

# CONSENSUS IN NETWORKED MULTI-AGENT SYSTEMS

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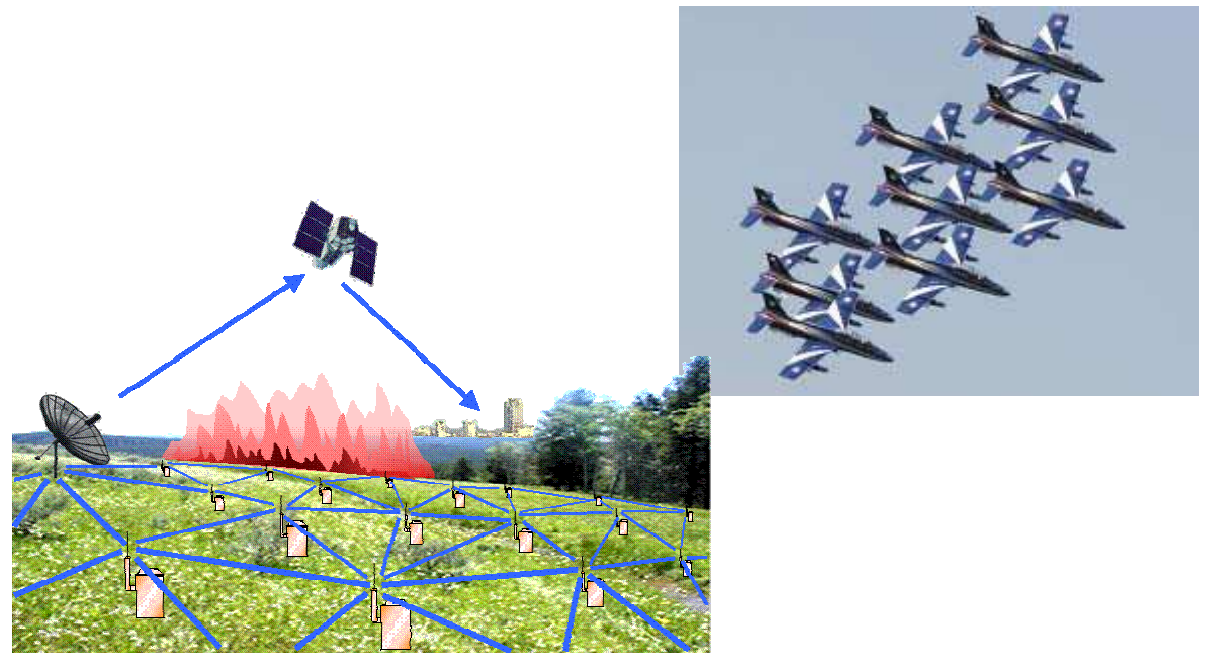
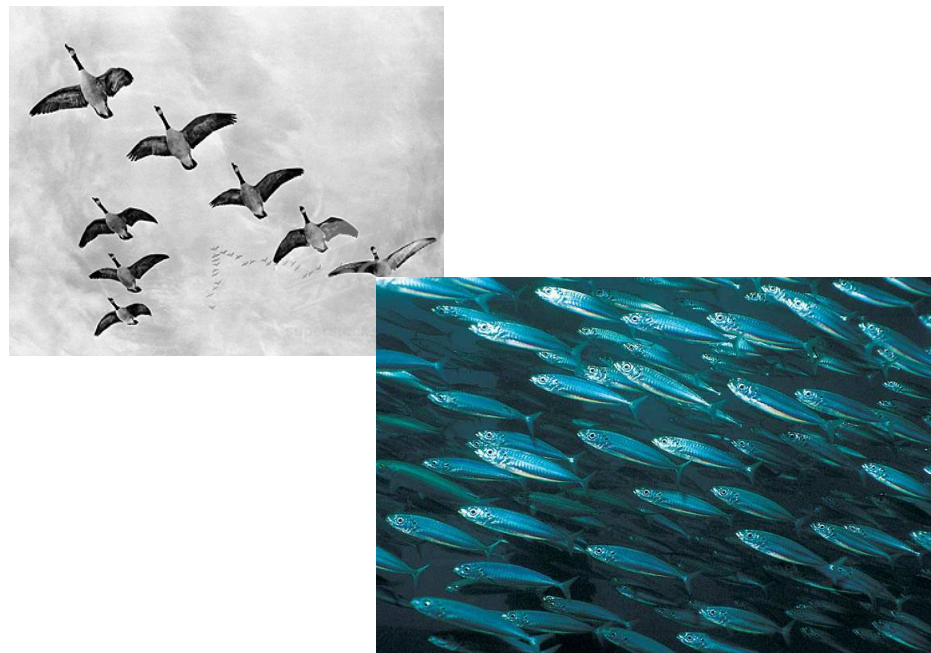
# CONSENSUS

Given a set of **agents** (=dynamical systems):

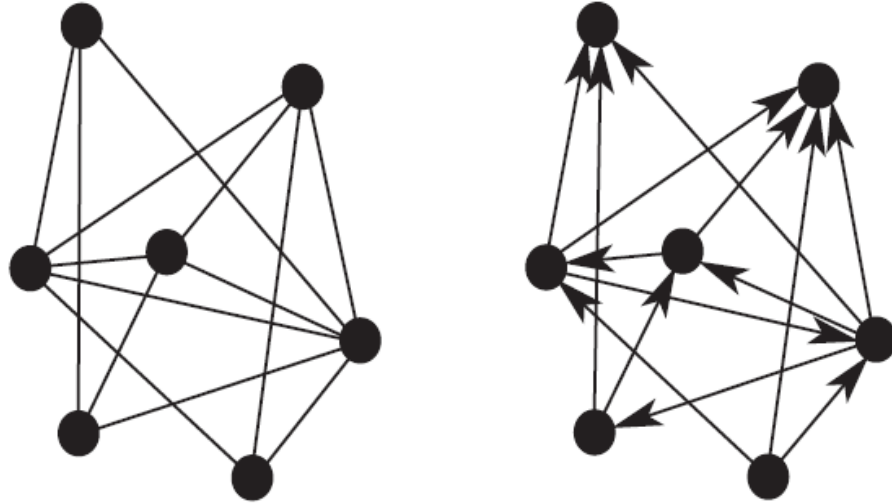
- "**consensus**" means to reach an **agreement** regarding a certain **quantity** of interest, that depends on the **state** of all agents;
- "**consensus algorithm**" (or protocol) is the **interaction rule** that specifies the **information exchange** between agent.

*Analyzing the consensus phenomena...*

*...designing the consensus of multi-agent systems*



## CONSENSUS ON NETWORKS



Agents are connected through a **network**:

*communication only with neighbours*

Each node  $i = 1, 2, \dots, N$  hosts a system

$\dot{x}_i(t) = f(x_i(t), u_i(t))$ , continuous-time model

$x_i(t + 1) = f(x_i(t), u_i(t))$ , discrete-time model

$u_i(t) = \text{input} = \text{interaction with other systems}$

(unconstrained) **consensus**:  $x_1 = x_2 = \dots = x_N$  for  $t \rightarrow \infty$

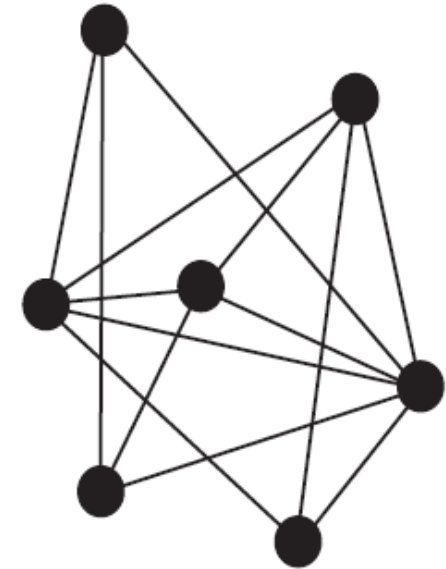
Can consensus be reached with **partial exchange of information** only (=local, distributed, decentralized)?

## Consensus in a network of $N$ integrator agents

Each node hosts the "simplest" linear system:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \dots, N, \quad x_i \text{ scalar}$$

Undirected, connected network,  $V_i = \{j | a_{ij} = 1\} = \text{neighbours of } i$



"diffusive" interaction: proportional to the **state unbalance**

$$\dot{x}_i(t) = u_i(t) = \sum_{j \in V_i} (x_j(t) - x_i(t))$$

The **collective dynamics** of  $x = [x_1 \ x_2 \ \dots \ x_N]^T$  are governed by

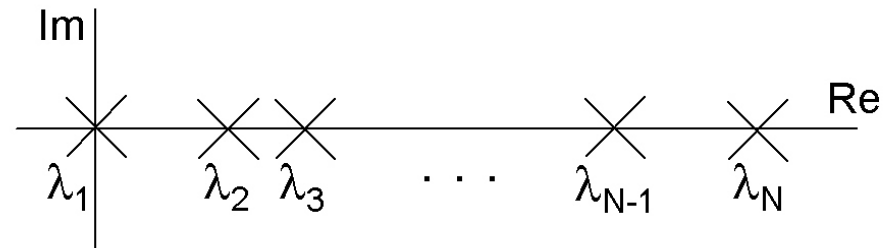
$$\dot{x} = -Lx$$

where  $L = \text{diag}(k_1, k_2, \dots, k_N) - A$  is the  $N \times N$  **Laplacian matrix** of the network.

**The collective dynamics depend on the graph topology!**

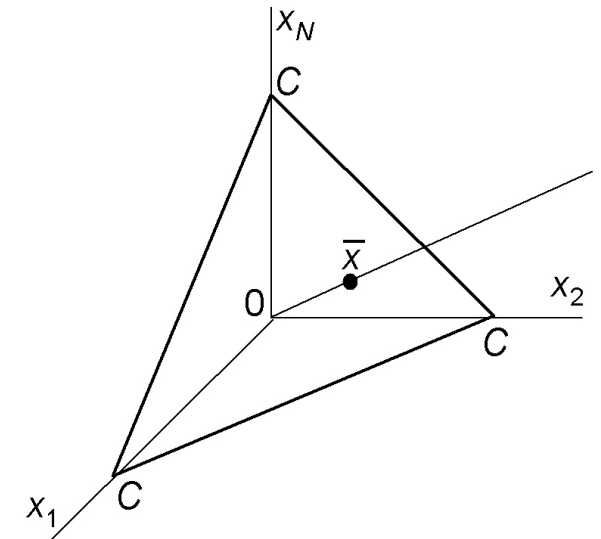
## Main results:

- $L$  is **symmetric, irreducible, zero-row-sum**, and  $(-L)$  is **Metzler**: eigenvalues are real and  $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$



- Possible **equilibria** are all the (infinitely many) states with  $x_1 = x_2 = \dots = x_N$ .
- Since  $\sum_i \dot{x}_i = 0$ , it turns out that  $\sum_i x_i(t) = \sum_i x_i(0) = C$  (constant).
- Thus the **only feasible equilibrium**  $\bar{x}$  has

$$\bar{x}_1 = \bar{x}_2 = \dots = \bar{x}_N = (1/N) \sum_i x_i(0)$$

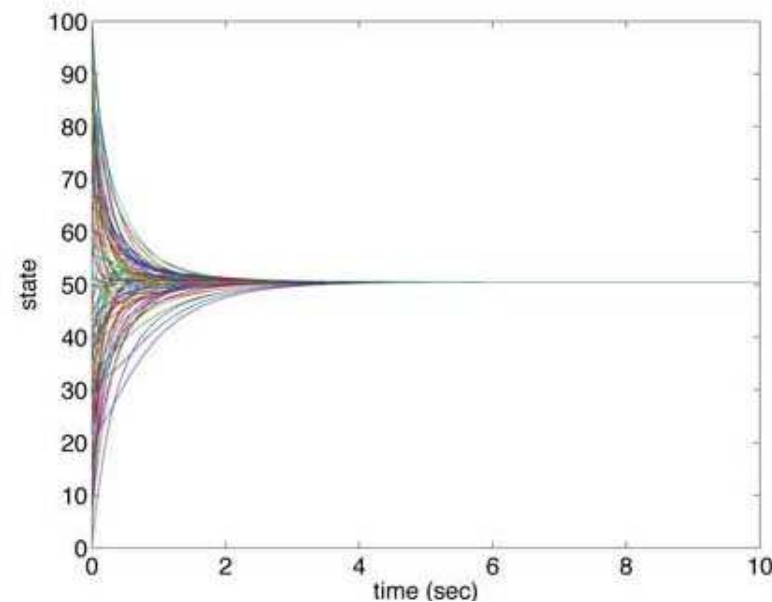
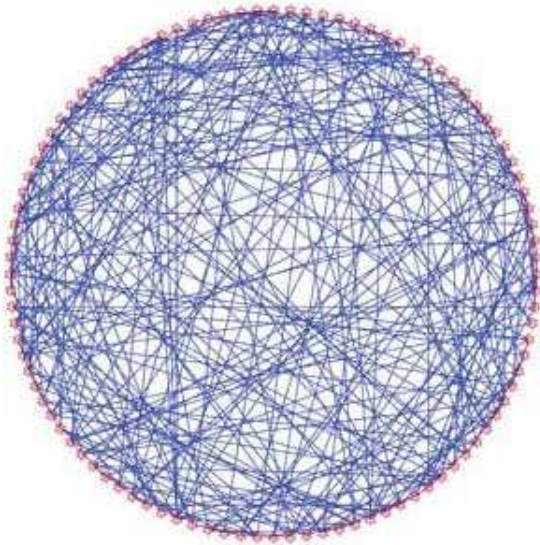


- $\lambda_1 = 0$  governs the (trivial) dynamics along the eigenvector  $[1 \ 1 \ \dots \ 1]^T$ .
- $-\lambda_N \leq \dots \leq -\lambda_2 < 0$  govern the **dynamics on the plane**  $\sum_i x_i(t) = C$ .

All non-dominant eigenvalues of  $(-L)$  are negative  $\Rightarrow$  asymptotic convergence to

$$\bar{x} = \left( \frac{1}{N} \sum_i x_i(0) \right) [1 \ 1 \ \dots \ 1]^T$$

- **consensus reached**: each system converges to the **mean value** of the initial states
- the **convergence speed** is dominated by the first sub-dominant eigenvalue, i.e.  $x_i(t) - \bar{x}_i \approx \exp(-\lambda_2 t)$ : the speed of convergence **depends on the topology** (the larger  $\lambda_2$ , the faster the convergence).

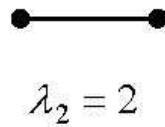


## Network topology and the speed of consensus

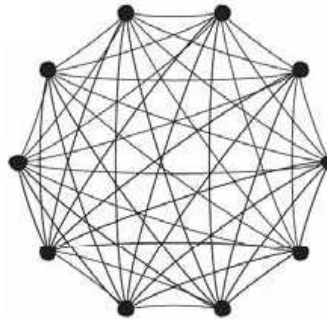
The larger  $\lambda_2$  ("*algebraic connectivity*"), the faster the convergence.

How does  $\lambda_2$  depend on the **network topology**?

the **two-node** net

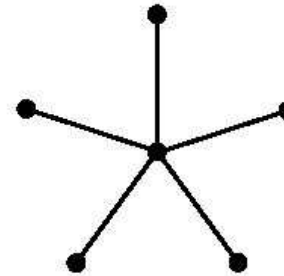


**complete** nets



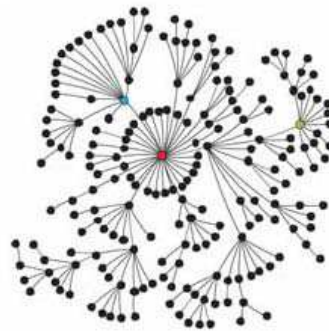
$$\lambda_2 = \dots = \lambda_N = N$$

**star** nets



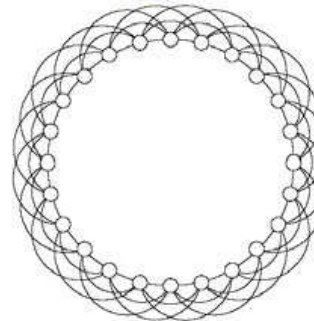
$$\lambda_2 = \dots = \lambda_{N-1} = 1$$
$$\lambda_N = N$$

**scale-free** nets



$$\lambda_2 \approx 1$$

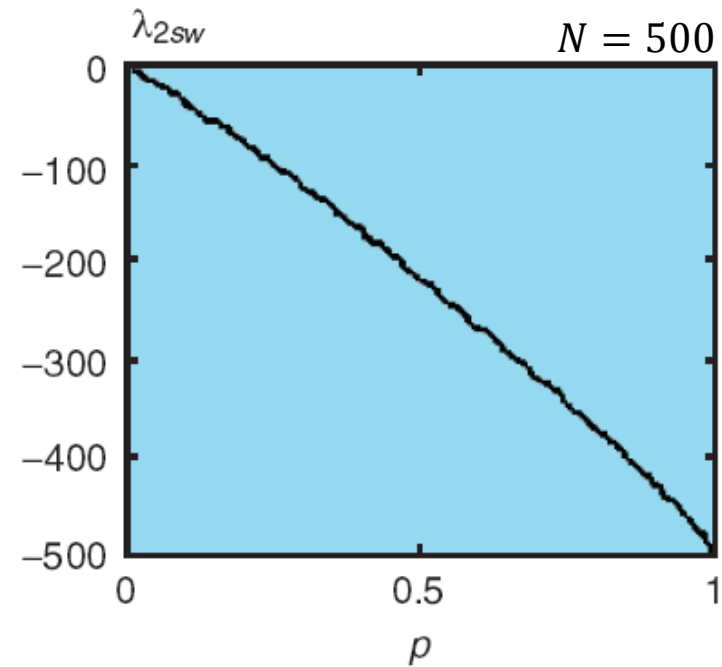
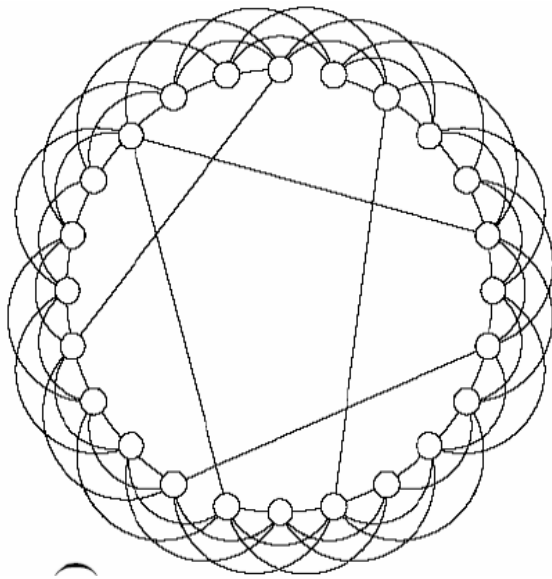
**Watt-Strogatz** loops



$$\lambda_2 \rightarrow 0, \frac{\lambda_N}{\lambda_2} \sim N^2 \text{ as } N \rightarrow \infty$$



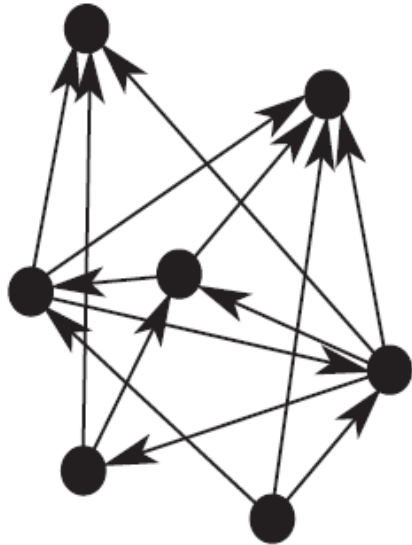
Given a Watts-Strogatz loop, the speed of convergence can be greatly increased by adding "long-distance" connections ([small-world network](#)).





Directed networks (Laplacian matrix:  $L_{ii} = k_i^{out}$ ,  $L_{ij} = -a_{ij}, i \neq j$ ):

If the network is **strongly connected**:



- **consensus reached**: each system converges to

$$\bar{x} = \alpha [1 \quad 1 \quad \dots \quad 1]^T$$

$\alpha = \sum_i w_i x_i(0)$  and  $\sum_i w_i = 1$ , thus  $\alpha$  is a **weighted average** of the initial states.

- $\alpha$  depends on the **network topology**:  $[w_1 \quad w_2 \quad \dots \quad w_N]$  is a left eigenvector of the Laplacian  $L$  for  $\lambda_1 = 0$ :

$$[w_1 \quad w_2 \quad \dots \quad w_N]L = 0$$

- If the network is **balanced** ( $k_i^{in} = k_i^{out}$  for all  $i$ ), then

$\alpha = \left(\frac{1}{N} \sum_i x_i(0)\right)$ : convergence to the **average initial state**

- The **speed** of convergence is dominated by the first sub-dominant eigenvalue of  $L_S = (L + L^T)/2$ .

## Possible extensions:

- different forms of the **coupling function**  $u_i = h(x)$
- network design for **optimizing consensus performance**
- other (less trivial) **agent models** (linear or nonlinear)
- **time-varying network topology** (e.g., mobile agents)
- ...
- consensus on oscillatory behaviour: **synchronization**