Homoclinic and heteroclinic connections

A trajectory connecting two saddles is called heteroclinic connection.

\[ \begin{array}{c}
\text{heteroclinic connection} \\
\uparrow \\
\text{heteroclinic loops}
\end{array} \]

A trajectory connecting a saddle to itself is called homoclinic connection (or homoclinic orbit or homoclinic loop).

\[ \begin{array}{c}
\text{double homoclinic loop} \\
\text{homocl. in } \mathbb{R}^3
\end{array} \]

The region inside the homoclinic loop is an invariant set.

(a) attracting homoclinic loop

(b) neutral

(c) repelling

Inside the loop in case (a)
Heteroclinic and homoclinic bifurcations

In general, a system with an heteroclinic or homoclinic connections is structurally unstable.

Ex. 1 Tritrophic food chain

\[ \begin{align*}
\dot{x}_1 &= r x_1 \left(1 - \frac{x_1}{K}\right) - \frac{q x_1}{b + x_1} x_2 \\
\dot{x}_2 &= \frac{e q x_1}{b + x_1} x_2 - m x_2 \\
x_3 &= \text{const.} \quad \frac{e x_2}{d + x_2} x_3
\end{align*} \]
Andronov's theorem in $\mathbb{R}^2$

Close to the saddle: $e^{\lambda t}$ against $e^{-\mu t}$

$\sigma < 0 \left[ \log \mu > \log \lambda \right] \quad \sigma > 0 \left[ \log \mu < \log \lambda \right]$

The homoclinic loop attracts

The cycle obtained through perturbation is stable

The homoclinic loop is repelling

The cycle obtained through perturbation is unstable

Extension to heteroclinic loops [Reyn 1929]

If a parameter is perturbed the loop breaks and a cycle appears (if the perturbation has the right sign)

Moreover

$\Sigma \log \mu_i > \Sigma \log \lambda_i \Rightarrow$ stable cycle

$\Sigma \log \mu_i < \Sigma \log \lambda_i \Rightarrow$ unstable cycle
Shilnikov's theorem in $\mathbb{R}^3$

- Real saddle
  $\sigma = \lambda - \mu$

- Complex saddle
  $\sigma = \lambda - \mu$

Eigenvalue closest to imaginary axis

- Real
  - Positive ($\sigma > 0$) $\Rightarrow$ stable cycle
  - Negative ($\sigma < 0$) $\Rightarrow$ unstable cycle
- Complex
  - Positive real part
    - $\Rightarrow$ $\infty$ cycles
  - Negative real part
    - $\Rightarrow$ $\infty$ cycles

- The theorem holds provided the saddle is not degenerate (11 conditions! Champneys-Kuznetsov 1994)

- The theorem has been extended to $\mathbb{R}^n$
Torus destruction is a second route to chaos. It is characterized by a single (but global) bifurcation, namely an heteroclinic bifurcation.

\[ \downarrow p + \delta p \]

Torus and period 2 cycle

\[ \downarrow p + \delta p \]

Pinched torus

\[ \downarrow p + \delta p \]

Collision: heteroclinic loop

\[ \text{CHAOS} \]

\[ \Rightarrow \]

\[ p + \delta p \]