Structurally stable systems

\[ \dot{x}(t) = f(x(t)) \]

**Problem:** what happens if \( f \) is slightly modified?

\[ f \]
\[ f + \delta f \]
\[ x \]
\[ x \]

We will consider the following special, but important, case:

\[ \dot{x}(t) = f(x(t), p) \]

\( \uparrow \) parameters

and vary the parameters to check if the qualitative behavior of the system changes.

**Definition 1 (structural stability)**

A system \( \dot{x} = f(x, \bar{p}) \) is structurally stable if there exists \( \varepsilon > 0 \) such that the state portraits of \( \dot{x} = f(x, p) \) are topologically equivalent to the state portrait of \( \dot{x} = f(x, \bar{p}) \)

\[ \forall p : \| p - \bar{p} \| < \varepsilon \]

In practice: a small perturbation of the parameters does not change the qualitative behavior of a structurally stable system.
Tacoma's bridge

suspended bridge
\( p = \text{wind speed (constant)} \)

\[ p = \tilde{p} - \varepsilon \]
\[ p = \tilde{p} \]
\[ p = \tilde{p} + \varepsilon \]

The system with \( p = \tilde{p} \) is not structurally stable.

The system for \( p = \tilde{p} - \varepsilon \) or \( p = \tilde{p} + \varepsilon \) is structurally stable.

\( \tilde{p} \)
wind

bifurcation point

\( E \)
\( E^* \)
bifurcation curve

\( \tilde{p}^* \)
p
Bifurcation curves

bifurcations of codimension 1, 2, 3
The system is structurally stable if attractors, repellors and saddles (and their stable and unstable manifolds) are "separated".

In fact, in such a case, a small variation of the parameters implies a small variation of the invariant sets, which remain separated, so that the portrait of the system remains qualitatively the same.

Conclusion: bifurcation as collision of invariant sets
Bifurcations as collisions

1 state
1 parameter

2 states
1 parameter

2 states
1 parameter
Local bifurcations

Full collisions of attractors, repellors or saddles

Example

One eigenvalue is on the stability boundary when the collision occurs.
Global bifurcations

Collision of stable and unstable manifolds

\[ \tilde{p} - \varepsilon \quad \tilde{p} \quad \tilde{p} + \varepsilon \]

At \( \tilde{p} \) there is a bifurcation because any small perturbation implies a qualitative change of the state portrait.

Approaching \( \tilde{p} \) the manifolds \( W^s_1 \) and \( W^s_2 \) become closer and closer and finally collide for \( p = \tilde{p} \). For \( p = \tilde{p} \) there is a saddle to saddle connection.

The bifurcation is not "announced" by an eigenvalue approaching the stability boundary.

These bifurcations are more difficult to detect.
Catastrophic bifurcations

For $p = \bar{p} - \delta$, the system can be in the upper stable equilibrium.

For $p = \bar{p} + \delta$, the system can only be in the lower equilibrium.

For a microscopic variation of a parameter we have a macroscopic variation of the equilibrium.

In practice, for a small variation of the parameter we will have a macroscopic transition from one equilibrium to another.

The macroscopic transition is called catastrophic transition and the bifurcation is called catastrophic.

Examples: hurricanes, explosions, revolutions, crashes, strokes.

In this case, increasing $p$ we have a transition from an equilibrium (A) to a cycle.