The Optimal Reclamation of Eutrophic Water Bodies

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ABSTRACT

The reclamation of a eutrophic water body is set as an optimal control problem. The aim is to take account of some basic biological and economic features which were disregarded in previous analyses of the problem. To this end, an optimization model is developed where phytoplankton and nutrients are the state variables, the efficiency of the wastewater treatment system is the control variable, and minimizing the discounted stream of both environmental and treatment cost is the objective. Through a singular perturbation argument a reduced order dynamic model is obtained, to which Pontryagin’s principle is applied. The optimal solution is such that (1) an optimal equilibrium for phytoplankton is achieved in the long run, and (2) both phytoplankton and the efficiency of the treatment system strictly decrease with time.

1. INTRODUCTION

The reclamation of eutrophic water bodies (lakes, seas, and artificial reservoirs) is a problem of major concern, facing several developed economies where large amounts of pollutants are discharged as a by-product of consumption and production. As a result, there are a great number of water bodies in which the concentrations of phosphorus, nitrogen, and other chemicals (the so-called nutrients) are so high as to maintain a high biomass of microscopic algae (phytoplankton). Any water body experiencing this phenomenon is called eutrophic and generally suffers from a loss of transparency, high dissolved oxygen deficit, and the death of zooplankton and fish. The ensuing environmental damage \( P \) is hardly measurable; nonetheless, one can safely expect that \( P \) increases with the concentration \( x \) of algae, possibly in a faster than a linear way, i.e., \( P(x), P'_x(x), P''_x(x) > 0 \) for every \( x > 0 \).

As made clear above, there is a dynamic link between the concentrations \( x \) of algae and \( n \) of nutrients, the latter being affected by the nutrient discharge of the influents to the eutrophic water body. It is therefore

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apparent that for, e.g., a eutrophic lake to recover, the nutrient discharge
must be reduced. This is usually accomplished by installing a wastewater
treatment system in the upstream catchment, thus improving the quality of
influents. The portion $u$ of nutrients removed by the treatment system is
called the efficiency and is a key variable in our analysis, as it determines the
treatment cost $R(u)$ of a given (constant) nutrient load. We assume
$R(u), R_u(u), R_{uu}(u) > 0$ for every $u > 0$ and, since hundred percent removal
is technically infeasible,

$$\lim_{u \to 1} R(u) = +\infty$$

(for more details on this subject, see [1]).

In the framework outlined above, the optimal reclamation problem is that
of finding the best tradeoff between environmental cost $P(x)$ and treatment
cost $R(u)$. The issue was tackled by Mosetti [2], who concluded that
treatment must be as strong as possible so as to reach some optimal
equilibrium $(u^*, x^*)$ in minimum time. Although this result is indeed appealing,
Mosetti’s analytical framework is open to some criticisms. First, the
phytoplankton dynamics is not suitably modeled, since no account is taken of
either the nutrient dynamics or the well-known factors limiting the algal
growth. Secondly, treatment is tacitly supposed to affect the nutrient concen-
tration of the water body rather than that of its influents, which is technically
impossible. Finally, the cost of treatment is linearly related to efficiency, so
that complete removal of nutrients at a finite cost is allowed, which makes no
sense, as pointed out in the preceding paragraph. The first flaw was noticed
by Garnaev [3], who solved the same problem under the sounder assumption
that there is an upper bound to phytoplankton growth.

In this paper, the whole problem is carefully restated by the use of a
second order model, which is illustrated in Section 2. Nutrient $n$ and algae $x$
serve there as state variables, and algal growth is sensitive to light intensity
(hence to $x$ itself, as a consequence of the shading effect exerted by upper on
lower layers of algae), so that phytoplankton growth is limited. Moreover, the
efficiency $u$ of the wastewater treatment system is taken as a control
variable, while minimizing the discounted flow of $P(x) + R(u)$ over an
infinite time horizon is assumed to be the objective. In Section 3, the
phytoplankton time response is shown to be much faster than the nutrient
one, so that a first order model is obtained by means of a singular perturba-
tion argument. The resulting equation is linear and represents the slow motion of phytoplankton. The attendant optimal control problem is solved in
Section 4 by the use of Pontryagin’s principle. The sensitivity of the optimal
solution with respect to the nutrient discharge and the discount rate is
examined in the last section, where some final remarks are also given.
2. PROBLEM STATEMENT

We assume, as implicitly done in [2] and [3], that the water body is perfectly mixed, the nutrient concentration $N$ in the inflow is constant, and inflow and outflow rates $Q$ are equal and constant, thus implying a constant storage $V$. Then, the mass conservation of phytoplankton and nutrient is represented by the ordinary differential equations below:

\[
\frac{dn(t)}{dt} = -an(t) - \frac{Q}{V} n(t) + \frac{QN}{V} [1 - u(t)] - F(x(t), n(t)),
\]

\[
\frac{dx(t)}{dt} = -dx(t) - \frac{Q}{V} x(t) + eF(x(t), n(t)).
\]

where $an$ is the sedimentation rate of nutrient, $dx$ is the death rate of phytoplankton, $(Q/V)n$ and $(Q/V)x$ are the loss rates of nutrient and phytoplankton due to the outflow, $F(x, n)$ is the rate of nutrient uptake by algae, and $e$ is a conversion factor specifying the production of algal biomass per unit of uptake nutrient. Uptake can be assumed to be proportional to $xn$ through a factor which is a Monod function of the average light intensity $L$ in the water column, i.e.

\[
F(x, n) = \frac{bl}{c+L} mn,
\]

where $c$ is the half-saturation constant, namely the light intensity at which the nutrient uptake is half maximum ($F = bxn/2$). We do not assume a similar Monod function for nutrient, because, as suggested by Liebig’s law (see, e.g., [4]), we consider only the nutrient limiting algal growth (usually phosphorus), which is present in concentrations far from the saturation range for the Monod function.

The light intensity in the water body obviously decreases with increasing depth, this being strongly influenced by suspended matter like phytoplankton. A standard assumption for the average light intensity hitting the phytoplankton layer is

\[
L = \frac{fL_0}{f + x},
\]

where $L_0$ is the average light intensity in a clear water column and $f$ is the
half-saturation constant, i.e., the concentration of phytoplankton at which light intensity is half maximum \((L = L_0/2)\).

In view of \((3)\) and \((4)\), the model \((1), (2)\) can be rewritten as

\[
\frac{dn(t)}{dt} = pN[1 - u(t)] - \left( q + \frac{rx(t)}{s + x(t)} \right)n(t),
\]

\[
\frac{dx(t)}{dt} = \left( -m + \frac{ern(t)}{s + x(t)} \right)x(t),
\]

where

\[ m = Q/V + d, \]
\[ p = Q/V, \]
\[ q = Q/V + a, \]
\[ r = bfL_0/c, \]
\[ s = f + fL_0/c. \]

Then, the problem of the optimal reclamation of eutrophic water bodies can be formulated as follows: Given the system \((5), (6)\) and initial conditions \(n(0), x(0)\), minimize the present value of treatment and environmental cost, i.e.,

\[
\min \int_0^\infty \left[ P(x(t)) + R(u(t)) \right]e^{-\rho t}dt,
\]

where \(\rho\) is the discount rate and \(P\) and \(R\) are positive functions with

\[ P_x > 0, \quad P_{xx} > 0, \quad R_u > 0, \quad R_{uu} > 0, \]

and

\[
\lim_{u \to 1} R(u) = +\infty.
\]

Obviously, the problem is of interest only if \(x(0)\) is high enough. In particular, if \(x(0)\) is the equilibrium concentration of phytoplankton in the
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absence of treatment \((u = 0)\), then

\[
x(0) = \frac{eprN - mqs}{mq + mr}
\]

and the problem makes sense if the untreated nutrient load \(N\) is high.

3. A REDUCED ORDER MODEL

The relevant hydrologic and biological features of a eutrophic water body are qualified by the parameters \(e, m, p, q, r, s\) of the model (5), (6), while the intertemporal preferences of the optimizing authority are summed up by \(\rho\) in (7). Although those parameters may take remarkably different values from case to case, some standard ranges for them can be found in the literature (see, for instance, [5]–[7]). In particular, the reciprocals of \(m, n, p, q, r,\) and \(\rho\), all of which are measured in time units, generally satisfy the inequalities below:

\[
\frac{1}{m} < \frac{1}{q} < \frac{1}{r} < \frac{1}{p} < \frac{1}{\rho}.
\]

(10)

Usual values are a few days for \(1/m\), a few weeks for \(1/r\), a few months for both \(1/q\) and \(1/p\), and some years for \(1/\rho\). This large spectrum of time constants suggests that the model (5), (6) may undergo some simplification. Indeed, a direct comparison between the time constant \(1/q\) of nutrient \(\text{i.e. of Equation (5) with } x = 0\) and the time constant \(1/m\) of phytoplankton \(\text{i.e. if Equation (6) with } n = 0\) shows that (5), (6) is a slow-fast system, with nutrient as the slow component and phytoplankton as the fast one. Thus, the slow motion in (5), (6) can be described by a first order model, which may then replace (5) and (6) in the optimal control problem (5)–(7), since the discount rate \(\rho\) is very small, as stated in (10). In other words, it is the slow variations of both phytoplankton and nutrient rather than the fast transients of phytoplankton that influence the objective function (7). Moreover, since only phytoplankton enters (7) through \(P(x)\), \(x\) is to be chosen as the state variable in the reduced order model.

It is worthwhile recalling (see, for example, [8] and [9]) that, to derive a reduced order model, one must first compute the stable equilibrium manifold of the fast system. In our case, this amounts to setting \(dx/dt = 0\) in (6), thus
obtaining the two solutions

\[ x' = 0, \quad (11a) \]

\[ x^* = \frac{en}{m} - s. \quad (11b) \]

By considering the sign of the right hand side of (6), one can readily conclude that

\[ x = x' \text{ is a stable equilibrium of the fast system} \]

for \( n < ms/er \), \quad (12a)

\[ x = x^* \text{ is a stable equilibrium of the fast system} \]

for \( n > ms/er \). \quad (12b)

To determine the dynamics of the system on these stable manifolds, one has to substitute (11) into (5) and take (12) into account. This yields

\[ \frac{dn(t)}{dt} = pN[1 - u(t)] \quad \text{for} \quad n(t) < \frac{ms}{er}, \quad (13) \]

\[ \frac{dn(t)}{dt} = \frac{ms}{e} - pN[1 - u(t)] - (q + r)n(t) \quad \text{for} \quad n(t) > \frac{ms}{er}. \quad (14) \]

Equation (13) is of no interest for the reclamation of eutrophic water bodies, because it solely allows for an endemic presence of phytoplankton. Conversely, (14) refers to the case of a high nutrient concentration. As explained above, it is convenient to restate (14) in terms of \( x \). To do so, substitute (11b) into (14), generating

\[ \frac{dx(t)}{dt} = -Ax(t) + B - Cu(t), \quad (15) \]

where

\[ A = q + r, \]

\[ B = Nepr/m - qs, \]

\[ C = Nepr/m. \]
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In this manner, we are able to restate the optimal control problem so that (7) is still the objective function while (15) replaces (5) and (6) as a dynamic constraint. Thus, the distinction between the slow motion of nutrient and the fast dynamics of phytoplankton has ultimately brought about a significant advantage.

4. OPTIMAL SOLUTION

The problem (7), (15) is solved by the use of Pontryagin’s maximum principle. For simplicity in notation, time dependence is omitted in the sequel. The current value Hamiltonian is

\[ H = P(x) + R(u) + \lambda(-Ax + B - Cu). \]  

(16)

Therefore, the stationarity condition \( H_u = 0 \) yields

\[ R_u(u) - \lambda C = 0, \]  

(17)

and the minimality condition is always met, since \( H_{uu} = R_{uu}(u) > 0 \). Differentiating (17) with respect to time generates

\[ \frac{du}{dt} = \frac{(\rho + A)R_u(u)}{R_{uu}(u)} - \frac{CP_x(x)}{R_{uu}(u)} \]  

(18)

on substituting from \( d\lambda/dt = -H_\lambda \) and (16). In addition, the transversality condition on \( \lambda \) may be stated as follows:

\[ \lim_{t \to +\infty} e^{-\rho t} \frac{R_u(u)}{C} = 0. \]  

(19)

Equations (15) and (18) form a second order nonlinear system in the \((x, u)\) plane. The assumptions (8) and the meaning of both \( x \) and \( u \) imply that when dealing with that system, attention can be restricted to the region where \( 0 \leq x \) and \( 0 \leq u < 1 \). The \( dx/dt = 0 \) isocline is defined by

\[ u = \frac{B}{C} - \frac{A}{C^x}. \]  

(20)
while the $\frac{du}{dt} = 0$ isocline is given by

$$ (\rho + A) R_s(u) - CP_s(x) = 0. $$

In the $(x, u)$ plane, (20) is a straight and downward-sloping line, whereas (21) is an upward-sloping curve, since both $P_s(x)$ and $R_s(u)$ are increasing in their arguments. As a consequence, there is a unique intersection between (20) and (21), hereafter denoted as $(x^*, u^*)$. It can be readily checked that $(x^*, u^*)$ is a saddle point.

It is also apparent that the optimal trajectory lies on the stable separatrix of the saddle point $(x^*, u^*)$, since no other trajectory of (15), (18) complies with both the transversality condition (23) and the abovementioned constraints $0 \leq x$ and $0 \leq u < 1$. The phase portrait of the system (15), (18) and the optimal trajectory are depicted in Figure 1 for a particular numerical example.

5. DISCUSSION AND CONCLUSIONS

The results obtained in Section 4 are rich in implications, as shown below. First, observe that the optimal solution was given the very suitable form of a
feedback policy, i.e.

\[ u^{\text{opt}}(t) = \varphi(x^{\text{opt}}(t)). \]

Since \( \varphi \) is strictly increasing with \( x \) (see Figure 1), if the water body is initially eutrophic, i.e. if \( x(0) \) is very high, the phytoplankton concentration will decrease from \( x(0) \) to \( x^* \) in accordance with the equation below [see (15)]:

\[
\frac{dx^{\text{opt}}(t)}{dt} = -Ax^{\text{opt}}(t) + B - C\varphi(x^{\text{opt}}(t)).
\]

At the same time, the efficiency of the wastewater treatment system will decrease as well from \( u^{\text{opt}}(0) = \varphi(x^{\text{opt}}(0)) \) to \( u^* \). In other words, the wastewater treatment system must be entirely exploited at the beginning, and the efficiency must be reduced during the reclamation period. Although this is roughly the same as in [2] and [3], our analysis points out that convergence to \((x^*, u^*)\) does not occur in minimum time, as was the case in [2] and [3], because in our setting the treatment cost turns infinite when \( u = 1 \).

The sensitivity of the optimal equilibrium to parameter variations can be easily analyzed. In particular, changes in both \( N \) (and hence in the nutrient load \( QN \)) and \( \rho \) (the discount rate) will be considered. In the first case, it may help to recall that the isocline \( du/dt = 0 \) of the system (15),(18) is given by

\[
(\rho + A)R_u(u) - CP(x) = 0, \tag{21}
\]

where \( \rho, A, \) and \( P \) are independent of \( N \). If we further assume that the treatment cost \( R(u) \) takes the form

\[ R(u) = QNR^*(u), \]

then (21) becomes

\[
Q(\rho + A)R_u^*(u) - \frac{e^{pr}}{m}P(x) = 0.
\]

and we can conclude that isocline \( du/dt = 0 \) is not affected by \( N \). Con-
versely, the isocline $dx/dt = 0$, which can be rewritten as

$$-(q + r)x + \left(\frac{N_{epr}}{m} - qs\right) - \frac{N_{epr}}{m} u = 0,$$

moves upwards when $N$ increases. Therefore, both $x^*$ and $u^*$ increase with the nutrient load, the same being true for both $P(x^*)$ and $R(u^*)$. Furthermore, (11b) entails that the equilibrium concentration $n^*$ of the nutrient also rises.

As for the influence of discounting on the optimal solution, it is worth remarking that the discount rate usually plays a prominent role in bioeconomic problems, with special reference to the management of renewable resources (see, e.g., [10]). However, this does not seem to hold in our case, since the optimal solution is weakly influenced by $\rho$. As a matter of fact, $\rho$ is negligible in the expression

$$\rho + A = \rho + q + r$$

which appears in Equation (18) (recall that $1/\rho$ is of the order of some years, while $1/q$ and $1/r$ are of the order of a few months and weeks respectively).

As a final remark, it is worthwhile to stress that our analysis is still very naive. First of all, the phytoplankton-nutrient model is a simplified paradigm of the complex dynamic phenomena taking place in any eutrophic water body. Second, we make use of an invariant model, thus assuming away important cyclic phenomena like algal blooms. Third, the efficiency of a complex wastewater treatment system is in practice related to the number of treatment plants in operation. Consequently, sudden and sharp decreases in efficiency are possible, whereas such increases are not. As a matter of fact, to increase efficiency, some plants have to be disconnected, whereas to increase efficiency, new plants have to be installed, which takes quite a long time. Overcoming the three criticisms above would require a more complex ecological model with periodically varying parameters and a model of investment in wastewater treatment plants. This is a demanding task indeed, which is left to future research.

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