Basics Equations for Filters Synthesis

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Scattering parameters for lumped-elements filters

For linear, lumped-element networks the scattering parameters can be always represented as a polynomial ratio of the complex frequency $p = \sigma + j\omega$, with $\omega = 2\pi f$:

$$
S_{11} = \frac{N_{11}(p)}{D(p)}, \quad S_{21} = S_{12} = \frac{N_{12}(p)}{D(p)}, \quad S_{22} = \frac{N_{22}(p)}{D(p)}
$$

Roots of $N_{11}$ ($N_{22}$) : reflection zeros

Roots of $N_{21}$ : transmission zeros

Roots of $D$ : poles (natural frequencies)

The above polynomials are called characteristic polynomials.
Properties of lossless networks

For synthesis purposes the 2-port network representing a filter is assumed lossless (i.e. composed by lossless elements)

The scattering matrix of a lossless network is unitary:

\[ S \cdot \tilde{S}^* = U_2 \]

\[ S_{11}(p) \cdot S_{11}^*(p) + S_{21}(p) \cdot S_{21}^*(p) = 1 \]
\[ S_{22}(p) \cdot S_{22}^*(p) + S_{12}(p) \cdot S_{12}^*(p) = 1 \]

As a consequence the following properties hold for the polynomials defining the scattering parameters:

\[ N_{11}(p) = N(p), \quad N_{22}(p) = (-1)^n N^*(p) \]
\[ N(p) \cdot N^*(p) + N_{21}(p) \cdot N_{21}^*(p) = D(p) \cdot D^*(p) \]

The latter expression is know as Feldtkeller equation
Properties of characteristic polynomials

- The roots of $D(p)$ are real or conjugate pairs. The real part must be negative (strict Hurwitz polynomial).
- The roots of $N(p)$ can be everywhere in the complex plane (if complex they occur as conjugate pairs).
- The roots of $N_{12}(p)$ can be on the imaginary axis (conjugate pair) or on the real axis (pairs with opposite values) or as a complex quad in the $p$ plane.
- The coefficients of all the polynomials are **real numbers**.
Steps of the synthesis process

- Assignment of selectivity specifications (Attenuation mask)
- Selection of a suitable *approximating function* and evaluation of its parameters in order to satisfy the specifications
- Evaluation of the characteristic polynomials from the approximating function
- Synthesis of the network from the polynomials
Synthesis in a transformed frequency domain

In order to simplify the design process, the synthesis can be performed in a **normalized low-pass domain**, analytically defined by a *frequency transformation*. The synthesized network is then **de-normalized** to the band-pass frequency domain with a suitable *circuit transformation*.
Let $s = \Sigma + j\Omega$ the normalized low-pass domain where the filter passband is defined $\Omega_B = -1 \leftrightarrow +1$. The following equation relates the normalized domain with a pass-band domain suitably defined:

$$s = \frac{f_0}{B} \left( \frac{p}{f_0} + \frac{f_0}{p} \right)$$

$f_0$ and $B$ are the frequency parameters defining the transformation:

$$f_0 = \sqrt{f_{p1} \cdot f_{p2}}, \quad B = f_{p2} - f_{p1}$$
Properties of the transformation (imaginary axis)

For \( s=j\Omega \) and \( p=j2\pi f \) it has

\[
\Omega = \frac{f_0}{B} \left( \frac{f}{f_0} - \frac{f_0}{f} \right)
\]

\[
f_{p1} \cdot f_{p2} = f_{s1} \cdot f_{s2} = f_0^2
\]
Conservation of response: $A(j\Omega) = A(j\omega)$

$B = 55$ MHz

$f_0 = 1000$ MHz
Circuital property of the transformation:
Conservation of Z and Y

Low pass

\[ Y = s \cdot g_c \]

\[ Z = s \cdot g_L \]

\[ s = \frac{f_0}{B} \left( \frac{p}{f_0} + \frac{f_0}{p} \right) \]

Band pass

\[ Y = \frac{f_0}{B} \left( \frac{p}{\omega_0} + \frac{\omega_0}{p} \right) \cdot g_c = p \cdot C_{eq} + \frac{1}{p \cdot L_{eq}} \]

\[ C_{eq} = \frac{g_c}{2 \pi B}, \quad L_{eq} = \frac{1}{\omega_0^2 \cdot C_{eq}} \]

\[ Z = \frac{f_0}{B} \cdot \frac{f_0}{B} \left( \frac{p}{\omega_0} + \frac{\omega_0}{p} \right) \cdot g_L = p \cdot L_{eq} + \frac{1}{p \cdot C_{eq}} \]

\[ L_{eq} = \frac{g_L}{2 \pi B}, \quad C_{eq} = \frac{1}{\omega_0^2 \cdot L_{eq}} \]
Normalized low-pass prototype

- Ladder network synthesized in the normalized domain assuming unitary reference load (generator side):

\[ g_1, g_3, \ldots \rightarrow \text{shunt resonators with } C_k = g_k / (R_1 \cdot \Omega_B) \]

\[ g_2, g_4, \ldots \rightarrow \text{series resonators with } L_k = R_1 \cdot g_k / \Omega_B \]
Specifications in the lowpass domain

\[ f_{s,1} \cdot f_{s,2} = f_0^2 \]

Good Specifications:

- Stopband: \( f_{s1} \) to \( f_{s2} \)
- Passband: \( f_{p1} \) to \( f_{p2} \)

Amplitude:

- Stopband: \( A_{\text{min}} \)
- Passband: \( A_{\text{max}} \)

Frequency:

- Stopband: \( f_{s1} \) to \( f_{s2} \)
- Passband: \( f_{p1} \) to \( f_{p2} \)

Phase Margin:

\[ \Omega_a = \frac{f_0}{B} \left( \frac{f_{s,1}}{f_0} - \frac{f_0}{f_{s,1}} \right) \]
Approximation of the ideal Lowpass response

General expression for attenuation in the normalized domain:

$$A(\Omega) = 1 + \varepsilon'^2 C_n^2(\Omega)$$

$C_n$ is called \textit{Characteristic Function} and define the approximation of ideal filter response. The parameter $n$ represents the \textit{order} of the characteristic function and corresponds to the number of resonators in the de-normalized network.

\textbf{Properties:}

- $-1 < C_n < 1$ for $-1 < \Omega < 1$
- $|C_n| = 1$ for $\Omega = \pm 1$
- $A(\pm 1) = 1 + \varepsilon'^2$
Characteristic function and S parameters

Assuming lossless condition ($|S_{11}|^2 + |S_{21}|^2=1$):

$$A(\Omega) = \frac{1}{|S_{21}|^2} = 1 + \varepsilon'^2 C_n^2(\Omega) \quad \Rightarrow \quad \varepsilon'^2 C_n^2(\Omega) = \frac{1}{|S_{21}|^2} - 1 = \left|\frac{S_{11}}{S_{21}}\right|^2$$

$$C_n^2(\Omega) = \frac{1}{\varepsilon'^2} \left|\frac{S_{11}(j\Omega)}{S_{21}(j\Omega)}\right|^2 \quad \Rightarrow \quad C_n(j\Omega) = \frac{1}{\varepsilon'} \frac{S_{11}(j\Omega)}{S_{21}(j\Omega)}$$

$\varepsilon'$ can be expressed as function of Return Loss (RL):

$$1 + \varepsilon'^2 = \frac{1}{1 - |S_{11}|_{\text{max}}^2} \quad \Rightarrow \quad \varepsilon'^2 = \frac{|S_{11}|_{\text{max}}^2}{1 - |S_{11}|_{\text{max}}^2} = 10 \frac{RL}{10}$$
The characteristic function in s domain

Being \( s = j\Omega \), it has:

\[
C_n^2(\Omega) = C_n(j\Omega) \cdot C_n(-j\Omega) \quad \Rightarrow \quad C_n^2(s) = C_n(s) \cdot C_n(-s) = C_n(s) \cdot C_n^*(s)
\]

As a function of the scattering parameters:

\[
C_n(s) = \frac{1}{\varepsilon'} \frac{S_{11}(s)}{S_{21}(s)}
\]

The S parameters can be expressed as ratio of *characteristic polynomials* (monic):

\[
S_{11}(s) = \frac{F(s)}{E(s)}, \quad S_{21} = \frac{P(s)/\varepsilon}{E(s)} \quad \Rightarrow \quad C_n(s) = \frac{\varepsilon}{\varepsilon'} \frac{F(s)}{P(s)}
\]
Properties of characteristic polynomials

Roots of $F(s)$: reflection zeros
Roots of $P(s)$: transmission zeros
Roots of $E(s)$: poles

For the unitary of S matrix:

$$P(s) \cdot P(-s) + \varepsilon'^2 \left[ F(s) \cdot F(-s) \right] = E(s) \cdot E(-s)$$

$E(s)$ is found from the roots with negative real part of right hand side (Hurwitz polynomial)

**In conclusion**: given $C_n(\Omega)$, the transmission zeros and the requested return loss at $\Omega=\pm 1$, the polynomials $P(s)$, $F(s)$, $E(s)$ can be analytically evaluated
All-pole characteristic functions \((P=\text{cost}, \ C_n=\alpha F)\)

- **Butterworth function:**
  \[
  C_n(\Omega) = \Omega^n
  \]
  Main property: maximum flatness for \(\Omega=0\), monotonic for \(|\Omega|>0\)

- **Chebycheff function:**
  \[
  C_n(\Omega) = \cos(n \cdot \cos^{-1}(\Omega)) \quad |\Omega| \leq 1
  \]
  \[
  C_n(\Omega) = \cosh(n \cdot \cosh^{-1}(\Omega)) \quad |\Omega| > 1
  \]
  Main property: oscillates between \(\pm 1\) for \(|\Omega|<1\) \((n\ \text{peaks})\), increases monotonically for \(|\Omega|>1\)
Comparison between Butterworth and Chebycheff

Attenuation of Chebycheff function is $6(n-1)$ dB larger than Butterworth function for $|\Omega| >> 1$
Evaluation of $\varepsilon'$ and $n$

- **Assigned parameters:**
  - Passband Return Loss in dB (RL)
  - Filter bandwidth (B) and center frequency ($f_0$)
  - Minimum attenuation (dB) in stopband ($A_M$)
  - Frequencies at the beginning of stopband $f_{s1}, f_{s2}$ with $f_{s1}, f_{s2} = (f_0)^2$.

Evaluation of $f_{s1}$ in the lowpass domain:

$$\Omega_s = \frac{f_0}{B} \left( \frac{f_{s1}}{f_0} - \frac{f_0}{f_{s1}} \right)$$

Evaluation of $\varepsilon' = \frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}}$ with $\Gamma_m = 10^{-(RL/20)}$

$$n \geq \frac{A_M + RL}{20 \log(\Omega_s)}$$  \hspace{1cm} \text{Butterworth characteristic}

$$n \geq \frac{A_M + RL + 6}{20 \log(\Omega_s) + 6}$$  \hspace{1cm} \text{Chebycheff characteristic}
Synthesis of lowpass prototype

- **Butterworth characteristic**
  \[ r_n = 1, \quad g_q = 2a_q \sqrt{\varepsilon}, \quad a_q = \sin \left( \frac{(2q-1)\pi}{2n} \right) \]

- **Chebycheff characteristic**
  \[ r_n = 1 \quad (n \text{ odd}), \quad r_n = \left[ \sqrt{1 + \varepsilon^2} - \varepsilon \right]^2 \quad (n \text{ even}) \]
  \[ g_1 = \frac{2a_1}{\gamma}, \quad g_q = \frac{4a_{q-1} \cdot a_q}{b_{q-1} \cdot g_{q-1}}, \quad a_q = \sin \left( \frac{(2q-1)\pi}{2n} \right) \]
  \[ \gamma = \sinh \left( \frac{1}{2n} \ln \left( \frac{\sqrt{1 + \varepsilon^2} + 1}{\sqrt{1 + \varepsilon^2} - 1} \right) \right), \quad b_q = \gamma^2 + \sin^2 \left( \frac{q\pi}{n} \right) \]
De-normalized (bandpass) network with only series or shunt resonators

For microwave frequencies implementation is requested to have only one type of resonator (series or shunt). This is obtained with the introduction of impedance (admittance) inverters.
Introduction of inverter determines additional degrees of freedom which allows the arbitrary assignments of some parameters (inverters or resonators parameter). The condition to be satisfied are expressed by the following equations:

**Series Network**

\[ K_{01} = \sqrt{\frac{B}{f_0} \frac{\omega_0 L_{s,1}}{g_1}} R_1 \]

\[ K_{q,q+1} = \frac{B}{f_0} \sqrt{\frac{\omega_0 L_{s,q} \cdot \omega_0 L_{s,q+1}}{g_q \cdot g_{q+1}}} \]

**Shunt Network**

\[ J_{01} = \sqrt{\frac{B}{f_0} \frac{\omega_0 C_{s,1}}{g_1}} G_1 \]

\[ J_{q,q+1} = \frac{B}{f_0} \sqrt{\frac{\omega_0 C_{s,q} \cdot \omega_0 C_{s,q+1}}{g_q \cdot g_{q+1}}} \]

The bandpass network is assumed symmetric respect the central inverter \((n \text{ even})\) or resonator \((n \text{ odd})\)
Universal coupling parameters

Coupling coefficient:

\[ k_{q,q+1} = \frac{K_{q,q+1}}{\sqrt{\omega_0 L_{s,q} \cdot \omega_0 L_{s,q+1}}} = \frac{J_{q,q+1}}{\sqrt{\omega_0 C_{s,q} \cdot \omega_0 C_{s,q+1}}} = \frac{B}{f_0} \sqrt{\frac{1}{g_q \cdot g_{q+1}}} \]

External Q:

\[ Q_E = \frac{\omega_0 L_{s,1}}{K_{01}^2/R_1} = \frac{\omega_0 C_{s,1}}{J_{01}^2/G_1} = \frac{g_1}{B/f_0} \]

\( \omega_0 L_{s,1} \) = Equivalent reactance of series resonators

\( \omega_0 C_{s,1} \) = Equivalent susceptance of shunt resonators