Synthesis of passband filters with asymmetric transmission zeros

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Passband filters with asymmetric zeros

- Placing asymmetric zeros respect the centre of the passband \((f_0)\) produce a response which is no more geometrically symmetric around \(f_0\).
- An asymmetric response allows a more flexible assignment of selectivity requirements, allowing at the same time to reduce the overall filter order.
- The synthesis techniques based on the lowpass - bandpass classical transformation cannot be directly employed (they implies a geometric symmetry in the bandpass domain).
Extension of the circuit components class

- In addition to capacitors, inductors, resistors and inverters, a new component is now introduced:

The frequency-invariant reactance (FIR) / susceptance (FIB):
\[ Z = jX, \quad Y = jB \]

- Circuits including this new component present network functions with more general properties. In particular, the response around zero frequency can be asymmetric.

- The synthesis of a lowpass prototype with FIR (FIB) components allows to obtain an asymmetric response (around \( f_0 \)) in the passband domain (after application of the classical lowpass - bandpass frequency transformation).
Are FIR significant from a physical point of view?

- Strictly speaking FIR are not physically realizable, so synthesized networks containing FIR are not meaningful.
- This is especially true around the zero frequency, where a FIR can not be even approximated with real component (concentrated or distributed).
- In the bandpass domain however, it is possible to obtain a reactance (susceptance) which does not present a relevant variation in a small range of frequencies.
- So a synthesized bandpass network containing FIR is significant from a practical point of view because it can be approximated with real components.
Strategic Plan for Positive and Positive-real functions

- Impedances (admittances) of networks with FIR (FIB) components are positive function in the complex frequency variable $s$ (not positive-real as in case of R,L,C networks).

- A rational function $f(s)$ in $s$ is a positive function if $\text{Re}\{f(s)\} \geq 0$ for $\text{Re}\{s\} \geq 0$. (It is a positive-real function if $f(s)$ is real for $s$ real)

- The main differences between the two class of functions are:
  - The coefficient of polynomials at numerator and denominator of a positive function are complex (real for positive-real functions)
  - The roots of such polynomials occur in complex conjugate pairs for positive-real function (not for positive function)
Characteristic polynomials for positive networks

Assuming to be in the normalized domain $s$, the characteristic polynomials define the scattering parameters of a lossless 2-port network:

$$S_{11}(s) = \frac{F(s)/\varepsilon_R}{E(s)}, \quad S_{21} = \frac{P(s)/\varepsilon}{E(s)}, \quad S_{22}(s) = \frac{F_2(s)/\varepsilon_R}{E(s)}$$

All polynomials are assumed monic. The coefficients $\varepsilon$ and $\varepsilon_R$ are real number which are related each other.

Conditions to be verified for positivity:

- Coefficients of $E$ and $F$ in general complex (same degree $n$)
- Roots of $E$ have negative real part (*Hurwitz* polynomial)
- Degree of $P(s) \leq n$
Unitary of $S$ matrix (Lossless condition)

$$S \cdot \tilde{S}^* = U \quad \Rightarrow \quad S_{11}(s)S_{11}(s)^* + S_{21}(s)S_{21}(s)^* = 1$$

$$S_{22}(s)S_{22}(s)^* + S_{12}(s)S_{12}(s)^* = 1$$

$$S_{11}(s)S_{12}(s)^* + S_{21}(s)S_{22}(s)^* = 0$$

Paraconjugation ($s=j\omega$): $Q(s)^*=Q^*(s^*)=Q^*(-s)$

$Q(s) = q_0 + q_1s + q_2s^2 + \ldots + q_ns^n$

$$\Rightarrow Q(s)^* = Q^*(-s) = q_0^* - q_1^*s + q_2^*s^2 - \ldots + q_n^*s^n \quad (n \text{ even})$$

$$-q_n^*s^n \quad (n \text{ odd})$$

The roots $zQ$ of $Q(s)^*$ are those of $Q(s)$ with the real part of opposite sign: $zQ=-zQ^*$.  \[ Q(s)^* = (-1)^n \prod_{k=1}^{n} (s + zQ_k^*) \]
Characteristic polynomials of lossless networks

Roots of $P$:
- Imaginary
- Complex pairs with opposite real part

Polynomial $P$ (degree $n_z$) must be multiplied by $j$ when $(n-n_z)$ is even

Roots of $F_2$:
- Equal to the negative conjugate of the roots of $F$:

$$zf_2 = -zf^* \quad \Rightarrow \quad F_2(s) = (-1)^n F(s)^*$$

Feldtkeller equation:

$$\frac{P(s)P(s)^*}{\varepsilon^2} + \frac{F(s)F(s)^*}{\varepsilon_r^2} = E(s)E(s)^*$$

$E(s)$ is defined once $P(s)$ and $F(s)$ are known
Relationship between $\varepsilon$ and $\varepsilon_r$

- When $n_z<n$:
  \[
  S_{21}(j\infty) = \frac{P(j\infty)}{\varepsilon E(j\infty)} = 0, \quad |S_{11}(j\infty)| = \frac{|F(j\infty)|}{\varepsilon_R|E(j\infty)|} = 1 \quad \Rightarrow \quad \varepsilon_R = 1
  \]

- When $n_z=n$ (fully canonical condition)
  \[
  \frac{P(j\infty)P(j\infty)^*}{\varepsilon^2 E(j\infty)E(j\infty)^*} + \frac{F(j\infty)F(j\infty)^*}{\varepsilon_r^2 E(j\infty)E(j\infty)^*} = 1 \quad \Rightarrow \quad \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon_r^2} = 1
  \]
  \[
  \varepsilon_r = \frac{\varepsilon}{\sqrt{\varepsilon^2 - 1}}
  \]

Let remember that $\varepsilon$ is determined by $RL=-10\log(|S_{11}(\pm j)|)$
The approximation problem: the characteristic function $C_n$ and polynomials $P, F$

$$A(\Omega) = 1 + \epsilon'^2 C_n^2(\Omega) = \frac{1}{|S_{21}(j\Omega)|^2} = \epsilon^2 \frac{E(j\Omega)E(-j\Omega)}{P(j\Omega)P(-j\Omega)} =$$

$$= 1 + \epsilon^2 \frac{F(j\Omega)F(-j\Omega)}{P(j\Omega)P(-j\Omega)} \Rightarrow C_n^2(\Omega) = \frac{\epsilon^2 |F(\Omega)|^2}{\epsilon'^2 |P(\Omega)|^2}$$

Applying the analytic continuation ($j\Omega \rightarrow s$):

$$C_n(s) = \frac{\epsilon}{\epsilon'} \frac{F(s)}{P(s)}$$

Given $C_n(\Omega)$ (order $n$ and trasmission zeros imposed), it is possible to compute the characteristic polynomials.
The generalized Chebycheff characteristic function

\[
C_n(\Omega) = \begin{cases} 
\cos \left[ (n - n_z) \cos^{-1}(\Omega) + \sum_{k}^{1,n_z} \text{Re} \left\{ \cos^{-1} \left( \frac{1 - \Omega \cdot \Omega_{z,k}}{\Omega - \Omega_{z,k}} \right) \right\} \right] & |\Omega| \leq 1 \\
\cosh \left[ (n - n_z) \cosh^{-1}(\Omega) + \sum_{k}^{1,n_z} \text{Re} \left\{ \cosh^{-1} \left( \frac{1 - \Omega \cdot \Omega_{z,k}}{\Omega - \Omega_{z,k}} \right) \right\} \right] & |\Omega| > 1 
\end{cases}
\]

\(\Omega_{z,k}\) are the assigned transmission zeros: \(zP_k = j\Omega_{z,k}\)

\(zP_k\) are the roots of \(P(s)\), which must be imaginary or complex pairs with opposite real part. \(C_n(\Omega)\) can be expressed in terms of the roots of \(P(s)\) and \(F(s)\)

\[
C_n(\Omega) = \frac{\mathcal{E}}{\mathcal{E}'} \frac{\prod_{k=1}^{n} (\Omega - zF_k / j)}{\prod_{k=1}^{n_z} (\Omega - zP_k / j)}
\]
Evaluation of polynomials $P(s)$, $F(s)$ given $C_n(\Omega)$

- Assign the order $n$ and the transmission zeros $zP_k$
- Evaluate $C_n(\Omega)$ (with $\Omega_{z,k}=zP_k/j$) for $\Omega_i=\Omega_1,\ldots,\Omega_N$, in the interval $-1<\Omega_i<1$ ($N>2n$)
- Generate the vector $F'(\Omega_i)=C_n(\Omega_i)\cdot\prod_{k=1}^{n_z}(\Omega_i-zP_k/j)$
- Find the coefficient of polynomial $F'(\Omega)$ by fitting $F'(\Omega_i)$ with a polynomial of order $n$
- Find the roots $\Omega_{F,k}$ of $F'(\Omega)$: $\Omega_{F,k}=zF_k/j \rightarrow zF_k=j\Omega_{F,k}$
- Generate the polynomials $F(s)$ and $P(s)$ from their roots $(zP_k, zF_k)$. Remember that $P(s)$ must be multiplied by $j$ if $n-n_z$ is even
Evaluation of $E(s)$ using lossless condition

From the imposed RL the constant $\varepsilon$ is computed:

$$
\varepsilon^2 = \frac{10^{-RL/10}}{1 - 10^{-RL/10}}
$$

If $n_z = n$ $\varepsilon_r$ is evaluated with the expression previously shown. Otherwise, $\varepsilon_r = 1$.

The polynomial $E_2(s)$ is then evaluated:

$$
E^2(s) = E(s)E(s)^* = \frac{P(s)P(s)^*}{\varepsilon^2} + \frac{F(s)F(s)^*}{\varepsilon_r^2}
$$

The roots of $E^2$ are computed and those with negative real part define the roots of $E(s)$

Finally $E(s)$ is obtained from its roots (it is monic)
More efficient method (for imaginary $zF_k$)

Let consider the following factorization of $E^2$:

$$E^2(s) = E(s)E(s)^* = \left( \frac{P(s)}{\varepsilon} + \frac{F(s)}{\varepsilon_r} \right) \cdot \left( \frac{P(s)^*}{\varepsilon} + \frac{F(s)^*}{\varepsilon_r} \right) = E_a \cdot E_b$$

The equality holds if:

$$\left( \frac{P(s)^* F(s) + P(s)F(s)^*}{\varepsilon \varepsilon_r} \right) = 0 \quad \Rightarrow \quad P(s)^* F(s) = -P(s)F(s)^*$$

The last equation is verified if the roots of $F$ are imaginary or symmetric with respect the imaginary axis.

The roots of $E(s)$ are obtained from the roots of $E_a$ (or $E_b$), by assigning all the real part negative (i.e. by changing the sign of those which are positive)
Example: $n=6$, $RL=26\text{dB}$, $z_P=\{1.12i, 1.31i\}$

\[ P=\{1.4224, -3.4564i, -2.0869\} \quad \varepsilon=0.7031 \]

\[ F=\{1, -1.0794i, 0.81597, -1.0023i, -0.0079246, -0.096402i\} \]

\[ E=\{1, 2.64-1.079i, 4.3-3.16i, 3.624-5.58i, 0.5864-5.316i, -1.1655-1.7338i\} \]
Group Delay Evaluation

- Group Delay in $\Omega$ domain:

$$\tau = -\frac{\partial}{\partial \Omega} \left[ \angle S_{21}(\Omega) \right] = -\frac{\partial}{\partial \Omega} \left[ \angle P(\Omega) - \angle E(\Omega) \right]$$

$$\angle P(\Omega) = \angle \prod_{k=1}^{n} (j\Omega - zP_k), \quad \angle E(\Omega) = \angle \prod_{k=1}^{n} (j\Omega - zE_k)$$

The phase of $P(\Omega)$ is independent of $\Omega$: in fact the roots $zP_k$ are on imaginary axis (0 contribute) or in pair with opposite real part (contributes of opposite sign). Then:

$$\angle S_{21}(\Omega) = -\angle E(\Omega) = -\sum_{k=1}^{n} \tan^{-1} \left( \frac{\Omega - \text{Im}(zE_k)}{\text{Re}(zE_k)} \right)$$

$$\tau = -\frac{\partial}{\partial \Omega} \left[ -\angle E(\Omega) \right] = \sum_{k=1}^{n} \frac{\text{Re}(zE_k)}{\left[ \text{Re}(zE_k) \right]^2 + \left[ \Omega - \text{Im}(zE_k) \right]^2}$$
Complex zero for phase equalizing

Group delay of the complex zero example (0.25-1.05i): Max value in passband: 14 sec
Attenuation worsens!

Attenuation/ReturnLoss

Normalized Frequency

dB

Normalized Frequency
Filtering networks presenting transmission zeros can be classified in two very general categories:

- **Crossed-coupled networks**: the transmission zeros are generated by means of multiple paths which allow the output signal to vanish at some frequencies.
- **Extracted pole networks**: each transmission zero (pure imaginary) is realized by means of a suitable impedance (admittance) which blocks the transmission between input at output at a specific frequency.

The networks synthesized in $\Omega$ are called **prototypes**. To obtain the network in the final bandpass domain $\omega$ it is necessary to perform a *de-normalization* process.

Prototypes with a number of transmission zeros equal to the number of poles are called **fully canonical**.
Cross-coupled prototype networks

- General topology: conventional representation

Given a set of polynomials \( F, P, E \) defining the generalized Chebyceff characteristic, it is always possible to synthesize a cross-coupled network with the above components. Such network is called normalized prototype.
Minimum path rule

- Given a cross-coupled topology, the maximum number of transmission zeros which can be accommodated is determined by the “minimum path rule”:

  “The maximum number of transmission zeros is equal to the prototype order \(n\) minus the number of nodes touched for going from the source to the load \(np\)”

  \[ n_z = n - np \]

- In the figure:
  - \(n = 9\), \(np = 4\)
  - \(n_z = 9 - 4 = 5\)
There are particular prototype networks with a specific topology which can be always synthesized from the given characteristic polynomials. These prototypes are called canonical.

Most important canonical prototypes:

- **Folded**
- **Transversal**
- **Wheel**
Notes on Canonical Prototypes

- Not all the couplings are different from zero (this is true only for transversal prototype)
- In folded and wheel prototypes there are two kinds of couplings: the direct (those connecting sequential nodes) and the cross (those between not-consecutive nodes).
- The number of not-zero cross couplings depends on the number of transmission zeros (according to the minimum path rule)
- In the folded prototype the cross couplings involving source and load are necessary when $n_z > n - 2$
- For fully canonical prototypes ($n_z = n$) a coupling between load and source is requested
Examples

\[ n=6, \, np=3 \rightarrow nz=3 \]

\[ n=7, \, np=4 \rightarrow nz=3 \]
Canonical prototypes with symmetric response

- Symmetric response in the normalized domain $\Omega$ is obtained with transmission zeros symmetrically placed around real axis
- The diagonal elements of $M$ are null
- The corresponding prototype does not include FIR (FIS) elements (positive-defined network)
- The canonical prototypes networks have specific properties:
  - **Folded**: oblique cross couplings are zero
  - **Wheel**: cross couplings terminating on the load vanish alternately
  - **Transversal**: couplings $M_{1,k}$ and $M_{k,n+2}$ have the same magnitude
Example

N=10, transmission zeros: $[\pm 1.23i, \pm 0.3 \pm 0.1i]$, RL=25
Circuit analysis of cross-coupled prototypes: formation of the $(n+2) \times (n+2)$ admittance matrix $Y$

\[
Y_{i,j} = jJ_{i,j} \\
Y_{1,1} = Y_{n+2,n+2} = 0 \\
Y_{i,i} \big|_{i \neq 0} = s + jb_i
\]
Normalized Coupling Matrix $\mathbf{M}$

$$
\mathbf{Y} = \begin{bmatrix}
0 & jJ_{01} & 0 & \ldots & 0 \\
 jJ_{01} & s + jb_1 & jJ_{12} & \ldots & jJ_{1n} \\
 jJ_{02} & jJ_{12} & s + jb_2 & \ldots & jJ_{2n} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & jJ_{n,n+1} & 0
\end{bmatrix} = s\mathbf{U}_n + j\mathbf{M}
$$

$\mathbf{M}$ = Normalized Coupling Matrix

$$
\begin{bmatrix}
0 & J_{01} & 0 & \ldots & 0 \\
 J_{01} & b_1 & J_{12} & \ldots & J_{1n} \\
 J_{02} & J_{12} & b_2 & \ldots & J_{2n} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & J_{n,n+1} & 0
\end{bmatrix}
$$
Evaluation of the scattering parameters from $\mathcal{M}$

- The $Y$ matrix is computed at frequencies $s=j\Omega_k$:
  \[ Y_k = j\Omega_k \mathbf{U}_n + j\mathbf{M} \]

  The $Z$ matrix is obtained by inverting $Y$:
  \[ Z_k = Y_k^{-1} \]

- The matrix $Z'$ (2x2) is extracted from $Z_k$ by cancelling all rows and columns except the first and last:
  \[ Z'_k = \begin{bmatrix} Z_{0,0} & Z_{0,n+1} \\ Z_{0,n+1} & Z_{n+1,n+1} \end{bmatrix} \]

- The scattering matrix of the prototype is computed from $Z'$:
  \[ S_k = (Z'_k - \mathbf{U}) \cdot (Z'_k + \mathbf{U})^{-1} \]
De-normalization of prototype networks

- De-normalization consists in the network transformation from the normalized domain $\Omega$ to the bandpass domain $f$, using the classical frequency transformation:
  $$\Omega = (f_0/B)(f/f_0 - f_0/f)$$

- At circuit level, this transformation is obtained by replacing the unit capacitance with a shunt resonator. If also the external loads are scaled from 1 to $G_0$ the correspondence between normalized and de-normalized components are as follows:

  \[ c' = \frac{G_0}{2\pi B}, \quad b' = b \cdot G_0, \quad J' = J \cdot G_0 \]
De-normalized bandpass network

- The de-normalized network is constituted by coupled resonators with the following coupling coefficients:

\[ k_{i,j} = \frac{J'_{i,j}}{\omega_0 c'} = \frac{B}{f_0} J_{i,j} = \frac{B}{f_0} M_{i,j} \]

- The resonant frequency of \( i \)-th shunt admittance results:

\[ f_{ris,i} = -\frac{B b_i}{f_0 2} + \sqrt{\left(\frac{B b_i}{f_0 2}\right)^2 + 1} = -\frac{B M_{i,i}}{f_0 2} + \sqrt{\left(\frac{B M_{i,i}}{f_0 2}\right)^2 + 1} \]

- External Q produced by the \( q \)-th resonator coupled to source (load):

\[ Q_{E,q} = \left(\frac{f_0}{B}\right) \cdot c' = \frac{1}{\left(\frac{B}{f_0}\right) J_{0,q}^2} = \frac{1}{\left(\frac{B}{f_0}\right) M_{0,q}^2} \]
Dependence of filter response on the coupling parameters

- Once the parameters $k_{i,j}$, $f_{ris,i}$ and $Q_{E,q}$ are defined, also the filter response is uniquely determined.
- This means that there are infinite networks, differing for the circuit component values but with same coupling parameters, which present the same response (identical scattering parameters).
- The circuit component values have however an influence on the voltages and currents along the filter; moreover, there could be some combinations of component values which result in a more easy implementation while other values may even not allow the physical realization of the filter.
De-normalized coupling matrix $M'$

- The matrix $M'$ is defined:
  \[ M' = \left( \frac{B}{f_0} \right) \cdot M \]

- The off main diagonal elements $M'_{q,k}$ represent the coupling coefficient $k_{q,k}$.

- The diagonal elements $M'_{q,q}$ determine the resonance frequencies of $q$-th node:
  \[
  f_{ris,q} = -\left( \frac{M'_{q,q}}{2} \right) + \sqrt{\left( \frac{M'_{q,q}}{2} \right)^2 + 1}
  \]

- The external Q if $q$-th resonator coupled to source (load) is given by:
  \[
  Q_{E,q} = \left( \frac{B}{f_0} \right) \frac{1}{\left[ M'_{0,q} \right]^2}
  \]
Evaluation of the de-normalized response from $M'$ including losses

- The $Y$ matrix vs. frequency (for $G_0=1$) can be written as:

$$Y(f) = \left(\frac{f_0}{B}\right)\left[\frac{1}{Q_0} U_n + j \left\{ \left(\frac{f}{f_0} - \frac{f_0}{f}\right) U_n + M' \right\} \right] \approx$$

$$= \left(\frac{f_0}{B}\right)\left[\frac{1}{Q_0} U_n + j \left\{ \text{diag} \left(\frac{f}{f_{\text{ris},i}} - \frac{f_{\text{ris},i}}{f}\right) + M'' \right\} \right]$$

where $Q_0$ is the unloaded Q of the resonators.

The approximated expression assumes that the frequency invariant elements are represented by de-tuned resonators (from $f_0$ to $f_{\text{ris},i}$, see next slide); $M''$ is then obtained from $M'$ by putting the element of the main diagonal to zero.
Approximated de-normalized resonators

\[ f_{ris,q} = -\left( \frac{M'_{q,q}}{2} \right) + \sqrt{\left( \frac{M'_{q,q}}{2} \right)^2 + 1} \]

The resonators in a cross-coupled bandpass filter are in general no synchronous. The approximation is typically acceptable for \((B/f_0) << 1\).

Note that the \(b'_q\) do not influence the coupling coefficients, which must be evaluated at \(f_0\).
Synthesis of canonical prototypes: the circuit approach

- Starting point: evaluation of Chain Matrix from characteristic polynomials:

\[
\begin{bmatrix}
A(s) & B(s) \\
C(s) & D(s)
\end{bmatrix}
= \frac{1}{jP(s)} \begin{bmatrix}
A(s) & B(s) \\
C(s) & D(s)
\end{bmatrix}
\]

\[2E(s) = A(s) + B(s) + C(s) + D(s), \quad 2F = A(s) + B(s) - C(s) - D(s)\]

\[A(s) = E_e(s) + F_o(s), \quad B(s) = E_o(s) - F_e(s)\]

\[C(s) = E_o(s) + F_o(s), \quad D(s) = E_e(s) - F_e(s)\]

Where the subscript \(e,o\) define the even and odd part of a polynomial:

\[Q(s) = Q_e(s) + Q_o(s)\]

\[Q_e(s) = \frac{Q(s) + Q(s)^*}{2}, \quad Q_o(s) = \frac{Q(s) - Q(s)^*}{2}\]
The synthesis is performed by subsequent extractions from the [ABCD] matrix of the prototype:

Suitable rules are available for the synthesis of specific canonical prototypes (see the works of Cameron on the *folded* prototype)
Example of synthesis (folded prototype)
The circuit synthesis does not produce in general a normalized prototype (i.e. the capacitances are not all equal to 1)

Using the conservation of the coupling coefficient $k_{i,j}$, it is easy to evaluate the elements of the coupling matrix $M_{i,j}$ resulting from synthesized components $J_{i,j}$, $c_i$, $c_j$:

$$M_{i,j} = \frac{J_{i,j}}{\sqrt{c_i \cdot c_j}}$$

The elements $M_{i,i}$ resulting from the synthesized frequency-invariant $b_i$ are given by:

$$M_{i,i} = \frac{b_i}{c_i}$$
Direct Synthesis of the Coupling Matrix

- This technique consists in the direct evaluation of the coupling matrix $M$ without resort to an explicit circuit synthesis.
- The method has been proposed by Cameron and allows the evaluation of the transversal canonical prototype.
- Details of the method, which rely on the relationship between the Coupling matrix and the short circuited Admittance Matrix of the transversal prototype, can be found in the literature.
- The computation procedure, can be easily automated in a computer program (input data: the characteristic polynomials).
Once a canonical prototype is available, it is possible to derive other topologies by performing suitable *transformations* of the synthesized coupling matrix.

The transformation must conserve the response of the network.

A class of topological transformations with such a property is represented by the *Similarity Transform* (Given’s Rotation).

Starting from the Transversal Prototype, specific transformations are available for obtaining the other canonical prototypes (folded, wheel).
SYNPROT: A software for the prototype synthesis