1 General formulas

1.1 The Maximum Variance Asymptotic upper bound (MVA)

First, write the $\alpha(t)$ function:

$$
\alpha(t) = - \frac{E(B(t)) - E(S(t + d))}{\sqrt{\text{var}(B(t)) + \text{var}(S(t + d))}}.
$$

(1)

$B(t)$ is the envelope of the relevant traffic flow $X(t)$. If an envelope is not available, use $X(t)$ directly. $S(t)$ is the service envelope of $X(t)$. $d$ is the delay threshold.

Second, find the absolute minimum of $\alpha(t)$, referred to as $\alpha_{\text{min}}$.

Third:

$$
P(D > d) = e^{-\frac{\alpha_{\text{min}}^2}{2}}.
$$

(2)

1.2 Service envelopes of relevant schedulers

For all schedulers, $C$ is the capacity of the output line.

1.2.1 First In First Out (FIFO) scheduler

$$
S(t) = Ct.
$$

(3)

The two moments characterization of this service envelope is:

$$
\begin{align*}
E(S(t)) &= Ct \\
\text{var}(S(t)) &= 0
\end{align*}
$$
1.2.2 Strict Priority (SP) scheduler

\[ S_i(t) = \max \left( 0, Ct - \sum_{j=1}^{i-1} B_j(t) \right). \] \hspace{1cm} (4)

An approximate two moments characterization of this service envelope is:

\[ E(S_i(t)) = Ct - \sum_{j=1}^{i-1} E(B_j(t)) \]
\[ \text{var}(S_i(t)) = \sum_{j=1}^{i-1} \text{var}(B_j(t)) \]

1.2.3 Earliest Deadline First (EDF) scheduler

The \( i \)th service class has a delay threshold equal to \( \delta_i \).

\[ S_i(t) = \max \left( 0, Ct - \sum_{j \neq i} B_j(t - \max (0, \delta_j - \delta_i)) \right). \] \hspace{1cm} (5)

An approximate two moments characterization of this service envelope is:

\[ E(S_i(t)) = Ct - \sum_{j \neq i} E(B_j(t - \max (0, \delta_j - \delta_i))) \]
\[ \text{var}(S_i(t)) = \sum_{j \neq i} \text{var}(B_j(t - \max (0, \delta_j - \delta_i))) \]

1.2.4 General Processor Sharing (GPS) scheduler

The \( i \)th service class has a weight \( w_i \).

\[ S_i(t) = w_i Ct + \sum_{j \neq i} \frac{w_i}{\sum_{k \neq j} w_k} \max (w_j Ct - B_j(t)). \] \hspace{1cm} (6)

An approximate two moments characterization of this service envelope is:

\[ E(S_i(t)) = w_i Ct + \sum_{j \neq i} \frac{w_i}{\sum_{k \neq j} w_k} (w_j Ct - E(B_j(t))) \]
\[ \text{var}(S_i(t)) = \sum_{j \neq i} \left( \frac{w_i}{\sum_{k \neq j} w_k} \right)^2 \text{var}(B_j(t)) \]

2 Closed form analysis

2.1 FIFO scheduler with linear-variance traffic

\( n \) individual independent traffic streams. Each traffic stream has the second-moment characterization:

\[ E(X(t)) = rt \]
\[ \text{var}(X(t)) = rbt \]

2.1.1 Delay threshold violation probability

\[ P(D > d) = e^{-2 \frac{C + \text{var}}{r} \cdot d}. \] \hspace{1cm} (7)
2.1.2 Average delay

\[ E(D) = \frac{nr b}{2C(C - nr)}. \] (8)

2.1.3 Admission control

\[ n \leq \frac{2C^2d}{2Cdr - b \ln p}. \] (9)

2.1.4 Resource provisioning

\[ C \geq \frac{nr}{2} + \sqrt{\left(\frac{nr}{2}\right)^2 - \frac{nb \ln p}{2d}}. \] (10)

2.2 FIFO scheduler with fractional Gaussian traffic (fGt)

One fGt traffic stream with the second-moment characterization:

\[ E(X(t)) = rt \]
\[ \text{var}(x(t)) = art^{2H}. \]

2.2.1 Delay threshold violation probability

\[ P(D > d) = e^{-\frac{1}{2H} \left[ \frac{1}{nr} \left( \frac{1}{nr} \right)^2 - \frac{1}{2H} (C - r)^{2H} \right] d^2}. \] (11)

2.3 SP scheduler with linear-variance traffic

For class \( i \), \( n_i \) individual independent traffic streams. Each traffic stream in class \( i \) has the second-moment characterization:

\[ E(X(t)) = r_i t \]
\[ \text{var}(x(t)) = r_i b_i t. \]

Define:

\[ A_i = \sum_{j=1}^{i} n_j r_j \quad \text{with} \quad i \geq 1, \]
\[ A_0 = 0 \quad \text{if} \quad i = 0; \]
\[ B_i = \sum_{j=1}^{i} n_j r_j b_j \quad \text{with} \quad i \geq 1, \]
\[ B_0 = 0 \quad \text{if} \quad i = 0. \]

2.3.1 Delay threshold violation probability

\[ P(D_i > d_i) = e^{-\frac{C - A_i}{B_i} (C - A_{i-1}) B_i - (C - A_i) B_{i-1} |d_i}. \] (12)
2.3.2 Admission control

Define:

\[ n_{i-k}^* = \frac{(Cd_i - b_i \ln p_i - d_i A_{i-1-k})}{r_{i-k}d_i} + \sqrt{-\ln p_i \left(2d_i B_{i-1-k} + 2d_i b_{i-k}(C - A_{i-1-k}) - b_{i-k}^2 \ln p_i\right)} \]

and define

\[ n_{i-k,\text{max}} = \frac{2d_i r_{i-k} C}{r_{i-k} d_i} \]

If \( \forall k < i : 0 \leq n_{i-k} \leq \min\left(n_{i-k}^*, n_{i-k,\text{max}}\right) \), then

\[ n_i \leq \left(\frac{B_{i-1}}{b_i}\right)^2 - \left(2C - 2A_{i-1} + \frac{B_{i-1}}{b_i}\right)^2 - 4B_{i-1} \frac{\ln p_i}{d_i} + \]

\[ -4r_i \left[2 \left(C - A_{i-1} + \frac{B_{i-1}}{b_i}\right) - \frac{b_i \ln p_i}{d_i}\right] \]

\[ -4r_i \left[2 \left(C - A_{i-1} + \frac{B_{i-1}}{b_i}\right) - \frac{b_i \ln p_i}{d_i}\right], \quad (13) \]

otherwise

\[ n_i = 0. \quad (14) \]

Particular case for two priorities

\[ n_1^* = \frac{(Cd_2 - b_1 \ln p_2) - \sqrt{-\ln p_2 (2d_2 b_1 (C - b_1^2 \ln p_2))}}{r_1 d_2} \]

and define

\[ n_{1,\text{max}} = \frac{2d_1 C^2}{2d_1 r_1 C - r_1 b_1 \ln p_1}. \]

If \( n_1 \leq \min\left(n_1^*, n_{1,\text{max}}\right) \), then

\[ n_2 \leq \left(\frac{B_{1}}{b_2}\right)^2 - \left(2C - 2A_1 + \frac{B_{1}}{b_2}\right)^2 - 4B_1 \frac{\ln p_2}{d_2} + \]

\[ -4r_2 \left[2 \left(C - A_1 + \frac{B_{1}}{b_2}\right) - \frac{b_2 \ln p_2}{d_2}\right] \]

\[ -4r_2 \left[2 \left(C - A_1 + \frac{B_{1}}{b_2}\right) - \frac{b_2 \ln p_2}{d_2}\right], \quad (15) \]

otherwise

\[ n_2 = 0. \quad (16) \]
2.3.3 Resource provisioning

\[ C_i \geq \frac{(2A_iB_{i-1} - A_{i-1}B_i - A_iB_i)d_i + \sqrt{A_i^2B_i^2d_i^2 - 2A_iB_i^2d_i^2 + A_i^2B_i^2d_i^2 + 2B_{i-1}B_i^2d_i \ln p_i - 2B_i^3d_i \ln p_i}}{2(B_{i-1} - B_i)d_i} \]  \tag{17}

Finally:

\[ C = \max_i C_i. \]  \tag{18}

2.4 EDF scheduler with linear-variance traffic

\[ A_i = \sum_{n \neq i} n_i r_{n} \]

\[ B_i = \sum_{n \neq i} n_i r_{n} b_{n} \]

\[ E_i = \sum_{n \neq i} n_i r_{n} \max(0, \delta_n - \delta_i) \]

\[ F_i = \sum_{n \neq i} n_i r_{n} b_{n} \max(0, \delta_n - \delta_i) \]

2.4.1 Delay threshold violation probability

\[ Pr(D_i > d_i) = \exp \left( -2(n_i r_i d_i (C - A_i) + E_i(B_i + n_i r_i b_i) + F_i(C - n_i r_i - A_i)) \frac{C - n_i r_i - A_i}{(B_i + n_i r_i b_i)^2} \right) \]  \tag{19}

2.4.2 Capacity planning

\[ C_i \geq \frac{A_i d_i - E_i + \sqrt{(E_i + n_i r_i d_i)^2 - 2 \ln p_i(F_i + n_i r_i b_i d_i)}}{2(F_i + n_i r_i b_i d_i)} (B_i + n_i r_i b_i) + \frac{A_i + n_i r_i}{2} \]  \tag{20}

Finally:

\[ C = \max_i C_i. \]  \tag{21}
2.4.3 Admission control

\[
\begin{align*}
n_i & \leq \frac{1}{r_i(2(b_i(Cd_i - A_i d_i + E_i) + B_i d_i - F_i) - b_i^2 \ln p_i)} \\
& \left( (A_ib_i + B_i - Cb_i)(A_i d_i - E_i - C d_i) + \\
& -2(B_i d_i - F_i)(A_i - C) + b_i B_i \ln p_i + \\
& + \sqrt{(-A_ib_i + B_i + Cb_i)^2((A_i d_i - E_i - C d_i)^2 + 2(B_i d_i - F_i) \ln p_i)} \right)
\end{align*}
\]

(22)

2.5 GPS scheduler with linear-variance traffic

\[
A_i = \sum_{j \neq i} \frac{w_i}{\sum_{k \neq j} w_k} (w_j C - n_j r_j)
\]

\[
B_i = \sum_{j \neq i} \left( \frac{w_i}{\sum_{k \neq j} w_k} \right)^2 n_j r_j b_j
\]

2.5.1 Delay threshold violation probability

\[
Pr(D_i > d_i) = \exp \left( -\frac{2n_i r_i (w_i b_i C + b_i A_i + B_i)(w_i C + A_i - N_i r_i) d_i}{(n_i r_i b_i + B_i)^2} \right)
\]

(23)

2.5.2 Admission control

\[
n_{i, \text{min}} = \frac{2d_i(w_i C)^2}{r_i(2d_i w_i C - b_i \ln p_i)}
\]

\[
n_{i, \text{max}} = \frac{d_i(w_i C + A_i)(w_i b_i C + b_i A_i + B_i) + b_i B_i \ln p_i}{r_i(2d_i(w_i b_i C + b_i A_i + B_i) - b_i^2 \ln p_i)} + \\
+ \frac{\sqrt{d_i(w_i b_i C + b_i A_i + B_i)^2(d_i(w_i C + A_i)^2 + 2B_i \ln p_i)}}{r_i(2d_i(w_i b_i C + b_i A_i + B_i) - b_i^2 \ln p_i)}
\]

if \(d_i((w_i C + A_i)^2 + 2B_i \ln p_i) > 0\) \(n_i \leq \max(n_{i, \text{min}}, n_{i, \text{max}})\)

otherwise \(n_i \leq n_{i, \text{min}}\)

3 Bounded variance network calculus

Given a scheduler where a reference traffic flow \(X(t)\) has service envelope \(S(t)\), the variance of the output traffic of the scheduler has the following envelope

\[
var(X_{\text{out}}(t)) = \max(var(X(t)), var(S(t))).
\]

(24)
4 Analytical calculation of end-to-end delay

The end to end delay of a traffic stream traversing a sequence of $H$ schedulers, is given by

$$d_{e2e} = d_1 + d_2 + \cdots + d_H,$$

(25)

where $d_i$ is the delay at scheduler $i$. Therefore, the probability density of end to end delay is

$$f_{d_{e2e}}(t) = f_{d_1}(t) * f_{d_2}(t) * \cdots * f_{d_H}(t),$$

(26)

where $*$ means convolution. If traffic flows have a linear variance envelope, we can write

$$f_{d_i}(t) = k_i e^{-k_i t},$$

(27)

thus, the Laplace transform of $f_{d_i}(t)$ is

$$F_{d_i}(s) = \frac{k_i}{k_i + s}.$$  

(28)

The Laplace transform of $f_{d_{e2e}}(t)$ is

$$F_{d_{e2e}}(s) = \prod_{i=1}^{H} \frac{k_i}{k_i + s},$$

(29)

thus,

$$f_{d_{e2e}}(t) = \mathcal{L}^{-1} \left\{ \prod_{i=1}^{H} \frac{k_i}{k_i + s} \right\},$$

(30)

which can be calculated with the partial fraction expansion method.

4.1 Case of $k_i \neq k_j$ for $i \neq j$

We assume that $k_i \neq k_j$ for $i \neq j$. In this case:

$$\mathcal{L}^{-1} \left\{ \prod_{i=1}^{H} \frac{k_i}{k_i + s} \right\} = \sum_{i=1}^{H} G_i e^{-k_i t},$$

(31)

where

$$G_i = \left\{ \prod_{j=1}^{H} \frac{k_j}{k_j + s} \right\}_{s=-k_i}^{(s + k_i)}.$$

(32)
4.1.1 Case of \( H=2 \)

In this case, we have

\[
G_1 = \frac{k_1 k_2}{-k_1 + k_2}
\]

and

\[
G_2 = \frac{k_1 k_2}{k_1 - k_2},
\]

thus:

\[
f_{d_{2e}}(t) = \frac{k_1 k_2}{-k_1 + k_2} \left( e^{-k_1 t} - e^{-k_2 t} \right).
\]

4.2 Case of \( k_i = k_j = \kappa \) for \( i \neq j \)

In this case:

\[
f_{d_{2e}}(t) = \frac{\kappa H t^{H-1}}{(H - 1)!} e^{-\kappa t}.
\]

5 Variance envelope of relevant traffic sources

5.1 On-Off source

The On-Off source has two states, On and Off. In the On state it transmits at \( P \) (bit/s) and in the Off state it does not transmit. The rate of transition from the On to the Off state is \( \mu (s^{-1}) \) and the inverse rate is \( \lambda (s^{-1}) \). The average value of the cumulative traffic generated by the source is:

\[
E(X(t)) = \frac{\lambda}{\lambda + \mu} P t.
\]

The variance of the cumulative traffic is:

\[
\text{var}(X(t)) = 2 \frac{\lambda \mu}{(\lambda + \mu)^3} P^2 t - 2 \frac{\lambda \mu}{(\lambda + \mu)^3} P^2 \left( 1 - e^{-(\lambda + \mu)t} \right).
\]

This source admits a simple linear variance envelope:

\[
\text{var}(X(t)) \leq 2 \frac{\lambda \mu}{(\lambda + \mu)^3} P^2 t.
\]

References