An MPC algorithm for offset-free tracking of constant reference signals

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Abstract

A model predictive control (MPC) technique for tracking piecewise constant reference signals is presented. The controlled system is assumed to be linear, time invariant and subject to constraints on both inputs and states. In order to reject constant disturbances and to guarantee offset-free regulation also in presence of model/plant mismatch, the control loop is enlarged with integrators, and the system is described in “velocity form”. The tracked reference signal is regarded as one of the optimization variables, so that feasibility is guaranteed also for unfeasible desired references. Stability results are given, and a simulation example is reported.

I. INTRODUCTION

In the development of efficient industrial MPC algorithms, two main issues must be considered: the offset-free problem and the unfeasible reference problem. The offset-free problem requires to develop methods which can guarantee asymptotic zero error regulation for piecewise constant and feasible reference signals, while the unfeasible reference problem aims at finding suitable solutions when the nominal constant reference signal cannot be reached due to the presence of state and/or control constraints.

Regarding the offset-free problem, many solutions have been proposed so far. The most popular one consists of augmenting the model of the plant under control with an artificial disturbance, which must be estimated together with the system state. This disturbance can account for possible model mismatch or for the presence of real unknown exogenous signals. Depending on its assumed dynamics, many algorithms have been developed, e.g. [13], [14], [16], [17]. Another approach directly relies on the Internal Model Principle [4], where an internal model of the reference, i.e. an integrator, is directly included in the control scheme and fed by the output error. Then, the MPC algorithm is designed to stabilize the ensemble of the plant and the integrator. This strategy has been followed in [10], [11] where also more general exogenous signals and nonlinear, non square systems have been considered. A third solution to the offset-free problem consists of describing the system in the so-called velocity form, see [19], [18], [21], where the enlarged state is composed by the state increments and the output error, while the manipulated variable is the control increment.

Although all these solutions to the offset-free problem share the common idea to include, implicitly or explicitly, an integral action in the control loop, each one has its own advantages and drawbacks. In particular, the use of the velocity form does not require the use of a state estimator even when the plant state is available and does not require to compute the steady state target for the plant state and control variables in order to properly formulate the optimization problem considered in the MPC formulation. On the other hand, and to the best of the authors knowledge, no stability results have been established for MPC algorithms based on the velocity form and characterized by a terminal cost and a terminal constraint, terms usually considered to guarantee stability and convergence, see [12]. This is due to the difficulty to find an auxiliary control law and an associated terminal set where the constraints on the state and input of the system are fulfilled, see [18].

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The unfeasible reference problem has been discussed in e.g. [3], [20], [15]; more recently, a solution has been proposed in [2], [1], [8], [9]. In these papers, the MPC cost function is complemented by a term explicitly penalizing the distance between the required (possibly unfeasible) reference signal and an artificial, but feasible, reference, which turns out to be one of the optimization variables. Stability and convergence results are proved both in nominal conditions and for perturbed systems, [2], [1], [8], [9]. In this paper, an MPC algorithm for linear systems described in velocity form is presented. Relying on the approach described in [2], [1], [8], [9] for the solution of the unfeasible reference problem, it guarantees stability properties and the convergence of the controlled output to the feasible reference or to the nearest artificial reference, i.e. the solution of the offset-free problem. The main point in the derivation of the method is related to the definition of an auxiliary stabilizing control law and of a region where its use can guarantee the fulfillment of the original state and control constraints. This is achieved by resorting to the methods described in [6] for the computation of the Maximum Output Admissible Sets and is discussed in Section II. The MPC algorithm, together with the analysis of its properties, is presented in Section III, while Section IV is devoted to discuss a simulation example. Finally, some conclusions are reported in Section V.

Notation. We say that a matrix is Schur if all its eigenvalues lie in the interior of the unit circle. We use the short-hand \( v = (v_1, \ldots, v_s) \) to denote the column vector \( v = [v_1^T, \ldots, v_s^T]^T \) with \( s \) (possibly vector) components. For a discrete-time signal \( s_t \) and \( a, b \in \mathbb{N}, a \leq b \), we denote \( (s_a, s_{a+1}, \ldots, s_b) \) with \( s_{[a:b]} \). Finally, \( I_n \) denotes the identity matrix of dimensions \( n \times n \), and we denote by 0 the matrix with zero entries, without specifying its dimension when clear from the context.

II. Statement of the Problem

Consider a discrete-time, linear, time-invariant system, described by

\[
\begin{align*}
x_{t+1} &= Ax_t + Bu_t \\
y_t &=Cx_t
\end{align*}
\]

where \( x_t \in \mathbb{R}^n \) is the state, \( u_t \in \mathbb{R}^m \) is the input, and \( y_t \in \mathbb{R}^m \) is the output. The inputs and state variables are subject to constraints, i.e. \( x_t \in \mathbb{X} \) and \( u_t \in \mathbb{U} \) for all instants \( t \), where \( \mathbb{X}, \mathbb{U} \) are compact and convex neighbors of the origin.

In this paper we consider the problem of designing a state-feedback control system, based on MPC, for tracking a given constant reference signal \( \hat{y} \in \mathbb{R}^m \), i.e., that asymptotically steers the system output \( y_t \) to the desired value \( \hat{y} \).

The following standard assumptions are made.

Assumption 1:

(i) The pair \((A, B)\) is reachable.

(ii) The input-output system (1) has no invariant zeros in 1, i.e., \( \text{rank}(S) = n + m \), where

\[
S = \begin{bmatrix} \mathbb{I}_n - A & -B \\ -C & 0 \end{bmatrix}
\]

Note that Assumption 1 (ii) guarantees the existence and the uniqueness of the steady-state pair \((\hat{x}, \hat{u})\) \(\in \mathbb{R}^{m+n}\) such that the system output corresponds to the desired value \( \hat{y} \).

A. The velocity form

In order to solve the tracking problem, the system is enlarged with \( m \) integrators and described in the so-called velocity form. Specifically, denoting by \( \tilde{y} \) a generic tracking target (which can, in principle, be different from \( \hat{y} \), for reasons which will become clear later on), the corresponding steady state condition is denoted \((\tilde{x}, \tilde{u})\). Letting \( \delta x_t = x_t - x_{t-1}, \epsilon_t = y_t - \tilde{y}, \text{and} \delta u_t = u_t - u_{t-1}, \) system (1) can be reformulated as follows.
\[
\begin{align*}
\delta x_{t+1} &= A\delta x_t + B\delta u_t \\
\varepsilon_{t+1} &= CA\delta x_t + \varepsilon_t + CB\delta u_t
\end{align*}
\] (2)

Define \( \xi_t = (\delta x_t, \varepsilon_t) \) and
\[
\tilde{A} = \begin{bmatrix} A & 0 \\ CA & I_m \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ CB \end{bmatrix}
\]

In this way system (2) can be written in compact form as
\[
\xi_{t+1} = \tilde{A}\xi_t + \tilde{B}\delta u_t
\] (3)

The following proposition can be easily proved.

**Proposition 1:** If Assumption 1 holds, then the pair \((\tilde{A}, \tilde{B})\) is reachable. \(\blacksquare\)

In view of the reachability property of \((\tilde{A}, \tilde{B})\), it is possible to compute the gain \(\tilde{K}\) such that \(\tilde{F} = \tilde{A} + \tilde{B}\tilde{K}\) is Schur. Note that such a gain \(\tilde{K} \in \mathbb{R}^{(m+n) \times n}\) is composed by two matrix blocks \(K_\delta \in \mathbb{R}^{m \times n}\) and \(K_\varepsilon \in \mathbb{R}^{m \times m}\) such that
\[
\tilde{K} = \begin{bmatrix} K_\delta & K_\varepsilon \end{bmatrix}
\]

The dynamics of system (3), under the state-feedback control law
\[
\delta u_t = \tilde{K}\xi_t
\] (4)

is given by
\[
\xi_{t+1} = \tilde{F}\xi_t
\] (5)

**B. The maximal output admissible set**

In the following, system (3) will be used to design an MPC algorithm with stability and tracking properties. To this end, (4) will be the auxiliary control law used to guarantee stability, see [12]. However, with respect to more standard formulations, the velocity form (3) with state and control increments \(\delta x_t\) and \(\delta u_t\) poses the problem of properly reformulating the constraints on the original variables \(x_t\) and \(u_t\) in terms of constraints on the state \(\xi_t\) of the closed-loop system (5).

To this regard, the main problem of this section is to define a suitable invariant set for the trajectory \(\xi_t\) and at the same time a set of output reference values \(\bar{y}\), such that it is guaranteed that the original input and state variables \(u_t\) and \(x_t\), respectively, lie in the feasibility sets \(\mathcal{U}\) and \(\mathcal{X}\). The first step towards this goal is the following result.

**Proposition 2:** Under Assumption 1, the following equation holds
\[
\begin{bmatrix} x_t \\ u_{t-1} \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} + C_\xi \xi_t = C_x\bar{y} + C_\xi \xi_t
\] (6)

where
\[
C_y = -S^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \quad C_\xi = \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_m \end{bmatrix} \Theta_\Delta^+ \begin{bmatrix} I_{n+m} \\ \vdots \\ F_{2n+m-1} \end{bmatrix}
\]

The symbol \((\cdot)^+\) indicates pseudo-inversion and \(\Theta_\Delta \in \mathbb{R}^{((n+m)\cdot(2n+m)) \times (2n+m)}\) is the observability matrix of the pair \((F_\Delta, C_\Delta)\), i.e.,
\[
\Theta_\Delta = \begin{bmatrix} C_\Delta \\ C_\Delta F_\Delta \\ \vdots \\ C_\Delta F_{2n+m-1} \end{bmatrix}
\]
where

\[ F_\Delta = \begin{bmatrix} 0 & I_n & 0 \\ -BK_\delta & A + B(K_\delta + K_\varepsilon C) & B \\ -K_\delta & K_\delta + K_\varepsilon C & I_m \end{bmatrix} \in \mathbb{R}^{(2n+m)\times(2n+m)} \]

and

\[ C_\Delta = \begin{bmatrix} -I_n & I_n & 0 \\ 0 & C & 0 \end{bmatrix} \in \mathbb{R}^{(n+m)\times(2n+m)} \]

In view of Proposition 2, the issue of computing an invariant set where \((\xi, \bar{y})\) must lie in order to guarantee that constraints \((x_t, u_t) \in X \times U\) are verified for all \(t\) can be cast as the problem of computing the maximal output admissible set (MOAS) for the following auxiliary system:

\[
\begin{align*}
\begin{bmatrix} \xi_{t+1} \\ \bar{y} \\ x_t \\ u_{t-1} \end{bmatrix} &= \begin{bmatrix} \bar{F} & 0 & \bar{C} \\ 0 & I_m & \bar{C} \end{bmatrix} \begin{bmatrix} \xi_t \\ \bar{y} \\ \bar{y} \end{bmatrix} \\
\end{align*} \tag{7a}
\]

\[
\begin{align*}
\begin{bmatrix} \bar{F} & \bar{C} \\ 0 & I_m \end{bmatrix} \begin{bmatrix} \xi_t \\ \bar{y} \end{bmatrix} + \begin{bmatrix} C_\varepsilon & C_\gamma \\ \bar{C} & \bar{C} \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \begin{bmatrix} \xi_t \\ \bar{y} \end{bmatrix} \\
\end{align*} \tag{7b}
\]

where (7a) and (7b) play the role of a state equation and of an output equation, respectively.

Letting

\[ F^* = \begin{bmatrix} \bar{F} & 0 \\ 0 & I_m \end{bmatrix}, \quad C^* = \begin{bmatrix} C_\varepsilon & C_\gamma \end{bmatrix} \]

under the following

Assumption 2:

i) the pair \((F^*, C^*)\) is observable,

ii) \(X \times U\) is a close polytope,

then a polytopic inner approximation \(\Diamond\) to the MOAS can be computed in a finite number of steps [6], which is defined as follows:

\[
\Diamond = \{(\xi, \bar{y}) \in \mathbb{R}^{n+2m} : C_\varepsilon \bar{F}^t \xi + C_\gamma \bar{y} \in X \times U \text{ and } C_\gamma \bar{y} \in X_\varepsilon \times U_\varepsilon \}
\tag{8}
\]

where \(X_\varepsilon\) and \(U_\varepsilon\) are close and compact sets satisfying \(X_\varepsilon \oplus B_\varepsilon^{\text{dim}}(0) \subseteq X\) and \(U_\varepsilon \oplus B_\varepsilon^{\text{dim}}(0) \subseteq U\), where \(\oplus\) denotes the Minkowski sum, \(B_\varepsilon^{\text{dim}}(0)\) defines a ball of radius \(\varepsilon\) in the \(\mathbb{R}^{\text{dim}}\) space, and \(\varepsilon\) can be arbitrarily small. The definition (8) enjoys a fundamental property: \(\Diamond\) results to be the set of initial conditions \((\xi, \bar{y})\) for the dynamic system (7a) such that, during the transient, it is guaranteed that \((x_t, u_t) = C_\varepsilon \bar{F}^t \xi + C_\gamma \bar{y} \in X \times U\), and the allowed steady state values \((\bar{x}, \bar{u}) = C_\gamma \bar{y}\) belong to the set \(X_\varepsilon \times U_\varepsilon \subseteq \text{Int}(X \times U)\). The latter is fundamental for the following result. For details on the computation of \(\Diamond\), please see [6].

III. THE MPC PROBLEM

In this section we introduce the MPC solution to the tracking problem stated in the previous section. Similarly to [8], we remark that an arbitrary desired reference output \(\hat{y}\) may easily lead to infeasible standard MPC optimization problems. In case this occurs, we assume (in line with the reference governor approach [5]) that the value \(\bar{y}\) to be considered as the output reference trajectory in the MPC problem at time \(t\) is different from \(\hat{y}\). Specifically, it will be computed as the nearest point to \(\hat{y}\) (according to a given measure), which at the same time guarantees feasibility. As such, \(\bar{y}\) will be regarded as an argument of the optimization problem itself, rather than a fixed parameter.

These considerations lead to state the following MPC optimization problem to be solved at each time instant \(t\):

\[
V_N^*(\delta x_t, y_t, \hat{y}) = \min_{\bar{y}, \delta u_{[t:T+N-1]}} V_N(\bar{y}, \delta u_{[t:T+N-1]} ; \delta x_t, y_t, \hat{y}) \tag{9}
\]
where

\[ V_N = \| \tilde{y}_t - \hat{y} \|_F^2 + \sum_{k=1}^{t+N-1} \left( \| \xi_k \|_Q^2 + \| \delta u_k \|_R^2 + \| \tilde{\xi}_{t+N} \|_P^2 \right) \]

subject to the dynamic constraint (3), to the constraints

\[ x_t + [I_n \ 0] \sum_{r=1}^{k} \xi_{t+r} \in \mathbb{X} \]

for all \( k = 1, \ldots, N - 1 \),

\[ u_{t-1} + \sum_{r=0}^{k} \delta u_{t+r} \in \mathbb{U} \]

for all \( k = 0, \ldots, N - 2 \), and to the terminal constraint

\[ \begin{bmatrix} \xi_{t+N} \\ \tilde{y}_f \end{bmatrix} \in \mathbb{O} \]

As discussed, in the minimization problem (9), \( \hat{y} \) and \( \tilde{y}_t \) are the fixed desired reference value and the “artificial” setpoint actually tracked at instant \( t \), respectively. Furthermore, the input sequence \( \delta u_{t:t+N-1} \) and \( \tilde{y}_f \) are the decision variables of the problem. The solution to (9) is denoted \( \delta u_{t:t+N-1}|_{t}, \tilde{y}_f|_{t} \).

Note that, in view of the definition of \( \xi \) and \( \epsilon \), the stated problem requires to optimize, at any time instant, not only the future control increments, but also the part of the current state \( \xi(t) \) which depends on \( \tilde{y} \).

The weighting matrices \( Q \in \mathbb{R}^{(n+m)\times(n+m)} \) and \( R \in \mathbb{R}^{m\times m} \) are symmetric and positive definite, and \( P \in \mathbb{R}^{(n+m)\times(n+m)} \) is the positive definite solution of the equation

\[ \tilde{F}^T P \tilde{F} - P = -(Q + \tilde{K}^T R \tilde{K}) \]

In view of the dimensions of \( \tilde{F} \), \( P \in \mathbb{R}^{(n+m)\times(n+m)} \). We can decompose it as follows

\[ P = \begin{bmatrix} P_{xx} & P_{xy} \\ P_{yx} & P_{yy} \end{bmatrix} \]

where \( P_{xx} \in \mathbb{R}^{n\times n} \), \( P_{xy} \in \mathbb{R}^{n\times m} \), and \( P_{yy} \in \mathbb{R}^{m\times m} \).

Matrix \( T \in \mathbb{R}^{m\times m} \) is a further tuning knob that must satisfy the constraint

\[ T - P_{yy} > 0 \]

Some comments are in order.

Constraints (10) and (11) are equivalent to require that \( x_{t+k} \in \mathbb{X} \) for all \( k = 1, \ldots, N - 1 \) and \( u_{t+k} \in \mathbb{U} \) for all \( k = 0, \ldots, N - 2 \). Furthermore, (12) implies that if, for all times \( k \geq t + N \), the system (3) is controlled using the auxiliary state-feedback control law (4) then, in view of the invariance properties of \( \mathbb{O} \), \( x_{t+N+i} \in \mathbb{X} \) and \( u_{t+N+i-1} \in \mathbb{U} \) are verified for all \( i \geq 0 \).

It is now possible to prove that, for any feasible initial state, the proposed method asymptotically steers the system to the desired reference value \( \hat{y} \) if \( \hat{y} \) is admissible, i.e., if \( (\hat{x}, \hat{u}) \in \mathbb{X} \times \mathbb{U} \) while, if \( \hat{y} \) is not admissible, the system output is driven to the admissible output \( \hat{y} \) that minimizes the cost \( \| y - \hat{y} \|_F^2 \).

The main convergence result can now be proved along the lines depicted in [8].

**Theorem 1:** Assume that (i) Assumption 1 is verified and that (ii) the set \( \mathbb{O} \) is invariant and satisfies the definition (8). Then, if at time \( t = 0 \) a feasible solution to (9) exists for a given set-point value \( \hat{y} \), the resulting MPC controller asymptotically steers the system to the admissible set-point \( \hat{y} \), while respecting the constraints \( (x_t, u_t) \in \mathbb{X} \times \mathbb{U} \) for all \( t \geq 0 \), and \( \hat{y} \) is the solution of

\[ \hat{y} = \arg \min_{C, \hat{y} \in \mathbb{X} \times \mathbb{U}} \| y - \hat{y} \|_F^2 \]
IV. EXAMPLE

Consider the following benchmark problem, originally proposed in [7] and [8]. The nominal model matrices in (1) are

\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Note that, differently from [7] and [8], here both states correspond to outputs. The constraints on the states and inputs are \( \|x\|_\infty \leq 5 \) and \( \|u\|_\infty \leq 3 \), respectively. System (1) is used for the synthesis of the controller. Assume also that the plant is affected by both model perturbations and disturbances: the real transition and output matrices are \( A_{\text{real}} \) and \( B_{\text{real}} \), respectively, and an additional constant unknown disturbance \( d \) is present. Therefore the evolution of the system obeys to

\[
\begin{align*}
x_{t+1} &= A_{\text{real}}x_t + B_{\text{real}}u_t + d \\
y_t &= Cx_t
\end{align*}
\] (16)

where

\[ A_{\text{real}} = \begin{bmatrix} 0.8 & 1.1 \\ 0.05 & 0.9 \end{bmatrix}, \quad B_{\text{real}} = \begin{bmatrix} 0.55 & 0.05 \\ -0.05 & 0.9 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad d = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \]

We set \( Q = \text{diag}(0.1, 0.1, 1, 1) \), \( R = \text{diag}(0.1, 0.1) \), and \( N = 10 \). Matrices \( \bar{K} \) and \( P \) have been computed according to the LQ synthesis criterion:

\[ K = \begin{bmatrix} -1.69 & -1.71 & -1.22 & 0.30 \\ -0.05 & -0.10 & 0.02 & -0.73 \end{bmatrix}, \quad P = \begin{bmatrix} 0.44 & 0.34 & 0.24 & -0.06 \\ 0.34 & 0.54 & 0.24 & 0.01 \\ 0.24 & 0.24 & 1.41 & 0.10 \\ -0.06 & 0.01 & 0.10 & 1.37 \end{bmatrix} \]

Matrix \( P_{yy} \) is therefore

\[ P_{yy} = \begin{bmatrix} 1.41 & 0.10 \\ 0.10 & 1.37 \end{bmatrix} \]

Matrix \( T \) has been set to \( T = 10P_{yy} \).

In the simulations, the reference trajectory for \( y_2 \) is constant and equal to zero. The first output \( y_1 \), instead, has a piece-wise constant reference trajectory, taking values 5, \(-5.5\) and 3. The results are shown in Fig. 1. In Fig. 2 we show the trajectories of the input variables.

Importantly, note that every admissible reference signal is tracked without error and that the effects of the model perturbation and of the unknown disturbance have been rejected, thanks to the particular offset-free MPC formulation proposed in this paper. On the other hand, the setpoint \(-5.5\) it is not admissible and the output is steered to the closest admissible value.

Although constraint satisfaction is not guaranteed for perturbed systems by the results presented in this work, they are accidentally satisfied in this example.

For the sake of completeness, in Fig. 3 and Fig. 4, the results when the real plant is identical to the nominal one are shown (here the constraint fulfillment is a-priori guaranteed).

V. CONCLUSIONS

In this paper, an MPC algorithm solving the offset-free tracking and the infeasible reference problems has been developed for systems expressed in velocity form. Stability results have been obtained by suitably defining the auxiliary control law and the terminal set used in the problem formulation. Further effort will focus on simplifying the equation (6) and on investigating the generality of the Assumption 2. Further research will also be devoted to extend this approach to the case of perturbed, as well as distributed systems.
Fig. 1. Trajectories of the real plant output variables $y_1$ (above) and $y_2$ (below). Grey solid line: real desired references ($\hat{y}_1, \hat{y}_2$); black dash-dotted line: artificial references ($\bar{y}_1, \bar{y}_2$); black solid line: outputs of the system ($y_1, y_2$).

Fig. 2. Trajectories of the inputs applied to the real system. Black solid line: first input; black dash-dotted line: second input.

APPENDIX A: PROOF OF PROPOSITION 2

Define $v_t = u_{t-1}$, $\chi_t = x_{t-1}$. Considering the control law (4), we write system (1) as follows

$$
\chi_{t+1} = x_t
$$

$$
x_{t+1} = Ax_t + B(v_t + K_\delta(x_t - \chi_t) + K_\epsilon(Cx_t - \bar{y}))
$$

$$
v_{t+1} = v_t + K_\delta(x_t - \chi_t) + K_\epsilon(Cx_t - \bar{y})
$$

(17)
Fig. 3. Trajectories of the output variables when nominal process is simulated $y_1$ (above) and $y_2$ (below). Grey solid line: real desired references ($\hat{y}_{1,2}$); black dash-dotted line: artificial references ($\bar{y}_{1,2}$); black solid line: outputs of the system ($y_{1,2}$).

Fig. 4. Trajectories of the inputs to the nominal system. Black solid line: first input; black dash-dotted line: second input.

given $\bar{y}$, $(\bar{x}, \bar{u})$ is the corresponding state/input steady state pair. Define $\Delta x_t = x_t - \bar{x}$, $\Delta x' = x'_t - \bar{\chi}$, $\Delta v_t = v_t - \bar{u}$, and recall that $Cy = \bar{y}$. From (17) we obtain

$$
\begin{bmatrix}
\Delta x'_{t+1} \\
\Delta v_{t+1} \\
\Delta v_{t+1}
\end{bmatrix} = F_{\Delta} \begin{bmatrix}
\Delta x' \\
\Delta v_t \\
\Delta v_t
\end{bmatrix}
$$

(18)
Note that
\[ \xi_t = \begin{bmatrix} x_t - x_{t-1} \\ y_t - \bar{y} \end{bmatrix} = C_\Delta \begin{bmatrix} \Delta x_t \\ \Delta v_t \end{bmatrix} \]  
(19)

We can write, from (18) and (19)
\[ \begin{bmatrix} \xi_t \\ \xi_{t+1} \\ \vdots \\ \xi_{t+2n+m-1} \end{bmatrix} = \Theta_\Delta \begin{bmatrix} \Delta x_t \\ \Delta v_t \end{bmatrix} \]  
(20)

In order to solve system (20) we need to show that a pseudo-inverse of \( \Theta_\Delta \) can be computed, i.e., that the pair \((F_\Delta, C_\Delta)\) is observable. To do so, we resort to the PBH-test, and prove that \( \tilde{\Theta}_\Delta(s) = \begin{bmatrix} sI_{n+m} - A_\Delta \\ C_\Delta \end{bmatrix} \)
has full column-rank \( 2n + m \) for all \( s \in \mathbb{C} \).

Assume that, for each value of \( s \in \mathbb{C} \), there exists a vector \( v = (v_1, v_2, v_3) \) such that \( \tilde{\Theta}_\Delta(s)v = 0 \). This implies that
\[ sv_1 = v_2 \]  
(21a)
\[ B(K_\delta v_1 - (K_\delta + K_e C)v_2 - v_3) + (sI_n - A)v_2 = 0 \]  
(21b)
\[ K_\delta v_1 - (K_\delta + K_e C)v_2 + (s - 1)v_3 = 0 \]  
(21c)
\[ v_1 = v_2 \]  
(21d)
\[ Cv_2 = 0 \]  
(21e)

First, using (21c) in (21b) we obtain that
\[ (sI_n - A)v_2 - sBv_3 = 0 \]  
(22)

Secondly, using (21d) in (21a) we obtain that
\[ (s - 1)v_1 = 0 \]  
(23)

Note that (23) can be verified either (I) if \( \lambda = 1 \), or (II) \( \lambda \neq 1 \) and \( v_1 = 0 \).

- In case (I), (22) and (21e) lead to
\[ S \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = 0 \]
from Assumption 1 \( S \) has full rank \( n + m \), and then \( v_2 = 0 \), and \( v_3 = 0 \). From (21d), also \( v_1 = 0 \).
- In case (II), from (21d) also \( v_2 = 0 \) and, from (22), \( v_3 = 0 \).

Therefore, both in case (I) and in case (II) the only solution, compatible with \( \tilde{\Theta}_\Delta(s)v = 0 \), is \( v = 0 \) for all \( s \in \mathbb{C} \). This proves that \( \tilde{\Theta}_\Delta(s) \) has full column-rank.

In view of (20) and the full column-rank of \( \Theta_\Delta \), and recalling that \( \xi_t \) evolves according to (5), we obtain that
\[ \begin{bmatrix} x_{t-1} - \bar{x} \\ x_t - \bar{x} \\ u_{t-1} - \bar{u} \end{bmatrix} = \Theta_\Delta^+ \begin{bmatrix} I_{n+m} \\ \bar{F} \\ \vdots \\ \bar{F}2n+m-1\xi_t \end{bmatrix} \]
from which (6) clearly follows.
APPENDIX B: PROOF OF THEOREM 1

The proof of Theorem 1 follows the same rationale as the proof of Theorem 1 in [8]. Specifically, it is based on the following steps:

I) we prove that the only closed-loop stable equilibrium point such that $\xi_t = 0$, compatible with the minimization problem (9), is the one corresponding to $y_t = \hat{y}$;

II) we prove recursive feasibility and that the system converges asymptotically to an equilibrium point, i.e., that $\xi_t \to 0$ as $t \to \infty$;

III) from the previous steps we infer that the system asymptotically converges to the unique equilibrium point, which is compatible with (9), i.e., $\tilde{y}_t \to \hat{y}$ as $t \to \infty$.

Step I

First note that, for all admissible points $(\xi_t, y_t) = (\delta x_t, y_t - \bar{y}_t, \tilde{y}_t)$, the point $(0, 0, \bar{y})$, where $\bar{y}$ fulfills (15), is the equilibrium condition to (3) (corresponding to $\bar{y}_t = \hat{y}$ and $\delta u_k = 0$ for all $k$) such that the cost function $V_N$ is globally minimized. Recall that it is admissible since $C_{y}\hat{y} \in X_e \times U_e$, see (8).

Consider a generic admissible reference $\bar{y} \neq \hat{y}$, such that $(\bar{x}, \bar{u}) = C_{y}\bar{y} \in X_e \times U_e$, corresponding to the equilibrium $(\delta x_t, y_t - \bar{y}_t, \tilde{y}_t) = (0, 0, \bar{y})$ for the system (7a) (i.e., corresponding to an equilibrium to (3) in case $\delta u_k = 0$ for all $k \geq 0$). The cost, associated to such equilibrium condition is

$$V_N(\bar{y}, 0; 0, \bar{y}, \bar{y}) = ||\bar{y} - \bar{y}||_T^2 = ||\bar{y} - \hat{y}||_T^2 + ||\hat{y} - \bar{y}||_T^2 + 2(\hat{y} - \bar{y})^T T(\bar{y} - \hat{y})$$

(24)

Instead, consider the solution corresponding to the trajectory starting from $(\delta x_t, y_t - \bar{y}_t, \tilde{y}_t) = (0, 0, \bar{y})$ (i.e., from the point $(\bar{x}, \bar{u})$, which does not correspond to an equilibrium point for (5), in view of the fact that the reference output is now $\bar{y} \neq \bar{y}$), and evolving according to the auxiliary control law (4). If $\bar{y}$ is defined according to $\bar{y} = \bar{\lambda} \bar{y} + (1 - \bar{\lambda}) \hat{y}$, $\bar{\lambda} \in [0, 1)$, then

a) the corresponding equilibrium is admissible, i.e., $C_{y}\bar{y} \in X_e \times U_e$, in view of the convexity of $X_e \times U_e$

b) since $C^* (0, 0, \bar{y}) \in X_e \times U_e \subset \text{Int}(X \times U)$, in view of continuity arguments, if $(1 - \bar{\lambda})$ (and hence $||\bar{y} - \bar{y}||$) is sufficiently small it also holds that $C^*(F^*)(0, \bar{y} - \bar{y}, \bar{y}) \in X \times U$ for all $t \geq 0$, and the initial condition $(0, \bar{y} - \bar{y}, \bar{y})$ for (7a) belongs to $\mathbb{O}$, and hence it is admissible.

The cost associated to such auxiliary control law is

$$\bar{V}_N = ||\bar{y} - \bar{y}||_T^2 + \sum_{k=0}^{N-1} \{||\bar{F}^k(0, \bar{y} - \bar{y})||_Q^2 + ||\bar{K}\bar{F}^k(0, \bar{y} - \bar{y})||_K^2\} + ||\bar{F}^N(0, \bar{y} - \bar{y})||_P^2$$

(25)

In view of (13), we obtain that the latter is equal to

$$\bar{V}_N = ||\bar{y} - \bar{y}||_T^2 + ||(0, 0, \bar{y} - \bar{y})||_T^2 = ||\bar{y} - \bar{y}||_T^2 + ||\bar{y} - \bar{y}||_{P_{yy}}^2$$

(26)

where

$$||\bar{y} - \bar{y}||_T^2 = ||\bar{y} - \bar{y}||_T^2 + ||\bar{y} - \bar{y}||_T^2 + 2(\bar{y} - \bar{y})^T T(\bar{y} - \bar{y})$$

We obtain, since $P_{yy} < T$, that

$$V_N - \bar{V}_N \geq (||\bar{y} - \bar{y}||_T^2 - ||\bar{y} - \bar{y}||_T^2 - ||\bar{y} - \bar{y}||_T^2 + 2(\bar{y} - \bar{y})^T T(\bar{y} - \bar{y})$$

(27)

Note that

$$a = ||\bar{y} - \bar{y}||_T = \bar{\lambda}||\bar{y} - \bar{y}||_T$$

$$b = ||\bar{y} - \bar{y}||_T = (1 - \bar{\lambda})||\bar{y} - \bar{y}||_T$$

$$c = a + b = ||\bar{y} - \bar{y}||_T$$

Since $a, b, c > 0$, if $a + b = c$ then

$$a^2 + b^2 < c^2$$

(28)
Furthermore, since $\tilde{y} - \hat{y} = (1 - \lambda)(\tilde{y} - \hat{y})$ and defining $f(y) = \|y - \tilde{y}\|_{T}^{2}$, we have that

$$2(\hat{y} - \tilde{y})^{T}T(\tilde{y} - \hat{y}) = (1 - \lambda) \frac{df}{dy} \bigg|_{y=\hat{y}} (\tilde{y} - \hat{y}) \geq 0$$  \hspace{1cm} (29)$$
in view of the optimality of $\hat{y}$ with respect to $f(y)$ in the admissible set. From (27), (28), and (29) it holds that

$$\tilde{V}_{N} < V_{N}(\tilde{y},0;0,\tilde{y},\hat{y})$$

from which it follows that, for all $\tilde{y} \neq \hat{y}$, $(0,0,\tilde{y},\hat{y})$ is not an equilibrium point for the closed loop system.

**Step II**  
Recursive feasibility and convergence of $\xi_{t}$ to zero can be proved resorting to standard MPC arguments.  
**Feasibility:** Consider that, at time $t$, a solution to (9) is $(\delta u|_{t},d_{t+1}^{N}|_{t},\xi_{t}|_{t})$. Then it is easy to see that a feasible solution to (9) at time $t+1$ is $(\delta u|_{t+1+N}|_{t+1},\xi_{t+1}|_{t+1})$, where it is defined $\delta u|_{t+1+N}|_{t} = K\xi_{t+1}|_{t}$, where $\xi_{t+1}|_{t}$ stems from (3) with inputs $\delta u|_{t+1+N}|_{t}$ and initial condition $\xi_{t}|_{t}$ (note that the initial condition to (3) at time $t+1$ is indeed equal to $\xi_{t+1}|_{t} = (\delta x_{t+1},y_{t+1} - \tilde{y}|_{t})$ = since $\tilde{y}_{t} = \tilde{y}|_{t}$ is a feasible reference also at time $t + 1$).

**Convergence:** Consider the above-mentioned feasible solution at time $t + 1$. Following standard arguments for the proof of the convergence of MPC [12] we obtain that

$$V_{N}(\delta x_{t+1},y_{t+1},\hat{y}) - V_{N}(\delta x_{t},y_{t},\hat{y}) \leq - \|\xi_{t}\|_{Q}^{2}$$

In view of the positive definiteness of $V_{N}$ and of matrix $Q$, it holds that $\xi_{t} \to 0$ as $t \to \infty$.

**Step III**  
In view of Step II, convergence of $\xi_{t}$ is guaranteed. Since, in view of Step I, the only equilibrium to (3) compatible with (9) is the one corresponding to $y_{t} = \hat{y}$, then it also holds that $y_{t} \to \hat{y}$ as $t \to \infty$.

**REFERENCES**


