Harmonics and subharmonics

\[
\begin{align*}
\text{ Harmonics } & : \quad \omega_k = k \omega \\
& \quad T_k = \frac{T}{k}
\end{align*}
\]

Example.

She looks at the waves and counts \( n_1 \).
He looks at the boat and counts \( n_2 \).

\[
\frac{n_1}{n_2} = \begin{cases} 
4 & \quad 2 \quad \text{period doubling} \\
8 & \quad 4 \quad \text{period doubling} \\
16 & \quad 8 \quad \text{period doubling}
\end{cases}
\]

\[
T_2 = \begin{cases} 
2T_1 & \quad \text{period doubling} \\
4T_1 & \quad \text{period doubling} \\
8T_1 & \quad \text{period doubling}
\end{cases}
\]
Flip bifurcation

before \((\bar{p} - \varepsilon)\)  

after \((\bar{p} + \varepsilon)\)

From the left: a stable cycle of period \(T\) bifurcates into an unstable cycle of period \(T\) and a stable cycle of period \(2T\).

This bifurcation is called flip or "period doubling".

When \(p\) is varied further, the period and the form of the stable cycle change.

Varying the parameter further, there can be other period doublings: \(T_3 \rightarrow 2T_3 \rightarrow T_2 \rightarrow 2T_2 \rightarrow T_3 \rightarrow 2T_3 \rightarrow \ldots\).

Sometimes a sequence of period doublings is referred to as "period 1, 2, 4, ..."

We can also have catastrophic flips.
How can a flip be detected?

\[ z(t+1) = P(z(t)) \] Poincaré map

The points \( z' \) and \( z^2 \) are very close to \( \hat{z} \) and visited alternatively:

\[ \delta^1 = z' - \hat{z} \]
\[ \delta^2 = z^2 - \hat{z} \]

\[ J \delta^1 = \delta^2 \]
\[ J \delta^2 = \delta^1 \]

\[ J^2 \delta^1 = \delta^1 \]

\[ J \text{ has an eigenvalue equal to } -1 \]
Logistic map: equilibria

\[ x(t+1) = r x(t) \left( 1 - x(t) \right) \Rightarrow x' = r x (1 - x) \]

if \( r \leq 4 \) : \([0, 1] \rightarrow [0, 1]\)

**Equilibria**

\[ x' = x \Rightarrow r x^2 + (1 - r) x = 0 \Rightarrow x = \begin{cases} 0 \\ \frac{r-1}{r} \quad (r > 1) \end{cases} \]

**Stability**

\[ \frac{df}{dx} \bigg|_{x=0} = r \Rightarrow x = 0 \text{ is stable for } r < 1 \text{ and unstable for } r > 1 \]

\[ \frac{df}{dx} \bigg|_{x= \frac{r-1}{r}} = 2 - r \Rightarrow x = \frac{r-1}{r} \text{ is stable for } 1 < r < 3 \text{ and unstable for } 3 < r < 4 \]

Remark: for \( 3 < r < 4 \), there are no stable equilibria

"interesting" behavior
An equilibrium is a period 2 solution. We have already seen that for $r = 3$ there is a period doubling. This means that for $r = 3 + \varepsilon$ we must have a cycle of period 2:

$\cdots \x_2 \rightarrow x_1 \rightarrow x_2 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$

$x_1$ and $x_2$ are equilibria of the so-called second iterate:

$x_1 = f(f(x_1)) \quad x_2 = f(f(x_2))$

The two solutions of the equation $x = f^{(2)}(x)$ i.e.

$x = f(f(f(x)))$

are $x_1$ and $x_2$. Finding $x_1$ and $x_2$ explicitly is not too difficult. Extending this method one can look for period 4, 8, ... solutions:

$x = f^{(4)}(x) \quad x = f^{(8)}(x) \quad \cdots$

The result is the following:

$r_\infty = 3.5699 \ldots$ (irrational)

$\Delta r_n = r_n - r_{n-1}$

$\Delta r_{n+1} = \frac{1}{\delta} \Delta r_n$

$\delta = 4.6692 \ldots$

Feigenbaum constant

Conclusion: an infinite sequence (cascade) of flips announces chaos.
Universal-ity

The Feigenbaum cascade is present and has the same $\delta$ (which is, therefore, a universal constant: the $\pi$ of dynamical systems) in many classes of discrete-time and continuous-time dynamical systems.

For example, in continuous-time systems the following often holds:

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, x_3) \\
\dot{x}_2 &= f_2(x_1, x_2, x_3) \\
\dot{x}_3 &= f_3(x_1, x_2, x_3)
\end{align*}
\]

\[\text{strange attractor}\]

**Universal-ity Theorem (1-dimensional quadratic map) bubble**

\[
x' = f(r, x)
\]

\[
\begin{align*}
1. \quad \frac{df}{dr} > 0 \\
2. \quad f([0, 1]) = [0, 1] \\
3. \quad f(0) = f(1) = 0 \\
4. \quad f(r) \text{ unimodal} \\
5. \quad f''(r, x^*) < 0 \quad \text{where } f'(r, x^*) = 0 \\
6. \quad f(r, x) \in C^3 \\
7. \quad \frac{f''}{f'} - \frac{3}{2} \left(\frac{f''}{f'}\right)^2 < 0
\end{align*}
\]

$\Rightarrow$ Feigenbaum cascade