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2 - (K_1 + K_2) \sin \varphi \\
\dot{\varphi} &= 0 \iff \sin \varphi = \frac{\omega_1 - \omega_2}{K_1 + K_2}
\end{align*}
\]

\[
\varphi = \frac{\omega_1 - \omega_2}{K_1 + K_2} \quad (\text{if } |\omega_1 - \omega_2| > K_1 + K_2)
\]

The behaviour is quasiperiodic.

\[
\omega^* = \lambda \omega_1 + (1 - \lambda) \omega_2
\]

\[
\lambda = \frac{K_2}{K_1 + K_2}
\]
Third order systems

The trajectory starting from 0 tends toward the torus. If the same happens for all points 0 close to the torus (also inside it) we say that the torus is an attractor.

Poincaré section

Inside the torus there is a limit cycle.

stable torus

unstable torus

In higher order systems we can also have saddle tori.
**Non existence of tori**

**Theorem 1** (non existence condition)

If the divergence \( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \ldots + \frac{\partial f_n}{\partial x_n} \) does not change sign in a domain \( \Omega \subset \mathbb{R}^n \) (and, at most, annihilates on a manifold of dimension \( n-1 \)) then there are no tori in \( \Omega \).

**Proof:** Some arguments used in Theorem 6 of lecture 6.

**Ex. 3** (Lorenz system)

\[
\begin{align*}
\dot{x}_1 &= \sigma (x_2 - x_1) \\
\dot{x}_2 &= r x_1 - x_2 - x_1 x_3 \\
\dot{x}_3 &= x_1 x_2 - b x_3
\end{align*}
\]

Divergence:
\[
\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = -\sigma - 1 - b < 0
\]

Thus, in the Lorenz system trajectories converge (because volumes contract) so that there cannot be an attracting or repelling torus (because inside the torus there should be a repelling or attracting cycle).
Problems

P. 1. \(\theta\) reconsider the system (see Ex. 2)
\[
\begin{align*}
\dot{\theta}_1 &= \omega_1 + \kappa_1 \sin (\theta_2 - \theta_1) \\
\dot{\theta}_2 &= \omega_2 + \kappa_2 \sin (\theta_1 - \theta_2)
\end{align*}
\]
Find a "conserved quantity" for this system
(i.e. a sort of energy function \(V\) such that \(V = \text{const.}\), i.e. \(V = 0\)).

P. 2. Consider the equations
\[
\begin{align*}
\mathbf{m} \ddot{r} &= \frac{\mathbf{h}^2}{m} \\
\dot{\theta} &= \frac{\mathbf{h}}{m r^2}
\end{align*}
\]
h = const. \(> 0\)

These equations describe (in polar coordinates)
the motion of a mass \(m\) subject to a central force of constant strength \(k > 0\).

(i) Show that the system has a solution
\[
\begin{align*}
\mathbf{r} &= \mathbf{R} \\
\dot{\theta} &= \omega
\end{align*}
\]
corresponding to uniform circular motion.

(ii) Find the frequency \(\omega_\text{r}\) of small radial oscillations about the circular orbit.

(iii) Show by a geometric argument that the motion is either periodic or quasiperiodic.