Transcritical bifurcation

For any \( p \) there are two equilibria: one stable and one unstable. The two equilibria collide at \( \bar{p} \) where they exchange stability.

Ex. 1 Prey–predator model (see Ex. 8 Lecture 1)

\[
\begin{align*}
\dot{x}_1 &= r x_1 \left(1 - \frac{x_1}{K}\right) - a \frac{x_1}{b+x_1} x_2 \\
\dot{x}_2 &= e a \frac{x_1}{b+x_1} x_2 - m x_2
\end{align*}
\]

prey

predator

\( e \) (efficiency)

transcritical bifurcation

\( e < e_{tc} \)

\( e > e_{tc} \)
Transcritical bifurcation

Transcritical bifurcations (of equilibria) can occur in systems of any order: even in first order systems.

The simplest form of a bifurcation is called normal form.

The normal form of the transcritical bifurcation is the following first order system:

\[
\dot{x} = p x - x^2
\]

Equilibria: \( \dot{x} = 0 \) \( \Rightarrow \) \( \begin{cases} \bar{x} = 0 \\ \bar{x} = p \end{cases} \)

Stability: \( J = \frac{\partial f}{\partial x} \bigg|_{\bar{x}} = p - 2x \bigg|_{\bar{x}} = \begin{cases} P \text{ stable} & \Rightarrow p < 0 \\ -P \text{ stable} & \Rightarrow p > 0 \end{cases} \)

- For \( p = 0 \) we have only one equilibrium, while for \( p \neq 0 \) we have two equilibria \( \Rightarrow \) for \( p = 0 \) the system is not structurally stable \( \Rightarrow p = 0 \) is a bifurcation.
- For \( p = 0 \) we have a collision of equilibria \( \Rightarrow \) bifurcation.
- For \( p = 0 \) here are eigenvalues on the stability boundary.
The node can be either stable or unstable: in the first case we have a catastrophic transition.

When \( p = \bar{p} \) the Jacobian matrices of the two equilibria must have the same eigenvalues, while for \( p < \bar{p} \) the node has eigenvalues of the same sign, and the saddle has positive and negative eigenvalues. This implies that for \( p = \bar{p} \) one eigenvalue of the node and one eigenvalue of the saddle hits the stability boundary.

**Examples**: avalanches, earthquakes

**Terminology**: the saddle-node bifurcation is also called **fold** or **tangent bifurcation**

**Normal form** (first order system):

\[
\dot{x} = p + x^2
\]
Pitchfork bifurcation

non catastrophic

Example

Normal forms

\[ \dot{x} = px - x^3 \]

non catastrophic supercritical

\[ \dot{x} = px + x^3 \]

catastrophic subcritical
From equilibria to cycles

Transcritical in \( \mathbb{R}^3 \)

Saddle-node in \( \mathbb{R}^2 \)

Pitchfork in \( \mathbb{R}^2 \)

The two cycles exchange their stability.

The two cycles disappear.

before

after
Problems

P. 1
Verify (using the normal form) that the saddle-node bifurcation is characterized by two zero eigenvalues (one associated with one equilibrium and one with the other equilibrium) coming from opposite sides of the imaginary axis, i.e.

\[ 3 \mathrm{i} \quad \text{1-st equil.} \quad 0 \quad \text{Re} \quad 1 \mathrm{i} \quad \text{2-nd equil.} \]

P. 2
Consider the following 1-st order system
\[ \dot{x} = r x \left( 1 - \frac{x}{K} \right) - h \]
where \( x \) is the amount of resource and \( h \) is the harvest rate. Assume that \( h \) is constant in time and discuss the behavior of the system for all \( h \geq 0 \).
Show that the system has two types of behavior depending upon the value of \( h \). Find the critical value of \( h \) and determine the kind of bifurcation involved.

P. 3
Show that the mechanical system described in the figure is described by
\[
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = \frac{1}{J} \left( P \cos x_1 - K x_1 - H \dot{x}_2 \right)
\end{cases}
\]
where \( K x_1 \) is the momentum of the spring and \( H \) is a friction coefficient. Study the equilibria of the system for small values of \( P \) and then consider higher and higher values of \( P \). Prove that a pitchfork bifurcation occurs.