Hybrid Cramér-Rao Bounds for Crustal Displacement Field Estimators in SAR Interferometry

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Abstract

This paper focuses on the performance achievable by spaceborne Synthetic Aperture Radar Interferometry (InSAR) in the estimation of line-of-sight crustal deformations from acquisitions over a distributed scatterer. Our model is suited for exploiting the Hybrid Cramér-Rao Bound (HCRB), where the unknowns are both deterministic parameters and stochastic variables. We take into account both target decorrelation and Atmospheric Phase Screen (APS). This approach leads to a viable evaluation of InSAR performance as a function of system configuration, target decorrelation, and APS variance.
I. INTRODUCTION

Spaceborne Synthetic Aperture Radar (SAR) interferometry is a well known, widely exploited technique to provide a quantitative estimate of crustal deformations, like subsidences and landslides [1]. The SAR sensor periodically revisits the same target from approximately the same point of view, providing complex images whose phase depends on both the target signature and the sensor-to-target optical path length. Millimetric variations in this last term may be measured via interferometric techniques, which are sensitive to phase changes. Depending on the application, such measurements may be related to various physical quantities [1], [2]. This paper considers the problem of estimating the Crustal Displacement Field (CDF), focusing on the widely adopted assumption of a homogenous distributed scatterer, where each pixel in a SAR image results from the superposition of many independent contributions within the resolution cell [2]. This is the case for forests, agricultural fields, soil or rock surfaces, and even ice shelves. Two different disturbances affect the quality of the estimates: the decorrelation undertaken by targets during the observation time (due to changes in the target reflectivity, thermal noise, uncompensated topography, etc. [3], [4]), and the fluctuation of the propagation delay (due to tropospheric disturbances and usually referred to as Atmospheric Phase Screen (APS) [4]). The ensemble of target decorrelation and APS results in non-Gaussian data, which complicates the a-priori assessment of the Crustal Displacement Field estimator performance for any given scenario.

II. INSAR DATA MODEL

Consider a dataset of \( N \) focused SAR images, \( y_n(r, x) \), where \( n \) indexes the images and \( (r, x) \) denotes Cartesian coordinates in the SAR range-azimuth plane. We assume that the images have been properly pixel-by-pixel co-registered, and that it is possible to relate each pixel to a specific location on the Earth’s surface. A suitable mathematical model for representing such image sets is the following two-dimensional convolution [2]

\[
y_n(r, x) = h(r, x) \ast \{ s_n(r, x) \exp(j\phi_n(r, x)) \},
\]

where \( y_n(r, x) \) is the \( n - \)th complex valued image (i.e., the observed data), \( h(r, x) \) represents the bandpass filtering effect of the SAR acquisition system, and \( s_n(r, x) \) is the target reflectivity. The distributed scatterer assumption allows us to regard \( s_n(r, x) \) as realizations of a complex, spatially uncorrelated, zero mean, stochastic process [2]. The target decorrelation from one observation to the
other is accounted for by the correlation coefficients $\gamma_{nm}$:

$$
E[s_n(r, x)] = 0,
$$

(2)

$$
E[s_n(r, x) s_m^*(r + \Delta r, x + \Delta x)] = \gamma_{nm} \delta(\Delta r, \Delta x).
$$

The major interest in (1) is in the phase term, $\phi_n(r, x) = \frac{4\pi}{\lambda} R_n(r, x)$, that results from the sensor-to-target optical path length in the $n$-th acquisition, $R_n(r, x)$, scaled by the carrier wavelength $\lambda$. The CDF causes variations in the target location, which can be discovered by measuring the phase differences between two images. Denoting the last image of the dataset as a reference, we may model the differential phases

$$
\phi_n(r, x) - \phi_N(r, x) = \\
\varphi_n(r, x; \theta) + \alpha_n(r, x) - \alpha_N(r, x),
$$

(3)

where $\theta$ is a vector of parameters that describes the CDF and $\{\varphi_n(r, x; \theta)\}_{n=1}^{N-1}$ is a set of known functions of $\theta$ (such as linear, exponential, periodic etc.). The term $\alpha_n(r, x)$ represents the APS at the time of the $n$-th acquisition. This term accounts for the variation in the optical path because of the stochastic fluctuation of the propagation delay, mainly caused by the wet refractivity of water vapor [4]. In our case of interest, where SAR images are taken with repeat intervals of a few days and the geometric resolution is on the order of a few meters, the APS turns out to be highly correlated over space, for many pixels, and uncorrelated from one acquisition to the other. Since the APSs appear in the phase model (3) as differences, it is convenient to deal directly with the APS differences, denoted $\omega_n = \alpha_n - \alpha_N$. Under the hypothesis\(^1\) that the fluctuation of the propagation delay may be modeled as a Gaussian, zero mean stochastic process with variance $\sigma^2_{\alpha}$, the APS differences, $\omega_n$, are normal, zero mean, with covariance matrix

$$
\{V_{\omega\omega}\}_{nm} = E[\omega_n \omega_m] = \sigma^2_{\alpha} \cdot (\delta_{n-m} + 1),
$$

(4)

where $\delta_{n}$ is the Kronecker symbol.

Our analysis focuses on a local scale, so we estimate the CDF on the basis of a small neighborhood of pixels, $\Omega$. This allows us to approximate both the APS and the CDF to be monitored as constant with

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\(^1\)The assumption of a Gaussian prior distribution is necessary to reach a simple analytical form. Otherwise, the final estimate accuracy would depend not only on the APS variance, but also on the shape of its distribution. As a first approximation, however, this assumption may be retained [5].
respect to the coordinates \((r, x)\). For this, we model the terms \(\varphi_n (r, x; \theta)\) as

\[
\varphi_n (r, x; \theta) = b_n \Lambda (r, x) + \varphi_n (\theta) \quad \forall \ (r, x) \in \Omega,
\]

where the term \(\Lambda (r, x)\) represents the contribution of the scene topography and \(b_n\) is the \(n - th\) normal baseline, i.e., the distance between the sensor positions in the \(n - th\) and \(N - th\) acquisitions [2]. The most suitable condition for estimating the CDF is met when the normal baselines are small, in such a way that we may neglect the contribution of the scene topography in (5). Hereafter, we will assume that this condition is met, which is reasonable for the next generation of spaceborne SARs [6]. However it is worth noticing that, as a first approximation, the assumption of small normal baselines may be relaxed by letting the correlation coefficients \(\gamma_{nm}\) account for the contribution of baseline decorrelation [3]. So far, (3) simplifies to

\[
\phi_n (r, x) - \phi_N (r, x) = \varphi_n (\theta) + \omega_n \quad \forall \ (r, x) \in \Omega,
\]

where \(\theta\) is the vector of unknowns, \(\varphi (\theta) = \begin{bmatrix} \varphi_1 (\theta) & \ldots & \varphi_{N-1} (\theta) \end{bmatrix}^T\) is a vector of known functions of \(\theta\), and \(\omega = \begin{bmatrix} \omega_1 & \ldots & \omega_{N-1} \end{bmatrix}^T\) is a vector of random variables with known prior distribution \(p (\omega)\).

A. Data statistics

Since the radar returns are assumed to result from the sum of several of i.i.d. contributions within the resolution cell, by virtue of the central limit theorem, the data, conditioned on both \(\theta\) and \(\omega\), may be regarded as having a zero-mean, multivariate normal distribution. To describe such a probability density function (pdf) in a compact fashion, it is useful to define the following:

- \(L\) is the number of pixels within the window \(\Omega\).
- \(y_n = \begin{bmatrix} y_n (r_1, x_1) & \ldots & y_n (r_L, x_L) \end{bmatrix}^T\) is a column vector of \(L\) pixels extracted from the \(n - th\) image.
- \(y = \begin{bmatrix} y_1^H & \ldots & y_N^H \end{bmatrix}^H\). The superscript \(H\) indicates Hermitian transposition.
- \(R_h\) is an \(L \times L\) matrix whose elements are the samples of the autocorrelation of the acquisition filter:

\[
\{R_h\}_{ij} = \int \int h (r + r_i - r_j, x + x_i - x_j) h^* (r, x) \, dr \, dx.
\]

For simplicity, we suppose that \(\text{rank} (R_h) = L\).

- \(C = E [yy^H | \omega, \theta]\) is the data covariance matrix conditioned on both the APS differences, \(\omega\), and the parameters to be estimated, \(\theta\).
From (1) and (2), the \( nm \)-th block of the data covariance matrix may be written as

\[
C_{nm} = E \left[ y_n y_m^H | \omega, \theta \right] = \gamma_{nm} R_h \exp \left( j ( \phi_n (\theta) - \phi_m (\theta) + \omega_n - \omega_m ) \right).
\]

The determinant of \( C \) is not affected by \( \phi \) nor \( \omega \), therefore:

\[
p(y|\omega, \theta) = \text{const} \cdot \exp \left( -y^H C^{-1} y \right).
\]

### III. THE HYBRID CRAMÉR-RAO BOUND

The data formulation described in the previous section is suited for exploiting the Hybrid Cramér-Rao Bound (HCRB) for lower bounding the estimator performances. The HCRB [7], [8], [9] applies in the case where some of the unknowns are deterministic and others are random; it unifies the deterministic and Bayesian CRB in such a way as to simultaneously bound the covariance matrix of the unbiased estimates of the deterministic parameters and the mean square errors on the estimates of the random variables [7], [8].

Let \( \hat{\theta} \) be an unbiased estimator of the deterministic parameters \( \theta \), and denote \( \hat{\omega} \) an estimator of the random variables \( \omega \). The HCRB assures that, for every estimator,

\[
E_{y, \omega} \left[ \begin{array}{c}
(\hat{\theta} - \theta)^T (\hat{\theta} - \theta) \\
(\hat{\omega} - \omega)^T (\hat{\omega} - \omega)
\end{array} \right] \geq J^{-1},
\]

where \( E_{y, \omega}[\ldots] \) denotes expectation with respect to the joint pdf of the data and the APS differences, \( p(y, \omega|\theta) \), and the inequality means that the difference between the left and the right sides of (9) is a nonnegative definite matrix. The Hybrid information matrix \( J \) is the sum of the standard Fisher Information Matrix (FIM), \( F \), averaged with respect to \( \omega \), and the prior information matrix \( I \). Define \( \Delta^Y_X \) as a matrix of the second order partial derivatives with respect to two multi-dimensional variables \( (x, y) \):

\[
\{ \Delta^Y_X \}_{nm} = \frac{\partial^2}{\partial x_n \partial y_m}.
\]

Then we have:

\[
J = E_{\omega} [F] + I,
\]
\[
\mathbf{F} = -E_{y|\omega}\left\{ \begin{bmatrix}
\Delta_\theta^y \log p (y|\omega,\theta) & \Delta_\omega^y \log p (y|\omega,\theta) \\
\Delta_\omega^y \log p (y|\omega,\theta) & \Delta_\omega^\omega \log p (y|\omega,\theta)
\end{bmatrix} \right\},
\]

\[
\mathbf{I} = -E_{\omega}\left\{ \begin{bmatrix}
0 & 0 \\
0 & \Delta_\omega^\omega \log p (\omega)
\end{bmatrix} \right\},
\]

where \(E_{y|\omega}[\ldots]\) denotes expectation with respect to \(p(y|\omega,\theta)\), and \(E_\omega[\ldots]\) denotes expectation with respect to \(p(\omega)\). Notice that we suppose that no priori information about \(\theta\) is available; therefore, the expression of the matrix \(\mathbf{I}\) in (13) accounts only for the APSs.

A. HCRB for InSAR

The computation of the FIM \(\mathbf{F}\) follows directly from (8) and (12):

\[
\mathbf{F} = \begin{bmatrix}
\mathbf{\Theta}^T \mathbf{X} \mathbf{\Theta} & \mathbf{\Theta}^T \mathbf{X} \\
\mathbf{X} \mathbf{\Theta} & \mathbf{X}
\end{bmatrix},
\]

where \(\mathbf{\Theta}\) is the matrix of the first order partial derivatives of \(\varphi\) with respect to \(\theta\),

\[
(\mathbf{\Theta})_{nm} = \frac{\partial \varphi_n (\theta)}{\partial \theta_m},
\]

and \(\mathbf{X}\) is the standard FIM for the estimating of the phases \(\varphi\). Letting \(\mathbf{\Gamma}\) be the matrix of the correlation coefficients, i.e., \(\{\mathbf{\Gamma}\}_{nm} = \gamma_{nm}\). from (7), it may be shown that:

\[
\{\mathbf{X}\}_{nm} = 2L \left( \gamma_{nm} \gamma_{nm}^{(-1)} - \delta_{n-m} \right),
\]

where \(\gamma_{nm}^{(-1)}\) is the \(nm-\)th element of \(\mathbf{\Gamma}^{-1}\).

From (4), and under the hypothesis of Gaussian APSs, the lower right block of the matrix \(\mathbf{I}\) is given by

\[
E_\omega [\Delta_\omega^\omega \log p (y)] = -\mathbf{V}_{\omega\omega}^{-1}.
\]

Finally, from (16), \(\mathbf{X}\) does not depend on \(\omega\), and thus \(E_\omega [\mathbf{F}] = \mathbf{F}\). So far, the hybrid information matrix may be written as

\[
\mathbf{J} = \begin{bmatrix}
\mathbf{\Theta}^T \mathbf{X} \mathbf{\Theta} & \mathbf{\Theta}^T \mathbf{X} \\
\mathbf{X} \mathbf{\Theta} & \mathbf{X} + \mathbf{V}_{\omega\omega}^{-1}
\end{bmatrix}.
\]
by estimators of \( \theta \) is bounded by

\[
E_{y, \omega} \left[ \left( \hat{\theta} - \theta \right) \left( \hat{\theta} - \theta \right)^T \right] \geq \left( \Theta^T \left( X^{-1} + V_{\omega \omega} \right)^{-1} \Theta \right)^{-1}.
\]

The kernel \( X^{-1} + V_{\omega \omega} \) in (19) has a precise meaning. The first term, \( X^{-1} \), is the standard CRB for the estimation of \( \varphi \); this term accounts for source decorrelation. The second term, \( V_{\omega \omega} \), represents the contribution of the APS differences, \( \omega \).

B. On the validity of the HCRB for InSAR applications

To give an intuitive idea of the conditions that must be met for the HCRB to represent a realistic bound in the practice, it is useful to consider separately the roles of \( X^{-1} \) and \( V_{\omega \omega} \) in (19). Let us assume that the APS noise power can be ignored with respect to the contribution coming from source decorrelation, represented by the term \( X^{-1} \). In this case, the bound for the variance of the CDF predicted by (19) defaults to the standard CRB, as expected since it ignores the statistical component \( \omega \). This bound is quite closely approached by maximum-likelihood estimators at sufficiently large signal-to-noise ratios, or when the number of available data is sufficiently large. In the framework of InSAR, this means that either the correlation coefficients, the number of images, or the estimation window must be large.

On the other hand, when the APS noise dominates with respect to the source decorrelation, the term \( X^{-1} \) can be ignored, and from (19) and (4), the HCRB for estimating \( \theta \) is proportional to the variance of the APS, \( \sigma_{\alpha}^2 \). However, this bound loses sensibility as the APS standard deviation approaches \( \pi \), since any CDF estimate would be seriously affected by the intrinsic \( 2\pi \) phase wrapping ambiguity associated with phase measurements. The fact that this aspect is not handled by the HCRB is an intrinsic limitation of the method, whereas other bounds have been proposed in literature that account properly for threshold effects [8]. Hence, as far as InSAR applications are concerned, the results predicted by the HCRB in (19) are useful as long as phase unwrapping is not a concern. This requires both the displacement field and the APSs to be sufficiently smooth in the range-azimuth dimension [2], [10].

IV. CLOSED FORM SOLUTIONS FOR INSAR

This last section provides a simple and practical example of an application of the proposed bound (19). For simplicity’s sake, we model the CDF as a Line of Sight (LOS) subsidence with constant velocity, \( v \), even though an extension of this analysis to a larger set of unknowns is immediate. In this case the set of the unknowns reduces to a single scalar, \( v \), and the phase functions \( \varphi_n(v) \) are given by:

\[
\varphi_n(v) = \frac{4\pi}{\lambda} (n - N) \delta t \cdot v
\]

(20)
\( \delta t \) being the time interval between two nearby acquisitions.

To properly characterize the source statistics, we assume that \( \gamma_{nm} \) is determined by an exponential temporal decorrelation [3] plus a thermal noise, uncorrelated from one acquisition to the other:

\[
\gamma_{nm} = \gamma_0 r^{|n-m| \delta t} + (1 - \gamma_0) \delta_{n-m}, \tag{21}
\]

where \( r \) is a parameter describing temporal decorrelation and \( \gamma_0 \) is related to the signal-to-noise ratio via \( \gamma_0 = \frac{SNR}{1 + SNR} \). From the decorrelation model (21), the HCRB for the estimate of the subsidence velocity is easily computed through (16) and (19); see Fig. 1 (left). However, to achieve deeper insight on how temporal decorrelation, thermal noise, and APSs impact the accuracy of the estimate, it is convenient to analyze these phenomena separately.

A. Temporal decorrelation

Neglecting the contribution of thermal noise, the decorrelation model (21) reduces to \( \gamma_{nm} = r^{|n-m| \delta t} \), for which the matrix \( \Gamma^{-1} \) is tridiagonal and computable in a closed form. Neglecting also the contribution of the APSs, from (16) and (19), and after some matrix manipulations we obtain

\[
\sigma_v^2 = \left( \frac{\lambda}{4\pi \delta t} \right)^2 \frac{1 - r^{2\delta t}}{2Lr^{2\delta t}} \frac{1}{N-1} \frac{\sigma_{temp}^2}{N-1}. \tag{22}
\]

B. Thermal noise and APS

If we neglect the temporal decorrelation, the decorrelation model (21) becomes \( \gamma_{nm} = \gamma_0 + (1 - \gamma_0) \delta_{n-m} \), for which the matrix \( \Gamma^{-1} \) is again achievable in a closed form. Computing the matrix \( \mathbf{X} \) through (16), and its inverse, the \( nm \)-th element of \( \mathbf{X}^{-1} \) is

\[
\{ \mathbf{X}^{-1} \}_{nm} = \frac{1 - \gamma_0 N \gamma_0 - \gamma_0 + 1}{\gamma_0^2 L} \frac{N}{2N} (\delta_{n-m} + 1) = \sigma_{\text{noise}}^2 (\delta_{n-m} + 1). \tag{23}
\]

Comparing the last expression to (4) we notice that the two matrices have exactly the same structure; thus, apart from a scale factor, thermal noise and APSs affect accuracy in the same way. From (19)

\[
\sigma_v^2 = \left( \frac{\lambda}{4\pi \delta t} \right)^2 \left( \frac{12}{N^3 - N} \sigma_a^2 + \sigma_{\text{noise}}^2 \right). \tag{24}
\]

C. Temporal decorrelation plus thermal noise and APSs

So far, the behavior of the HCRB curve may be qualitatively explained as a mixture of (24) and (22). When the number of images is large, the \( 1/(N-1) \) mechanism is dominant; thus, we may assume that
the HCRB is given by (22). On the other hand, when $N$ is small, the dominant contribution is given by thermal noise and APSs, and thus we expect the curve to be proportional to $1/(N^3 - N)$. To find the proportionality factor it suffices to compute the variance for $N = 2$ images. In this case, $\mathbf{X}$ becomes a scalar, and the result is immediate:

$$
\sigma_v^2(2) = \left( \frac{\lambda}{4\pi\delta t} \right)^2 \left( 2\sigma_a^2 + \frac{1 - \gamma_0^2 r^2 \delta t}{2\gamma_0^2 r^2 \delta t L} \right),
$$

and thus

$$
\sigma_v^2(N) = \sigma_v^2(2) \frac{6}{N^3 - N} \quad \text{for } N \text{ small}.
$$

The value of $N$ where the HCRB curve changes its behavior is determined by intersecting (22) and (26), yielding

$$
\tilde{N} \simeq \sqrt{\frac{6\sigma_v^2(2)}{\sigma_{\text{temp}}^2}}.
$$

This value corresponds to the intersection of the dashed lines in Fig. 1 (right).
V. Conclusions

This paper proposed a bound for the parametric estimation of the CDF through InSAR. This bound was derived by formulating the problem in such a way as to be handled by the HCRB. This methodology allows for a unified treatment of source decorrelation (target changes, thermal noise, volumetric effect, etc.) and APS under a consistent statistical approach. By introducing some reasonable assumptions, we could obtain some closed form solutions of practical use in InSAR applications. These solutions provide a quick performance assessment of an InSAR system as a function of its configuration (wavelength, resolution, SNR), the intrinsic scene decorrelation, and the APS variance. Although some limitations may arise at higher wavelengths, due to phase wrapping, the result may still be useful for the design and tuning of the overall system.

REFERENCES