Bistability and hysteresis

when there are two attractors
we say that the system is
"bistable"

Bistability is often present in a range of $p$

What happens if the parameter is varied step-by-step
over a range $[p_-, \bar{p}]$ larger than $[p, \bar{p}]$?

each time the parameter is
changed the system goes toward
a new equilibrium, following
the sequence $0, 1, 2, \ldots, 4, 5, 6, 0$
The corresponding graph is called
hysteresis or hysteretic loop.

conclusion
in order to have hysteresis at least
two bifurcations are needed.
A fishery model

\[ x(t) = \text{fish stock} \]

\[ E = \text{fishing effort} = \text{# boats} \]

\[ \dot{x} = r x \left( 1 - \frac{x}{K} \right) - \frac{a x}{b + x} \]

\[ f(x) \quad g(x) \]

For \( E = E_0 \) \( f'(0) = g'(0) E \)

For \( E = \bar{E} \) two equilibria collide

For \( E < E < \bar{E} \) there are three equilibria: the origin and two positive equilibria.

The highest equilibrium is stable because in its neighborhood \( E \bar{g}' > f' \) if \( x > x \)

The system has an hysteresis

\[ Y = \text{yield} = \text{production rate at equilibrium} \]

The maximum yield \( Y_{\text{max}} \) is very close to a catastrophe

( the tragedy of natural resources!)
Cusp

The analysis of a dynamical system with two parameters very often points out a structure called "cusp".

The cusp is characterized by 2 saddle-nodes with respect to one parameter and by 1 pitchfork with respect to the other parameter.

The normal form of the cusp is \( \dot{x} = p_1 + p_2 x - x^3 \)

If \((p_1, p_2) \in \text{cusp region}\) there are 3 equil. (2 stable and 1 unstable).

If \((p_1, p_2) \in \text{cusp region}\) there is only 1 equil. (which is stable).

Varying the parameters step-by-step one can bring the system from A to D smoothly. But there are also alternative paths passing through catastrophes.

Examples

- Cardiac block
- Zeeman's machine
- Acid rain and forest exploitation

\[(p_1, p_2)\]
Problems

P. 1.
Describe (in words) a phenomenon that you have observed which is an hysteresis.

P. 2.
Discuss the dynamics of the 1-st order system
\[ \dot{x} = p \cdot x + x^3 - x^5 \]
by plotting its equilibria \( \bar{x} \) versus \( p \). In particular, find out the bifurcations and determine if the system has hysteresis.

P. 3.
Consider the following model of a couple
\[ \begin{align*}
\dot{x}_1 &= -f_1 x_1 + R_i(x_2) + p_1 A_i \\
\dot{x}_2 &= -f_2 x_2 + R_j(x_1) + p_2 A_j
\end{align*} \]
where \( x_i \) is the feeling of individual \( i \) for the partner, \( R_i \) is the reaction to the love of the partner, \( A_i \) is the appeal of individual \( i \), and \( f_i \) and \( p_i \) are positive parameters (forgetting coefficient and sensitivity to appeal). Assume that the reaction function are of this type

\[ R_i \]

Show that the system can have multiple equilibria for small (positive) values of the appeals. Letting \( A_1 = A_2 = A \) say if there is an hysteresis with