Fundamental Performance Limits of TOA-based Cooperative Localization

M. Nicoli and D. Fontanella
Dip. di Elettronica e Informazione, Politecnico di Milano, Piazza L. da Vinci 32, I-20133 Milano, Italy

***Abstract*— Knowledge of node locations is desired in many types of wireless sensor network (WSN) technologies. Cooperative localization has been proved to provide self-calibration in the WSN even in presence of sparse reference nodes and noisy range estimates, as it relies on redundancy exploiting all the available inter-node measurements. In this paper we consider cooperative localization based on time-of-arrival (TOA) inter-node ranging. We analyze the Cramer-Rao bound on the positioning accuracy for one/two dimensional regular network topologies. The aim is to obtain some understanding of the behavior of localization accuracy for varying system settings, as a base for future evaluations of more realistic random network deployments.**

I. Introduction

Positioning is a topic of great interest nowadays as its application to emergency, security, environmental monitoring, and location-aware services, is expected to play an important role in the near-future wireless markets. An important area of applications is that of wireless sensor networks (WSN), flexible low-cost networks that are able to self-organize and adapt to changing conditions. In typical WSN applications, only few nodes have known positions (*reference nodes* or *anchors*), all the others (*unknown nodes*) have to be localized by the network itself, even those that are out of the range of any anchor. In such a situation, localization can be obtained through the cooperation of all nodes, using not only measurements from anchors but also measurements among pairs of unknown nodes [1]. If nodes are equipped with ultra-wide-band (UWB) transmission capabilities, accurate inter-node range measurements are achievable through time-of-arrival (TOA) estimation [2].

The Cramér-Rao bound (CRB) provides a useful means for the analysis of the limits of positioning accuracy. CRB results for cooperative localization, based on different types of inter-node measurements, can be found in [1]- [7]: the localization error is shown to depend on several elements, in particular on the measurement reliability (which may be impaired by multipath, non-line-of-sight, synchronization errors, etc.) and the topology of the network. The CRB analysis for TOA-based cooperative localization is here extended to include path-loss effects and it is focused on simple network topologies, such as regular one/two dimensional (1D/2D) sensor arrays. The aim is to obtain some understanding of the behavior of localization accuracy for varying system settings, e.g. network size, unknown/known node density, path attenuation, transmission coverage, etc. Analytical close form results show how the average location accuracy is related to the number of unknown nodes and to the number of reliable connections for each unknown node. We believe that this preliminary study can be the starting point for successive evaluations of more realistic WSN topologies (e.g., modeled as random networks) and for defining guidelines for deployment of practical WSN systems given location accuracy requirement.

II. Model

Consider a network composed of N nodes with locations $\theta_1,\ldots,\theta_N$, each defined by a pair of spatial coordinates $\theta_k = [x_k \ y_k]^{\mathrm{T}} \in \mathbb{R}^2$, $k=1,\ldots,N$. We assume that the first N_u nodes have unknown positions $\{\theta_k\}_{k=1}^{N_u}$, while the remaining $N_r = N - N_u$ are reference devices located in known positions $\{\theta_k\}_{k=N_u+1}^{N}$ (e.g., through GPS). In cooperative localization [1], the estimation of the $2N_u$ unknown parameters $\boldsymbol{\theta} = [\mathbf{x}^{\mathrm{T}} \ \mathbf{y}^{\mathrm{T}}]^{\mathrm{T}}$, with $\mathbf{x} = [x_1 \cdots x_{N_u}]^{\mathrm{T}}$ and $\mathbf{y} = [y_1 \cdots y_{N_u}]^{\mathrm{T}}$, is obtained from pairwise measurements $\{z_{k,\ell}\}$ made between any pair of (either known or unknown) nodes k and ℓ. Focusing on TOA's, $z_{k,\ell}$ represents the estimate of the distance between nodes k and ℓ:

$$d_{k,\ell} = |\theta_k - \theta_\ell| = \sqrt{\Delta x_{k,\ell}^2 + \Delta y_{k,\ell}^2} \tag{1}$$

with $\Delta x_{k,\ell} = x_k - x_\ell$, $\Delta y_{k,\ell} = y_k - y_\ell$. Let Ω_k be the subset of nodes with which node k makes measurements, $I_{\Omega_k}(\ell)$ is the visibility function such that $I_{\Omega_k}(\ell) = 1$ if $\ell \in \Omega_k$, or $I_{\Omega_k}(\ell) = 0$ if not. For $I_{\Omega_k}(\ell) = 1$, the measurement from node k to node ℓ can be modeled as follows:

$$z_{k,\ell} = h(\theta_k, \theta_\ell) + e_{k,\ell} = |\theta_k - \theta_\ell| + e_{k,\ell}, \tag{2}$$

where $h(\theta_k,\theta_\ell) = |\theta_k - \theta_\ell|$ and $e_{k,\ell} \sim \mathcal{N}\left(0, \sigma_{k,\ell}^2\right)$ is the measurement uncertainty, here assumed as Gaussian and uncorrelated to the other measurements' errors $\{e_{m,n}\}_{m\neq k, n\neq \ell}$. In this work, range estimates are considered unbiased, as we assume that both $z_{k,\ell}$ and $z_{\ell,k}$ are available for each pair of nodes k,ℓ, so that clock biases can be canceled (as for round-trip measurements). The analysis can be generalized to include synchronization errors following the approach in [3].

According to the CRB on TOA estimates [8], the variance $\sigma_{k,\ell}^2$ should be linearly related to the inverse of the signal to noise ratio (SNR) that is experienced over the radio link used to obtain that range estimate $z_{k,\ell}$. This SNR is typically modeled as proportional to $d_{k,\ell}^{-\alpha}$ where α is the path-loss exponent, thus it is reasonable to assume:

$$\sigma_{k,\ell}^2 = \sigma_0^2 \cdot (d_{k,\ell}/d_0)^\alpha = \sigma_0^2 \cdot (|\theta_k - \theta_\ell|/d_0)^\alpha, \tag{3}$$

where σ_0^2 is the range accuracy at a reference distance d_0.

Arrange all measurements into the vector $\mathbf{z} = [z_1 \cdots z_M]^{\mathrm{T}}$; based on the Gaussian assumption it is $\mathbf{z} \sim \mathcal{N}(\mathbf{h}(\boldsymbol{\theta}), \mathbf{Q}(\boldsymbol{\theta}))$, where $\mathbf{h}(\boldsymbol{\theta})$ is the vector collecting the terms $\{h(\theta_k, \theta_\ell)\}$, while $\mathbf{Q}(\boldsymbol{\theta})$ is the diagonal covariance matrix having as diagonal entries the variances $\{\sigma^2_{k,\ell}\}$ that depend on the unknown parameters $\boldsymbol{\theta}$ according to (3). The total number of measurements is $M \leq N(N-1) - N_r(N_r-1)$ and it is maximum in case of *full coverage*, i.e. when each node makes measurements with all others (excluding measurements between two anchors). We will assume, however, that each node has a limited coverage and makes measurements only to nodes located within a radius r from itself, i.e. $\Omega_k = \{\theta_\ell : |\Delta\theta_{k,\ell}| \leq r\}$. The ideal case of full coverage will be considered only as a reference for performance evaluation in Sec. IV (notice that in this case it is $r \to \infty$, but the reliability of the range estimate always reduces for increasing distance due to the path loss effects according to (3)).

III. Analysis of the CRB

The CRB for any unbiased estimate $\hat{\boldsymbol{\theta}}$ is [8]:

$$\mathrm{Cov}(\hat{\boldsymbol{\theta}}) = \mathrm{E}[(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^{\mathrm{T}}] \geq \mathbf{F}^{-1}, \qquad (4)$$

where $\mathbf{F}$ is the $2N_u \times 2N_u$ Fisher information matrix (FIM) for the $2N_u$ coordinates $\boldsymbol{\theta}$. For TOA-based cooperative localization, the CRB has been derived in [1]- [3] assuming that the estimator has no a priori knowledge about the dependence of the measurement accuracies on the locations $\boldsymbol{\theta}$. The analysis is here modified to include the path-loss model (3). The FIM inverse can be written as [8]:

$$\mathbf{C} = \mathbf{F}^{-1} = \begin{bmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xy} \\ \mathbf{C}_{xy}^{\mathrm{T}} & \mathbf{C}_{yy} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{xx} & \mathbf{F}_{xy} \\ \mathbf{F}_{xy}^{\mathrm{T}} & \mathbf{F}_{yy} \end{bmatrix}^{-1}, \qquad (5)$$

where $\mathbf{F}_{xx}$, $\mathbf{F}_{yy}$ and $\mathbf{F}_{xy}$ are the $N_u \times N_u$ FIM sub-blocks that can be derived as in [1] using the Gaussian assumption for the measurements $\mathbf{z}$. The derivation is modified in the Appendix so as to include the path-loss model (3), yielding:

$$[\mathbf{F}_{xx}]_{k,\ell} = \begin{cases} \sum_{j \in \Omega_k} \left(\frac{2\Delta x^2_{k,j}}{\sigma^2_{k,j} d^2_{k,j}} + \beta \frac{\alpha^2 \Delta x^2_{k,j}}{d^4_{k,j}} \right), & \text{if } k = \ell \\ -I_{\Omega_k}(\ell) \left(\frac{2\Delta x^2_{k,\ell}}{\sigma^2_{k,\ell} d^2_{k,\ell}} + \beta \frac{\alpha^2 \Delta x^2_{k,\ell}}{d^4_{k,\ell}} \right), & \text{if } k \neq \ell \end{cases} \qquad (6)$$

$$[\mathbf{F}_{yy}]_{k,\ell} = \begin{cases} \sum_{j \in \Omega_k} \left(\frac{2\Delta y^2_{k,j}}{\sigma^2_{k,j} d^2_{k,j}} + \beta \frac{\alpha^2 \Delta y^2_{k,j}}{d^4_{k,j}} \right), & \text{if } k = \ell \\ -I_{\Omega_k}(\ell) \left(\frac{2\Delta y^2_{k,\ell}}{\sigma^2_{k,\ell} d^2_{k,\ell}} + \beta \frac{\alpha^2 \Delta y^2_{k,\ell}}{d^4_{k,\ell}} \right), & \text{if } k \neq \ell \end{cases} \qquad (7)$$

$$[\mathbf{F}_{xy}]_{k,\ell} = \begin{cases} \sum_{j \in \Omega_k} \left(\frac{2\Delta x_{k,j} \Delta y_{k,j}}{\sigma^2_{k,j} d^2_{k,j}} + \beta \frac{\alpha^2 \Delta x_{k,j} \Delta y_{k,j}}{d^4_{k,j}} \right), & \text{if } k = \ell \\ -I_{\Omega_k}(\ell) \left(\frac{2\Delta x_{k,\ell} \Delta y_{k,\ell}}{\sigma^2_{k,\ell} d^2_{k,\ell}} + \beta \frac{\alpha^2 \Delta x_{k,\ell} \Delta y_{k,\ell}}{d^4_{k,\ell}} \right), & \text{if } k \neq \ell \end{cases} \qquad (8)$$

The parameter β has been introduced above to indicate the a-priori knowledge about the relation of the covariance $\mathbf{Q}(\boldsymbol{\theta})$ with the unknown parameters $\boldsymbol{\theta}$ at the estimator: it is $\beta = 1$ in case the model (3) is assumed, $\beta = 0$ for no a-priori knowledge. In the latter case the FIM equals the results given in [1], while for $\beta = 1$ a second term is added to each element of the FIM leading to a lower estimate error. For increasing SNR ($\sigma_0^2 \to 0$) this second term becomes negligible and the FIM elements (6)-(8) tend to those for $\beta = 0$. This result is confirmed by Fig. 1 that compares the location root

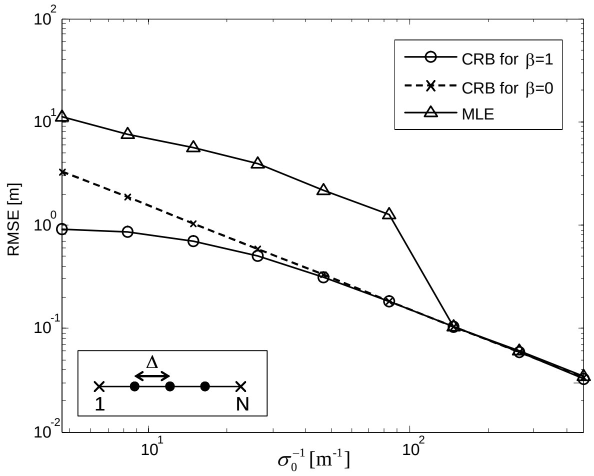

Fig. 1. CRB of the location accuracy and RMSE of the MLE for the network deployment exemplified in the box ('o' denotes unknown node, 'x' is anchor): $N_u = 3$, $N_r = 2$, $\alpha = 4$, $d_0 = 1$m, $\Delta = 5$m, $r \to \infty$.

MSE (RMSE) of the maximum likelihood estimation (MLE) averaged over 100 independent realizations with the two CRB bounds ($\beta = 0$ and $\beta = 1$) for different values of σ_0. The MLE requires a nonlinear unconstrained optimization and it is here implemented using a standard iterative gradient-search algorithm initialized according to [3].

The lower bound on the kth location estimate accuracy is:

$$\mathrm{MSE}_k = \mathrm{E}[||\hat{\theta}_k - \theta_k||^2] \geq [\mathbf{C}_{xx}]_{k,k} + [\mathbf{C}_{yy}]_{k,k}. \qquad (9)$$

that, averaged over the whole network, yields:

$$\mathrm{MSE} = \frac{\sum_{k=1}^{N_u} \mathrm{MSE}_k}{N_u} \geq \frac{\mathrm{tr}(\mathbf{C})}{N_u} = \frac{\mathrm{tr}(\mathbf{C}_{xx}) + \mathrm{tr}(\mathbf{C}_{yy})}{N_u}. \qquad (10)$$

In the following we investigate the above bounds for 1D/2D network topologies. The aim is to introduce analytical tools to be used in Sec. IV for studying the impact of the network deployment on cooperative localization performance.

A. Simplified Bounds

In 1D networks the N nodes are deployed along a straight line and each position is given by a single scalar coordinate $\theta_k = x_k \in \mathbb{R}$. In this case, the $N_u \times N_u$ FIM, $\tilde{\mathbf{F}}$, for the unknown parameter vector $\boldsymbol{\theta} = \mathbf{x}$ reduces to the matrix $\mathbf{F}_{xx}$ in (6) with $\Delta x^2_{k,\ell} = d^2_{k,\ell}$. For large SNR, this leads to:

$$\tilde{\mathbf{F}} = 2 \begin{bmatrix} \sum_{\ell \in \Omega_1} \frac{1}{\sigma^2_{1,\ell}} & -\frac{I_{\Omega_1}(2)}{\sigma^2_{1,2}} & \cdots & -\frac{I_{\Omega_1}(N_u)}{\sigma^2_{1,N_u}} \\ -\frac{I_{\Omega_2}(1)}{\sigma^2_{2,1}} & \sum_{\ell \in \Omega_2} \frac{1}{\sigma^2_{2,\ell}} & & -\frac{I_{\Omega_2}(N_u)}{\sigma^2_{2,N_u}} \\ \vdots & & \ddots & \\ -\frac{I_{\Omega_{N_u}}(1)}{\sigma^2_{N_u,1}} & -\frac{I_{\Omega_{N_u}}(2)}{\sigma^2_{N_u,2}} & \cdots & \sum_{\ell \in \Omega_{N_u}} \frac{1}{\sigma^2_{N_u,\ell}} \end{bmatrix}. \qquad (11)$$

From the above equation and from (3), we can see how the cooperative localization accuracy is related to the connectivity of the unknown nodes' network and to the reliability of the links towards the anchors. For convenience of notation, let us introduce the parameter $\gamma_{k,\ell} = I_{\Omega_k}(\ell)/\sigma^2_{k,\ell}$. Based on the assumptions in Sec. II, it is $\gamma_{k,\ell} = \gamma_{\ell,k}$. This parameter can be seen as an indicator of the quality of the connection between nodes k and ℓ. The FIM can also be written as $\tilde{\mathbf{F}} = \mathbf{F}_r + \mathbf{F}_u$, the sum of the localization information from reference nodes ($\mathbf{F}_r$) and from cooperation ($\mathbf{F}_u$). The first term, $\mathbf{F}_r = 2\,\mathrm{diag}\left(\sum_{\ell=N_u+1}^{N}\gamma_{1,\ell},\ldots,\sum_{\ell=N_u+1}^{N}\gamma_{N_u,\ell}\right)$, accounts for the links between each unknown node and the N_r reference nodes. On the other hand, the matrix

$$\mathbf{F}_u = 2\begin{bmatrix} \sum_{\ell=1}^{N_u}\gamma_{1,\ell} & -\gamma_{1,2} & \cdots & -\gamma_{1,N_u} \\ -\gamma_{2,1} & \sum_{\ell=1}^{N_u}\gamma_{2,\ell} & & -\gamma_{2,N_u} \\ \vdots & & \ddots & \\ -\gamma_{N_u,1} & -\gamma_{N_u,2} & \cdots & \sum_{\ell=1}^{N_u}\gamma_{N_u,\ell} \end{bmatrix} \tag{12}$$

depends on the connectivity graph of the unknown nodes' network [9]. In case of no reference nodes ($N_r = 0$), we have $\mathbf{F}_r = \mathbf{0}$ and $\tilde{\mathbf{F}} = \mathbf{F}_u$ is singular (all raws sum to zero): the estimation of the absolute coordinates is not possible in this case, only relative localization may be achieved. Remember, indeed, that in a 1D geometry absolute localization requires the use of $N_r \geq 1$ reference nodes for the existence of solution, and $N_r \geq 2$ for the uniqueness (to solve flip ambiguity)[1].

For 2D networks the FIM has a more complex structure with respect to (11). Still, a simplified lower bound for (9) can be obtained by observing that it is [8]:

$$\mathbf{C}_{xx} = (\mathbf{F}_{xx} - \mathbf{F}_{xy}\mathbf{F}_{yy}^{-1}\mathbf{F}_{yx}^{-1})^{-1} \geq \mathbf{F}_{xx}^{-1} \tag{13}$$

$$\mathbf{C}_{yy} = (\mathbf{F}_{yy} - \mathbf{F}_{yx}\mathbf{F}_{xx}^{-1}\mathbf{F}_{xy}^{-1})^{-1} \geq \mathbf{F}_{yy}^{-1} \tag{14}$$

with equality for $\mathbf{F}_{xy} = \mathbf{0}$ (i.e., the FIM is block-diagonal, the estimation of $\mathbf{x}$ and $\mathbf{y}$ can be decoupled into two separate 1D problems). Furthermore, for any pair of positive definite matrices $\mathbf{F}_{xx}$ and $\mathbf{F}_{yy}$ the series-parallel inequality holds: $\mathbf{F}_{xx}^{-1} + \mathbf{F}_{yy}^{-1} \geq 4\left(\mathbf{F}_{xx} + \mathbf{F}_{yy}\right)^{-1}$ [10]. From (6)-(8) and using the relation $\Delta x_{k,\ell}^2 + \Delta y_{k,\ell}^2 = d_{k,\ell}^2$, it follows that the elements of the matrix $\mathbf{F}_{xx} + \mathbf{F}_{yy}$ have the same expression as in (11) for large SNR. Thus we get the simplified bound:

$$\mathrm{MSE}_k \geq \left[\mathbf{F}_{xx}^{-1}\right]_{k,k} + \left[\mathbf{F}_{yy}^{-1}\right]_{k,k} \geq 4\left[\left(\mathbf{F}_{xx} + \mathbf{F}_{yy}\right)^{-1}\right]_{k,k} = 4\left[\tilde{\mathbf{F}}^{-1}\right]_{k,k} \tag{15}$$

Even though this bound may be non realistic for complex topologies, in Sec. IV it will be shown to be close to the CRB for large isotropic regular networks with nodes uniformly and equally distributed along the two coordinate axes.

IV. Analysis of Regular 1D/2D Networks

We start by considering a 1D uniform linear array (ULA) composed of N nodes uniformly spaced along a straight line at positions $\theta_k = x_k = (k-1)\Delta$, $k = 1,\ldots,N$, with inter-distance $\Delta = 5$m. Of these nodes, N_r are reference devices placed uniformly among the unknown nodes (see the deployment in Fig. 2, where 'o' denotes unknown node, while 'x' is anchor). According to the CRB of TOA estimation [2], the range accuracy is $\sigma_0 = c/(2\sqrt{2\cdot\mathrm{SNR}_0}B)$, where c is the speed of light, $\mathrm{SNR}_0 = 10$dB is the SNR at the reference distance $d_0 = 1$m and $B = 500$MHz is the effective signal bandwidth. The path-loss exponent is $\alpha = 4$. The coverage radius ranges from $r = \Delta$ (each node makes measurements with up to 2 neighbors) to $r \to \infty$ (full coverage).

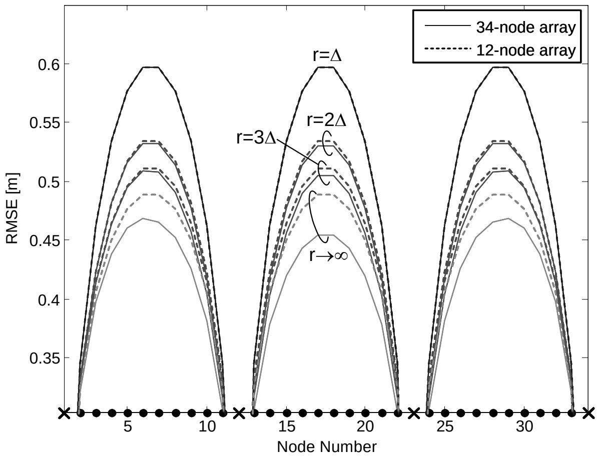

Fig. 2. CRB for a 12-node (dashed line) and a 34-node (solid line) sensor arrays, for different values of coverage radius r.

An initial study of regular 1D networks shows that a rough estimate of the localization accuracy can be obtained by reducing the problem to a single sub-array deployment for limited node coverage (i.e., low r or high α). In Fig. 2 the CRB on the location accuracy is plotted along the ULA for two different deployments: an array of $N = 34$ nodes with $N_r = 4$ (solid line); three separate (non cooperating) arrays having each $N = 12$ nodes and $N_r = 2$ reference nodes at the ends (dashed line). For $r \leq 3\Delta$ the performance of the 12-node array is almost the same as that of the 34-node array, as far as the density of the anchors is constant. This means that we can approximate the 30×30 FIM of the first deployment by a block-diagonal structure having over the main diagonal three 10×10 blocks equal to the FIM of the 12-node sub-array.

Based on the above result, we can now focus on a single regular array with $N_r = 2$ anchors at the two ends. Due to the uniform deployment, it is $d_{k,\ell} = |k-\ell|\Delta$, and the FIM elements are $[\mathbf{F}\ ]_{k,\ell} = f_{k,\ell}\cdot 2\cdot[(\Delta/d_0)^\alpha \cdot \sigma_0^2]^{-1}$ with

$$f_{k,\ell} = \begin{cases} \sum_{m=1,k-m\leq N_n}^{k}\frac{1}{m^\alpha} + \sum_{m=1,m\leq N_n}^{N-1-k}\frac{1}{m^\alpha}, & \text{if } k=\ell \\ -\frac{1}{|k-\ell|^\alpha}, & \text{if } 0<|k-\ell|\leq N_n \\ 0, & \text{if } |k-\ell|>N_n \end{cases}$$

where $N_n = \lfloor r/\Delta \rfloor$ is half the number of nodes within a distance r (i.e., the number of cooperating neighbors on each side of the node). It is worth noticing that, for large N, the FIM tends to be a Toeplitz matrix having elements: $f_{k,k} \approx t_0 = 2\sum_{m=1}^{N_n}\frac{1}{m^\alpha}$ and $f_{k,k+m} = t_m = -\frac{1}{m^\alpha}$, with $N_n = \lfloor (N-1)/2 \rfloor$ for full coverage. The strength of this assumption is shown in Fig. 3 where the true CRB (solid

[1]For conventional 1D (or 2D) localization, it is well known that existence and uniqueness of solution to the TOA-based localization problem is guaranteed when each node is linked to at least 2 (or 3) reference nodes. For cooperative localization the problem of solvibility has been investigated using graph rigidity theory in [11]. A sufficient condition is full connectivity (i.e., complete graph) with $N_r \geq 2$ (or $N_r \geq 3$).

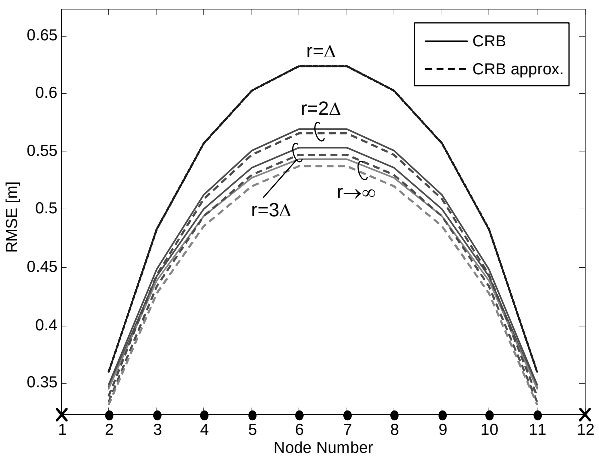

Fig. 3. True CRB and its approximation by a Toeplitz FIM for a 12-node ULA with different values of the coverage radius.

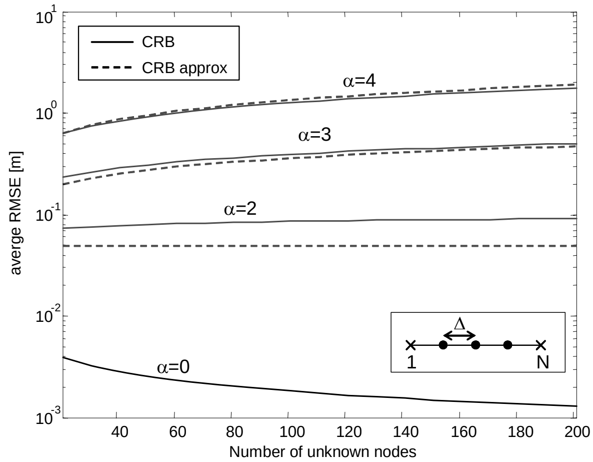

Fig. 4. Average CRB along the array vs. the number of unknowns for different values of path-loss exponent: true CRB (solid line) and approximation (19) (dashed line). The array geometry is represented in the box.

line) and the one computed with the Toeplitz approximation (dashed line) are plotted along a sensor array of $N = 12$ nodes for different values of r. The approximation is particularly good for low coverage radius (for $r = \Delta$ the two lines are perfectly superimposed). Notice also that the positioning accuracy improves for increasing node coverage, but not so significantly for $r > 2\Delta$ (due to the path-loss).

An extreme case is $\alpha = 0$ and $r \to \infty$ (all nodes cooperate with each other providing range measurements with constant variance σ_0^2, regardless of the inter-node distance), which leads to $t_0 = N - 1$ and $t_n = -1$. In such a situation the inverse FIM $\mathbf{C} = \tilde{\mathbf{F}}^{-1}$ can be easily calculated in closed form, leading to the following bound on the average location accuracy:

$$MSE \geq \frac{\mathrm{tr}(\mathbf{C})}{N_u} = \sigma_0^2 \frac{1}{2N}\left(1 + \frac{1}{N_r}\right). \tag{16}$$

For a fixed number of anchors, the above CRB decreases as $1/N$ for increasing network size, as the number of unknowns goes with N while the number of measurements grows with N^2. On the other hand, for $\alpha > 0$, the reliable measurements reduce and the MSE may increase with the network size. This is shown in the following by deriving a close form for the CRB for $\alpha > 0$ based on the Toeplitz FIM approximation.

The MSE bound (10) for 1D networks depends on the trace of the matrix $\mathbf{C}$ and thus on the sum of the inverse eigenvalues of $\tilde{\mathbf{F}}$. Let $\{\mu_i\}_{i=0}^{N_u-1}$ be the eigenvalues of $\tilde{\mathbf{F}}$ sorted by decreasing order, experimental results not included here have indicated that for limited coverage (large α or small r) the trace of $\mathbf{C}$ is dominated by the minimum eigenvalue μ_{N_u-1}: $\sum_{k=0}^{N_u-1}\frac{1}{\mu_k} \approx \frac{1}{\mu_{N_u-1}}$. Furthermore, for Toeplitz matrices the following lower bound holds [12]: $\mu_{N_u-1} \geq \frac{1}{2}(\min\{\lambda_{2i}\} + \min\{\lambda_{2i+1}\})$, where $\{\lambda_i\}_{i=1}^{2N_u-1}$ are the coefficients of the order-$2N_u$ discrete Fourier transform (DFT) of $\{t_i\}_{i=-(N_u-1)}^{N_u-1}$. These can be expressed as:

$$\lambda_i = t_0 + 2\sum_{m=0}^{N_n} t_m \cos\left(\frac{2\pi i m}{2N_u}\right) = 2\sum_{m=1}^{N_n}\frac{1}{m^\alpha}\left[1 - \cos\left(\frac{\pi i m}{N_u}\right)\right]. \tag{17}$$

It follows that $\min\{\lambda_{2i}\} = \lambda_0 = 0$, $\min\{\lambda_{2i+1}\} = \lambda_1$, with:

$$\lambda_1 = 2\sum_{m=1}^{N_n}\frac{1}{m^\alpha}\left[1 - \cos\left(\frac{\pi m}{N_u}\right)\right] \approx \sum_{m=1}^{N_n}\frac{1}{m^\alpha}\left(\frac{\pi m}{N_u}\right)^2 = \frac{\pi^2\Psi(\alpha)}{N_u^2} \tag{18}$$

where we used the truncated Taylor series expansion $\cos x \approx 1 - 1/x^2$ for $x \ll 1$ and we introduced the function of the path-loss exponent $\Psi(\alpha) = \sum_{m=1}^{N_n}\frac{1}{m^{\alpha-2}}$. It follows that:

$$\mathrm{MSE} \geq \frac{\mathrm{tr}(\mathbf{C})}{N_u} \approx \frac{1}{N_u}\cdot\frac{1}{\mu_{N_u-1}} \approx 2\sigma_0^2\left(\frac{\Delta}{d_0}\right)^\alpha \frac{1}{\Psi(\alpha)\pi^2}N_u. \tag{19}$$

This bound is validated in Fig. 4 by comparing the true CRB with the approximation (19) for $\alpha = 2, 3, 4$ and $r \to \infty$. The true CRB for $\alpha = 0$ is shown as a reference. Notice that the approximation (19) has been derived under the assumption of limited coverage and thus it is accurate only for large α (or small r). The term $\Psi(\alpha)$ gives the dependence of the positioning accuracy on the connectivity degree of the WSN: the higher is the number of reliable connections for each node (i.e., the effective number of cooperating neighbors), the larger is $\Psi(\alpha)$. For the limit case $\alpha = 2$ and $r \to \infty$ it is $\Psi(\alpha) = (N-1)/2 \approx N_u/2$ and the MSE is approximately constant for increasing N_u. On the other hand, for $\alpha > 2$ the effective number of connections decreases rapidly: it is $\Psi(\alpha) = 1.6$ for $\alpha = 4$, $\Psi(\alpha) = 1.2$ for $\alpha = 5$ and $\Psi(\alpha) \to 1$ for larger α. From (19) we can thus conclude that the average accuracy is linearly related to N_u for $\alpha > 2$, as confirmed by the results in Fig. 4. The same holds for limited coverage radius, e.g. for $r = 1$ which leads to $\Psi(\alpha) = 1\ \forall\alpha$.

The analysis is finally extended to regular 2D ULA, using the same parameters as for the previous simulations with $\alpha = 4$. Fig. 5 shows a network of $N = 25$ nodes, with $N_u = 21$ unknowns (indicated with 'o') and $N_r = 4$ anchors (indicated with 'x'). The ellipses (red line) plotted around the unknown nodes represent the CRB of the location accuracy obtained from the 2×2 covariance sub-matrices of (5); the black circles are calculated from (9), while the blue circles from (15). We

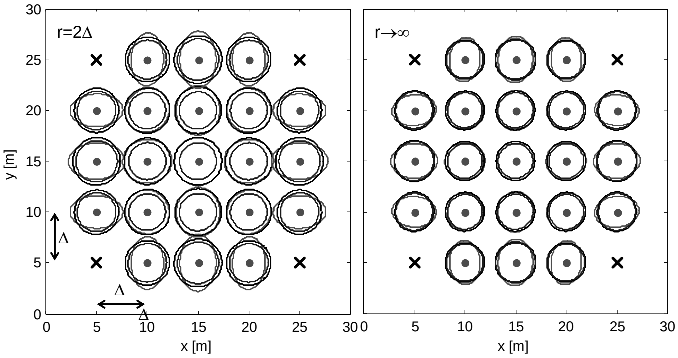

Fig. 5. Location uncertainty ellipses (red) and circles with radius MSE_k (blue) compared with the lower bound (15) (black) for $r = 2\Delta$ (left) and $r \to \infty$ (right), for a 2D regular WSN with $N = 4 \times 4$, $N_r = 2 \times 2$, $\Delta = 5$m. The error has been multiplied by 4 for visualization purposes.

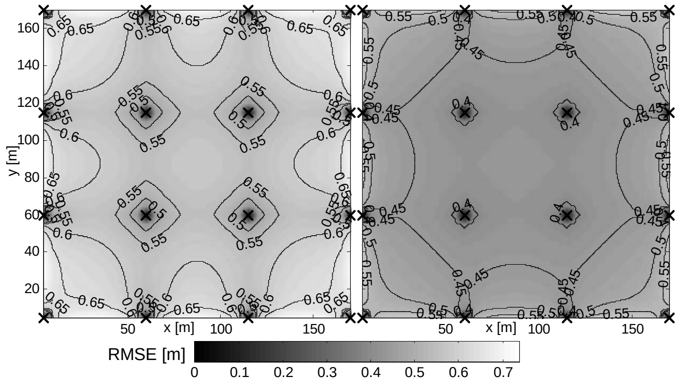

Fig. 6. CRB on the location accuracy for $r = 2\Delta$ (left) and $r \to \infty$ (right) for a 2D regular WSN with $N = 34 \times 34$, $N_r = 4 \times 4$, $\Delta = 5$m.

can observe that for a regular network the uncertainty ellipses tend to be circles, especially for central nodes that are not affected by border effects. This justifies the reduction of the 2D problem to the 1D discussion. Moreover, it is clearly visible that for growing coverage radius the approximation (15) tends to the true CRB. Fig. 6 shows the RMSE on a gray scale for $N = 34 \times 34$ and $N_r = 4 \times 4$. We can observe that the location accuracy tends to be uniform on the 2D area apart from locations close to the anchor nodes.

V. Conclusions

In this paper, we investigated the performance limits of TOA-based cooperative localization. We extended previous CRB analyses in the literature in order to account for the path-loss effects. We focused, in particular, on regular 1D-2D network topologies. For 1D deployments, analytical results, validated by numerical simulations, have shown how the location accuracy is related to the number of known (N_r) and unknown (N_u) nodes and to the network connectivity. In particular, for ULAs with fixed N_r, the CRB on the average location MSE increases linearly with N_u in case of limited connectivity among the nodes (e.g., for small coverage radius r or path-loss exponent $\alpha > 2$). The coefficient of proportionality, $\Psi(\alpha)$, has been calculated in closed form showing the dependence on the number of reliable connections for each node (i.e., the effective number of cooperating neighbors). An approximated CRB has been derived also for regular 2D networks. A further development will be a more in-depth investigation of 2D networks and the extension to more realistic WSN topologies modeled, e.g., as random networks.

Appendix

The FIM sub-blocks are calculated as follows [8]:

$$[\mathbf{F}_{\mu\nu}]_{k,\ell} = [\mathbf{H}_\mu^{\mathrm{T}} \mathbf{Q}^{-1} \mathbf{H}_\nu]_{k,\ell} + \frac{1}{2} \operatorname{tr}\left(\mathbf{Q}^{-1} \frac{\partial \mathbf{Q}}{\partial \mu_k} \mathbf{Q}^{-1} \frac{\partial \mathbf{Q}}{\partial \nu_\ell} \right) \quad (20)$$

where μ and ν represent either x or y, $\mathbf{Q} = \mathbf{Q}(\boldsymbol{\theta})$, $\mathbf{H}_x$ and $\mathbf{H}_y$ denote the two components of the $M \times 2N_u$ gradient $\mathbf{H} = \frac{\partial \mathbf{h}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = [\mathbf{H}_x \ \mathbf{H}_y]$. The element (k, ℓ) of each sub-block (20) is the sum of two terms: the first one is taken from the matrix $\mathbf{H}^{\mathrm{T}} \mathbf{Q}^{-1} \mathbf{H}$ and it is already known from [1]; the second one is obtained by computing the derivatives of $\mathbf{Q}$ with respect to x_k and y_k. Let $i = i(k, \ell) = 1, \ldots, M$ be the indexing function giving the position of the measurement (k, ℓ) in the vector $\mathbf{z}$, the derivatives can be calculated according to (3) as:

$$\left[\frac{\partial \mathbf{Q}}{\partial \mu_k}\right]_{i,i} = \begin{cases} \frac{\sigma_{k,\ell}^2 \alpha}{d_{k,\ell}^2} \Delta\mu_{k,\ell}, & \text{if } i = i(k,\ell) \\ \frac{\sigma_{\ell,k}^2 \alpha}{d_{\ell,k}^2} \Delta\mu_{\ell,k}, & \text{if } i = i(\ell,k) \\ 0, & \text{otherwise} \end{cases} . \quad (21)$$

This result together with (20) leads to the FIM values (6)-(8).

Acknowledgements

The authors wish to acknowledge the activity of the Network of Excellence NEWCOM++ of the European Commission (contract n. 216715) that motivated this work.

References

[1] N. Patwari et al., "Locating the nodes," *IEEE Signal Processing Mag.*, Vol. 22, No. 4, pp. 54-69, July 2005.

[2] S. Gezici et al., "Localization via ultra-wideband radios: a look at positioning aspects for future sensor networks," *IEEE Signal Processing Mag.*, vol. 22, pp. 70–84, July 2005.

[3] E.G. Larsson, "Cramer-Rao bound analysis of distributed positioning in sensor networks," *IEEE Signal Processing Letters*, Vol. 11, No. 3, pp. 334-337, March 2004.

[4] Y. Shen, H. Wymeersch, M.Z. Win, "Fundamental Limits of Wideband Cooperative Localization via Fisher Information," *in Proc. IEEE WCNC'07*, pp. 3951 - 3955, March 2007.

[5] Y. Qi, H. Kobayashi, H. Suda, "On time-of-arrival positioning in a multipath environment," *IEEE Trans. Veh. Technol.*, Vol. 55, No. 5, Sept. 2006.

[6] D. B. Jourdan, D. Dardari, and M. Z. Win, "Position error bound and localization accuracy outage in dense cluttered environments," *in Proc. IEEE Int. Conf. on Ultra-Wideband*, Waltham, MA, September 2006.

[7] J. N. Ash, R. L. Moses, "On the relative and absolute positioning errors in self-localization systems," *IEEE Trans. Signal Processing*, Vol. 56, No. 11, Nov. 2008.

[8] S. M. Kay, *Fundamentals of Statistical Processing, Volume I: Estimation Theory.* Englewood Cliffs, NJ: Prentice-Hall.

[9] C. Godsil, G. Royle, *Algebraic Graph Theory*. New York: Springer-Verlag, 2001.

[10] W. N. Anderson and R. J. Duffin, "Series and parallel addition of matrices," *J. Math. Anal. Appl.*, vol. 26, pp. 576–594, 1969.

[11] T. Eren et al., "Rigidity, computation, and randomization in network localization," *in IEEE Proc. INFOCOM'04*, Vol. 4, pp. 2673- 2684, March 2004.

[12] P.J.S.G. Ferreira, "Localization of the Eigenvalues of Toeplitz Matrices using additive decomposition embedding in circulants, and the Fourier transform," *in Proc. IFAC SYSID'94*, pp. 271-276, July 1994.