

SYNCHRONIZATION OF NETWORKED OSCILLATORS

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KURAMOTO MODEL (Kuramoto, 1984)

N "planar rotors", with **phase** ϕ_i and **natural frequency** ω_i : $\frac{d\phi_i}{dt} = \omega_i$

ω_i -s are **random**, extracted from a unimodal distribution with **mean** Ω

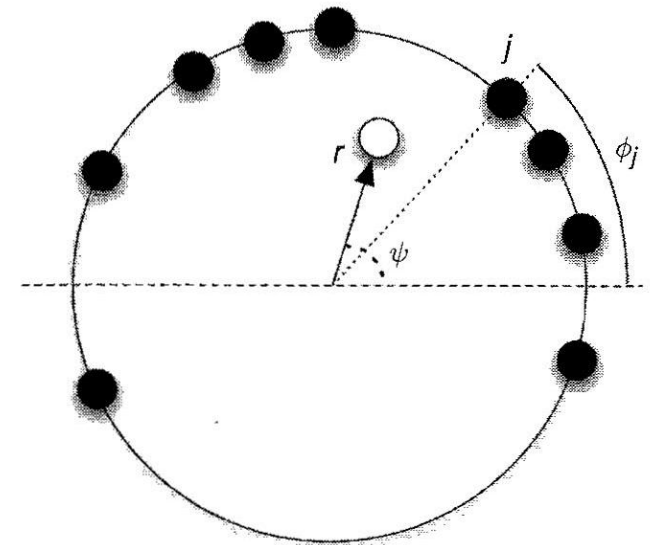
Nonlinear **coupling** ($V(i) =$ neighbours of i):

$$\frac{d\phi_i}{dt} = \omega_i + K \sum_{j \in V(i)} \sin(\phi_j - \phi_i)$$

Synchronization can be quantified by the "**order parameter**" $r(t)$ (centre of mass of the oscillators):

$$r(t)e^{i\psi(t)} = \frac{1}{N} \sum_{j=1}^N e^{i\phi_j(t)}$$

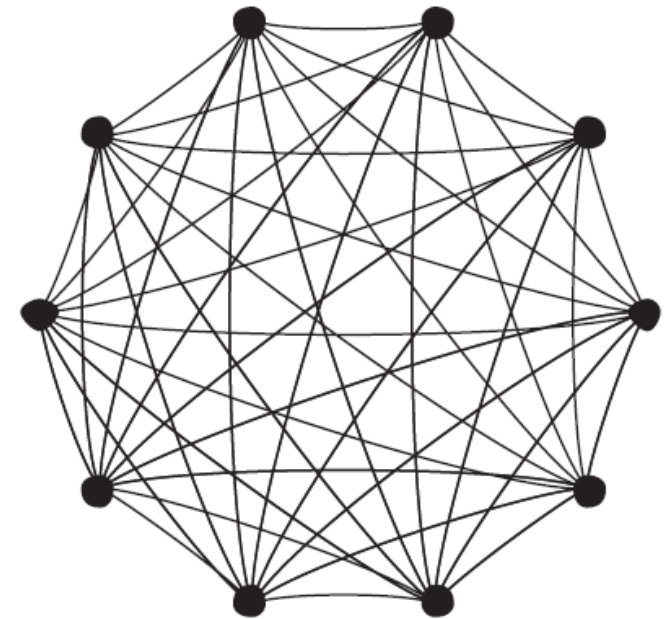
- $r(t) \rightarrow 0$: **no** synchronization
- $r(t) \rightarrow 1$: synchronization of **all** rotors (=identical $d\phi_i/dt$ -s)
- If $r(t)$ tends to a nonzero value, a **fraction** of oscillators is synchronized.



Mean-field Kuramoto model

- N oscillators on a **complete network**
- coupling strength $K = K^0/N$
- combining $\frac{d\phi_i}{dt} = \omega_i + \frac{K^0}{N} \sum_j \sin(\phi_j - \phi_i)$ with $r(t)e^{i\psi(t)} = \frac{1}{N} \sum_{j=1}^N e^{i\phi_j(t)}$ we obtain:

$$\frac{d\phi_i}{dt} = \omega_i + K^0 r \sin(\psi - \phi_i)$$

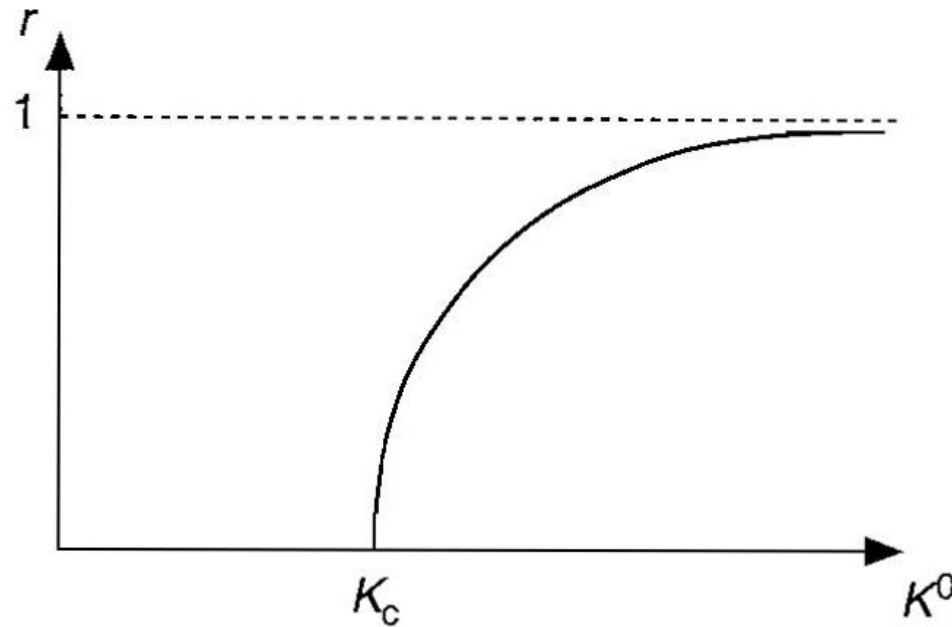


The coupling term depends on the "**mean phase**" ψ .

The coupling strength $K^0 r$ increases with r : **positive feedback**
(the largest the number of synchronized oscillators, the largest the chance to capture the remaining ones)

Numerical + analytical results: existence of a **threshold value** K_C

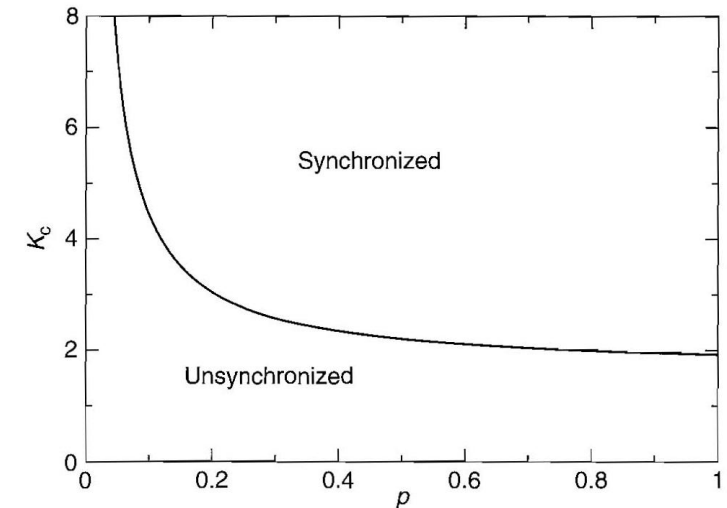
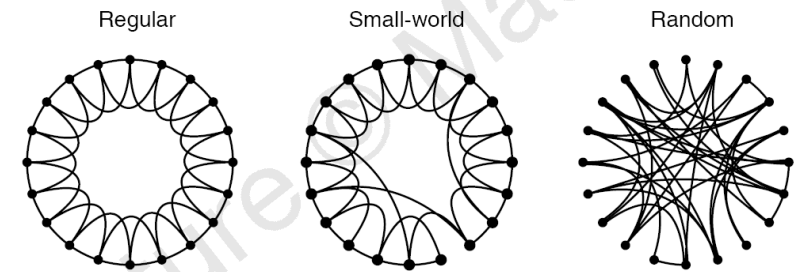
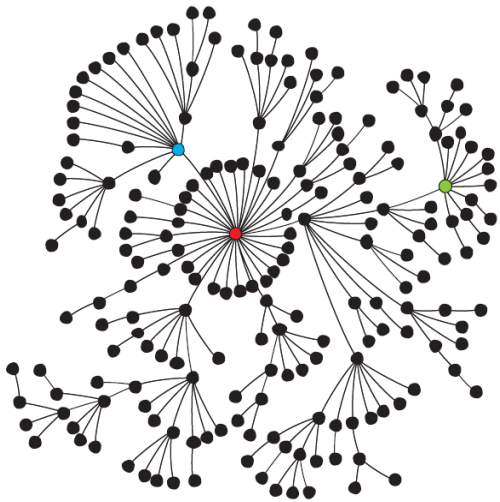
- $K^0 < K_C$: **no** synchronization ($r(t) \rightarrow 0$)
- $K^0 > K_C$: synchronization of larger and larger **fractions** of oscillators ($r(t) \rightarrow 1$ as $K^0 \rightarrow \infty$)
- when $r(t) \rightarrow r < 1$, the **de-synchronized oscillators** are those with largest **detuning** $|\omega_i - \Omega|$.



Kuramoto model on complex networks

Regular lattices: $K_C \rightarrow \infty$ as $N \rightarrow \infty$ (no synchronization)

Small-world networks (p = prob. of rewiring): K_C decreases with p



Heterogeneous networks (e.g., scale-free): $K_C \approx \frac{\langle k \rangle}{\langle k^2 \rangle}$

$K_C \rightarrow 0$ in the limit $N \rightarrow \infty$: **hubs drive the dynamics** and impose synchronization

COMPLETE SYNCHRONIZATION IN NETWORKED OSCILLATORS

Isolated nodes i ($i = 1, 2, \dots, N$) represent **identical**, autonomous, n -dimensional dynamical systems

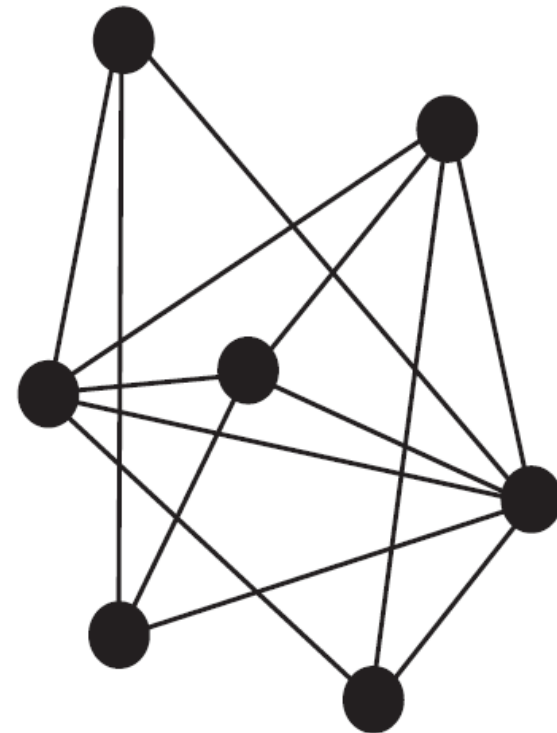
$$\dot{x} = f(x) \quad , \quad x \in R^n$$

Their behaviour (**when isolated**) is **oscillatory** (periodic or chaotic).

The coupling is linear (diffusive):

$$\dot{x}^{(i)} = f(x^{(i)}) + \sum_{j:a_{ij}=1} d(H(x^{(j)} - x^{(i)}))$$

- $d \geq 0$ is the **coupling strength**,
- H is a $n \times n$ nonnegative matrix (**diffusion profile**) specifying which variables interact



Example: Each node represents a geographic location (island, patch) which is the habitat of a three-trophic food chain (resource R , consumer C , predator P).

The isolated demographic dynamics are described by the classical Rosenzweig-MacArthur model:

$$\begin{aligned}\dot{R} &= rR\left(1 - \frac{R}{k}\right) - \frac{a_1 R}{1 + a_1 b_1 R} C \\ \dot{C} &= e_1 \frac{a_1 R}{1 + a_1 b_1 R} C - d_1 C - \frac{a_2 C}{1 + a_2 b_2 C} P \\ \dot{P} &= e_2 \frac{a_2 C}{1 + a_2 b_2 C} P - d_2 P\end{aligned}$$

In this case H is diagonal: it specifies which are the species that disperse and sets the relative dispersal rates, e.g:

- only R disperses (e.g., seeds transported by the wind): $H = H' = \text{diag}[1 \ 0 \ 0]$
- only C disperses (e.g., herbivores): $H = H'' = \text{diag}[0 \ 1 \ 0]$
- only P disperses (e.g., carnivores): $H = H''' = \text{diag}[0 \ 0 \ 1]$
- all variables disperse, at different rates: $H = H'''' = \text{diag}[1 \ 0.1 \ 0.01]$

The **overall dynamics** are governed by the $N \times n$ equations:

$$\dot{x}^{(i)} = f(x^{(i)}) - d \sum_{j=1,2,\dots,N} l_{ij} Hx^{(j)} \quad , \quad i=1,2,\dots,N$$

where

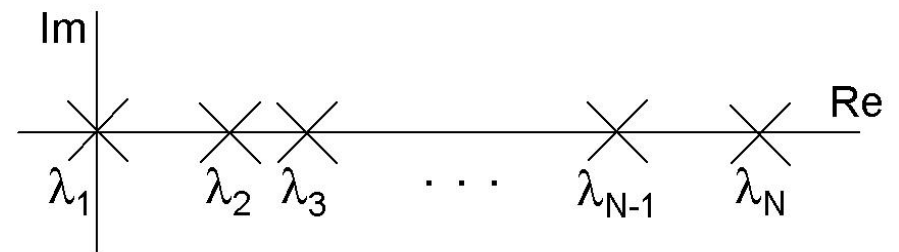
- if $i \neq j$, $l_{ij} = l_{ji} = -a_{ij} = -a_{ji}$ ($= -1$ if the link $i \leftrightarrow j$ exists, 0 otherwise)
- if $i = j$, $l_{ii} = -\sum_{j \neq i} l_{ij}$ ($=$ degree of i)

$L = [l_{ij}]$ is the Laplacian matrix of the **undirected network**:

- **real** and **symmetric** (thus diagonalizable)
- all the **off-diagonal** entries are **non-positive**
- all rows have **zero sum**
- **irreducible** if the network is **connected**

It follows that the spectrum of L has the form

$$\sigma(L) = \{0 = \lambda_1 < \lambda_2 \dots \leq \lambda_N\}$$



Complete synchronization

We have **complete synchronization** when

$$x^{(1)}(t) = x^{(2)}(t) = \dots = x^{(N)}(t) = \bar{x}(t) \quad \forall t$$

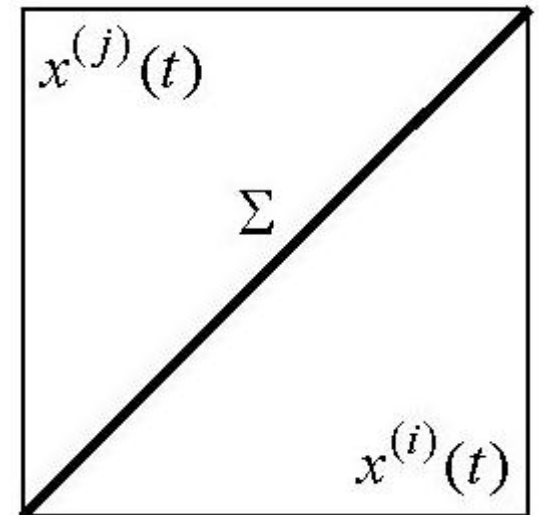
Since

$$\dot{x}^{(i)} = f(x^{(i)}) + \sum_{j:a_{ij}=1} d(H(x^{(j)} - x^{(i)})),$$

at synchronization all interaction terms $dH(x^{(j)} - x^{(i)})$ vanish: $\bar{x}(t)$ **must be a solution of the isolated system** $\dot{x} = f(x)$.

The **synchronized trajectory** lies in n -dimensional **subspace** Σ defined by

$$x^{(i)}(t) = \bar{x}(t), \quad i = 1, 2, \dots, N$$

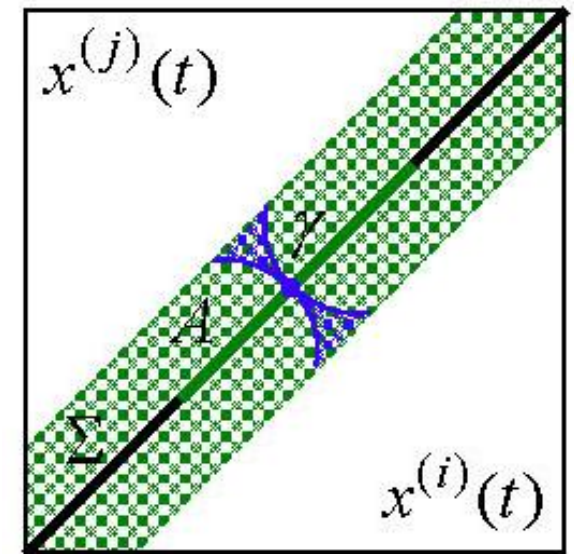


The synchronized solution $\bar{x}(t)$ has $N \times n$ Liapunov exponents:

- n are the exponents of the (periodic or chaotic) attractor of $\dot{x} = f(x)$ (perturbations within Σ)
- the remaining $(N-1) \times n$ are related to the perturbations transversal to Σ

The stability of the synchronized solution $\bar{x}(t)$ requires that all the $(N-1) \times n$ transversal Liapunov exponents be negative.

Technically, the above condition guarantees that the synchronized solution has a basin of attraction with positive measure (volume).



Master stability equation - Master stability function

Since L is diagonalizable, the **variational equation** (linearization) around $x^{(i)}(t) = \bar{x}(t)$ can be decomposed in N independent blocks, each one related to one single eigenvalue λ_i of L :

$$\dot{v}_i = J(\bar{x}(t))v_i - d\lambda_i H v_i \quad , \quad i = 1, 2, \dots, N$$

$J(x) = \partial f / \partial x$ is the Jacobian of $f(x)$.

- For $i = 1$ ($\lambda_1 = 0$):

$\dot{v}_i = J(\bar{x}(t))v_i$ relates to the **dynamics within the synchronization subspace** Σ

- For $i = 2, 3, \dots, N$ ($\lambda_i > 0$):

$\dot{v}_i = J(\bar{x}(t))v_i - d\lambda_i H v_i$ relates to the **dynamics within the i -th direction transversal to Σ**

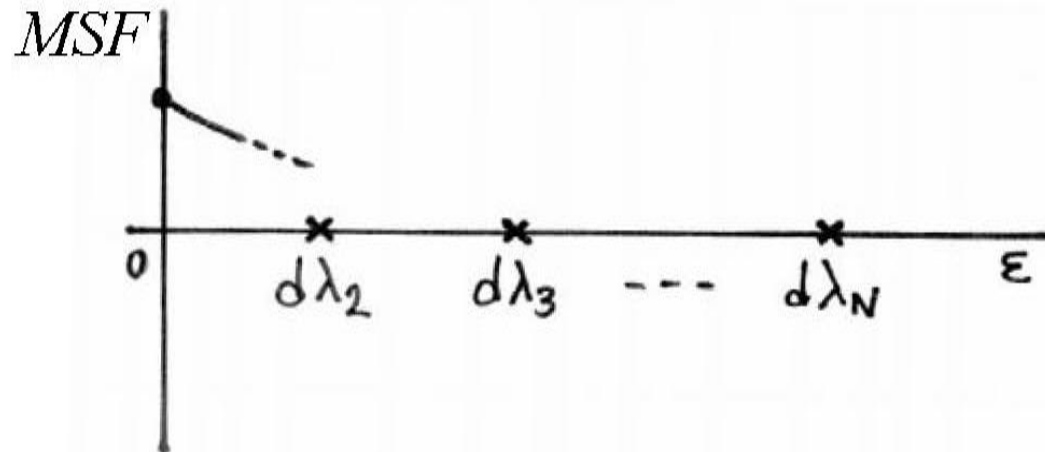
The stability of $\bar{x}(t)$ requires that, for each $i = 2, 3, \dots, N$, the maximal Liapunov exponent of the latter equation be negative.

Instead of the $N-1$ equations (n -dim) related to $0 < \lambda_2 \dots \leq \lambda_N$, we discuss one single parameterized equation (n -dim) ([Master Stability Equation – MSE](#)):

$$\dot{v} = J(\bar{x}(t))v - \varepsilon H v, \quad \text{where } \varepsilon = d\lambda \geq 0$$

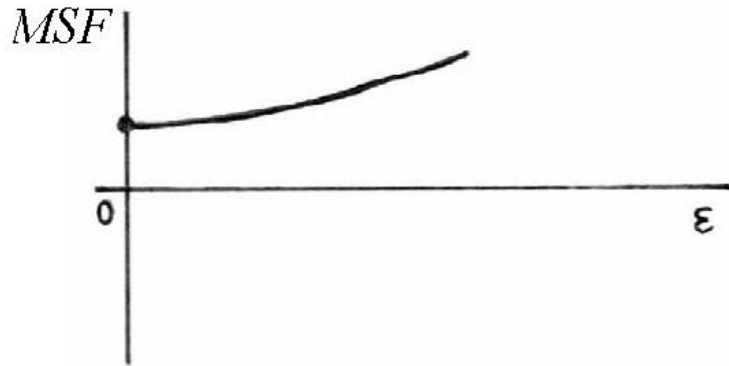
If MSF is the [maximal Liapunov exponent](#) of the MSE for a given ε , then the function $MSF = MSF(\varepsilon)$ is the [Master Stability Function – MSF](#).

[Synchronization requires \$MSF\(\varepsilon\) < 0\$ for all \$\varepsilon = d\lambda_i\$.](#)



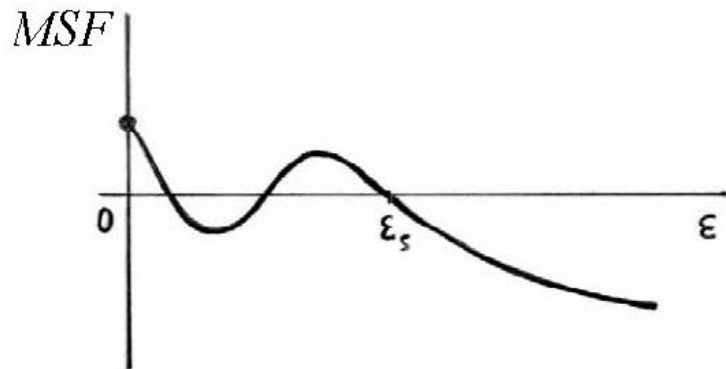
[Remark:](#) $MSF(0) > 0$ ($= 0$) if the solution $\bar{x}(t)$ of the isolated system $\dot{x} = f(x)$ is chaotic (periodic).

Remark: the MSF depends on the **system** (f) and **diffusion profile** (H), but **NOT** on the **network topology** (L).



MSF type I: $MSF(\varepsilon) > 0$ for all $\varepsilon > 0$

synchronization is **impossible** on all networks



MSF type II: $MSF(\varepsilon) < 0$ for all $\varepsilon > \varepsilon_s > 0$

synchronization is **possible** on all networks,
provided

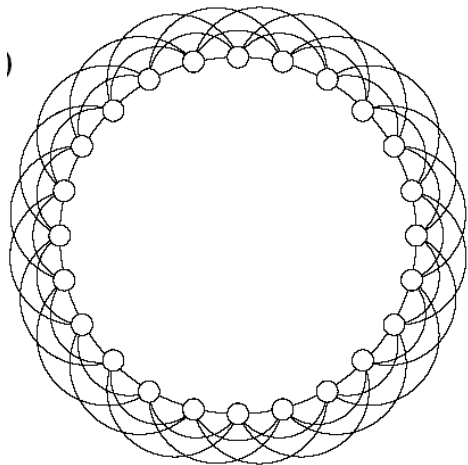
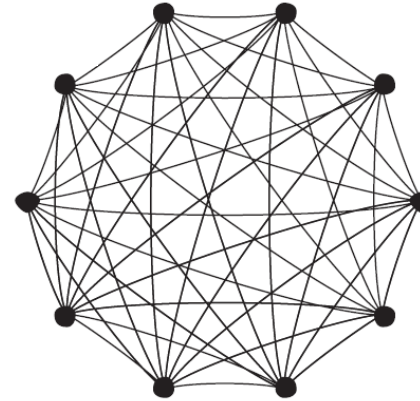
$$d > \frac{\varepsilon_s}{\lambda_2}$$

Given a **MSF type II**, synchronization is favoured in **networks with large λ_2** , because $\varepsilon_s / \lambda_2$ is small (=less coupling strength d needed).

Complete network:

$$\lambda_2 = N$$

Synchronization is favoured as N grows.



Watts-Strogatz "loop":

$$\lambda_2 \rightarrow 0 \text{ per } N \rightarrow \infty$$

Synchronization is more difficult as N grows.

In a Watts-Strogatz loop, synchronization can be greatly favoured by adding a few “long distance” connections (small-world network).

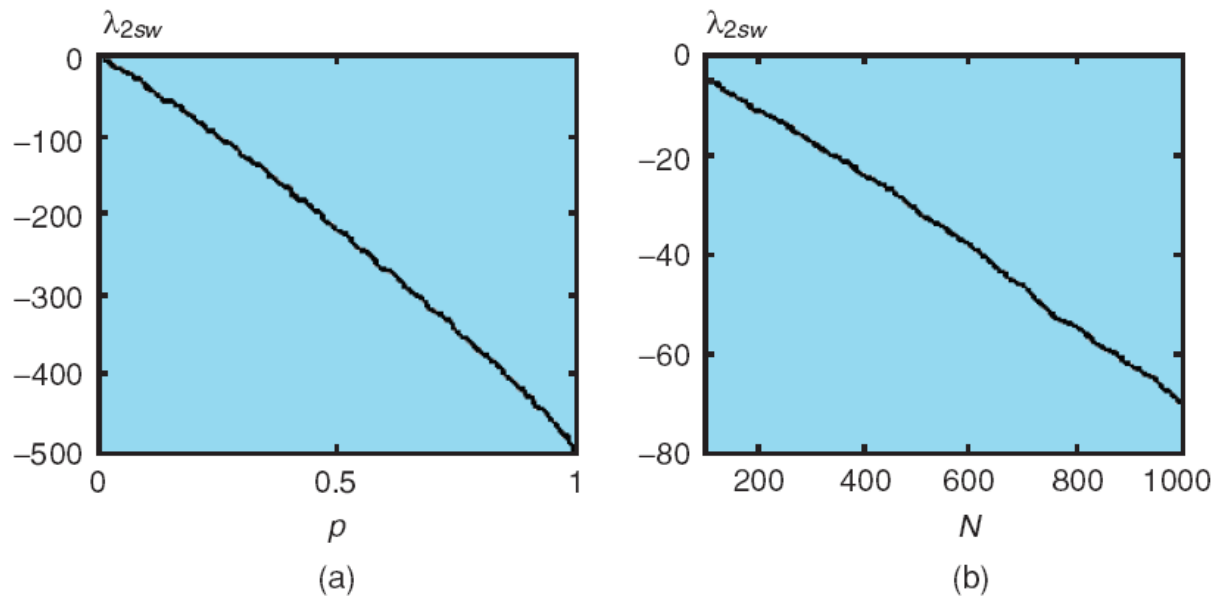
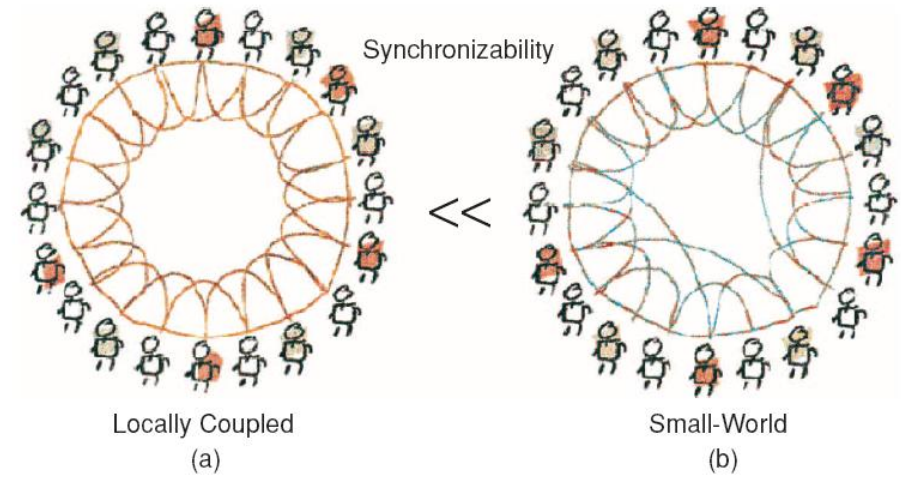


Figure 14. The second-largest eigenvalue λ_{2sw} of the coupling matrix of the small-world network (1) [41]. (a) λ_{2sw} as a function of the adding probability p with the network size $N = 500$. (b) λ_{2sw} as a function of the network size with adding probability $p = 0.1$.

In a **scale-free network** λ_2 is close to 1, similarly to a pure **star network** ($\lambda_2 = 1$).

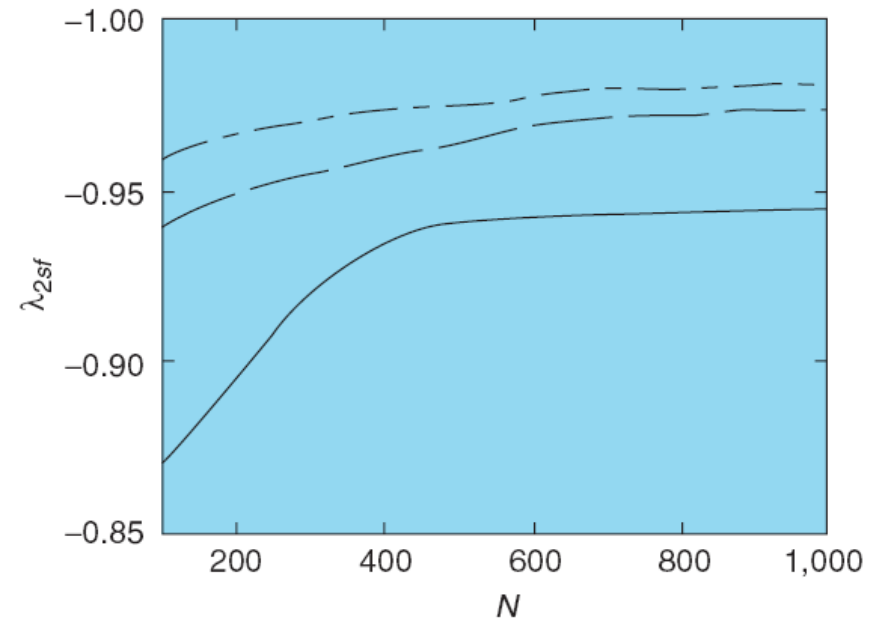
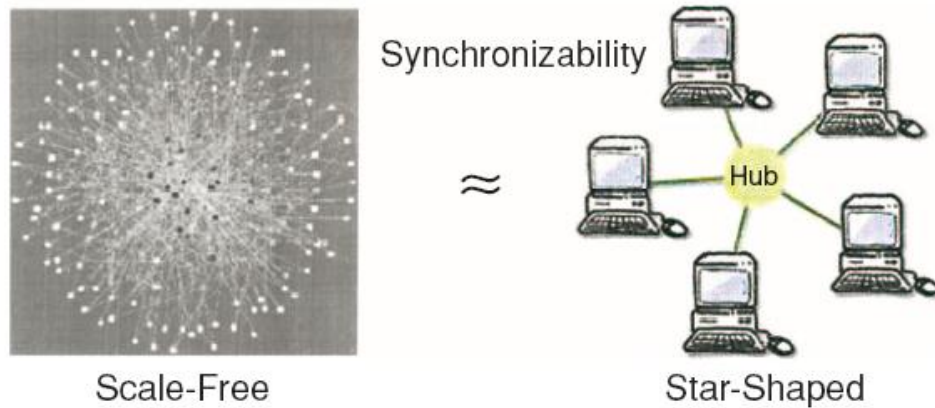


Figure 16. The second-largest eigenvalue of the coupling matrix of the scale-free network (1), for $m_0 = m = 3$ (-); $m_0 = m = 5$ (- -); and $m_0 = m = 7$ (-·) [42].

MSF type III:

synchronization **is possible** (with suitable d) if

$$\frac{\lambda_N}{\lambda_2} < \frac{\varepsilon_2}{\varepsilon_1}$$

Remark: it is not true that any network can be synchronized provided d is sufficiently large:

- in some networks, when d grows we have first **synchronization** and then **de-synchronization**
- some networks cannot be **synchronized** by any d

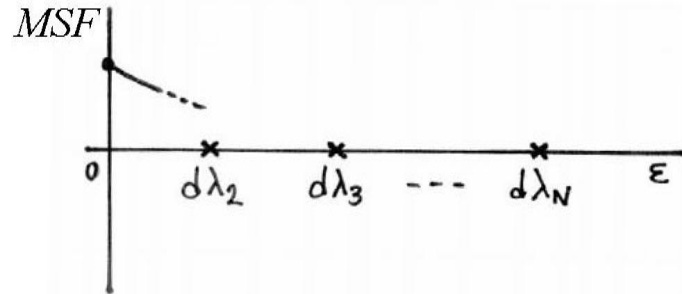
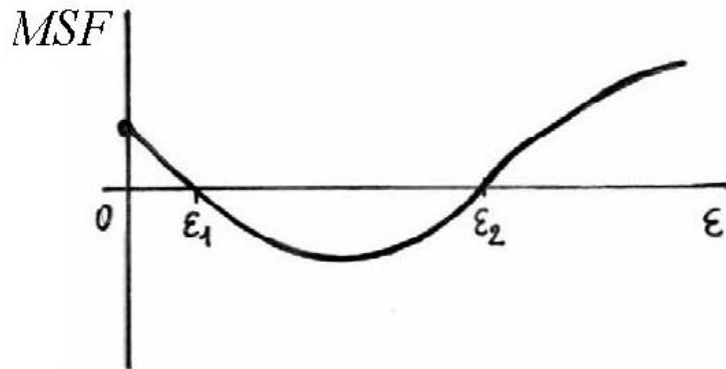
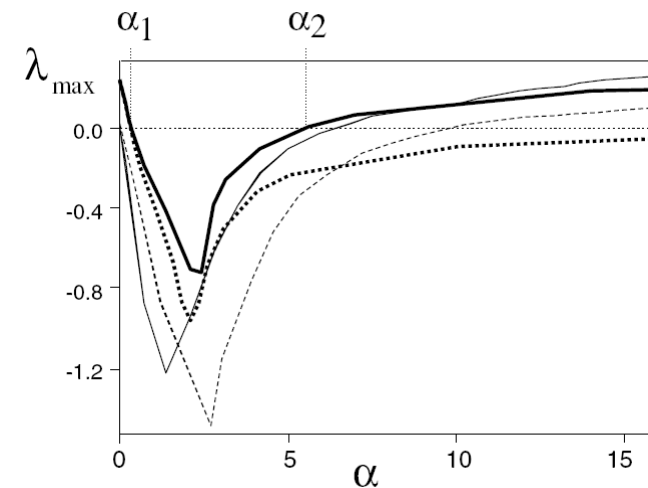


FIG. 1. Four typical master stability functions for coupled Rössler oscillators: chaotic (bold) and periodic (regular lines); with y coupling (dashed) and x coupling (solid lines). (All scaled for clearer visualization.) We concentrate on the x -coupled chaotic case with a negative region (α_1, α_2) .



Given a **MSF type III**, synchronization is favoured in **networks with small λ_N / λ_2** .

In a **Watts-Strogatz "loop"**, λ_N / λ_2 diverges with N ($\lambda_N / \lambda_2 \cong \alpha N^2 / (m(m+1))$).

Again, synchronization can strongly be favoured (at constant N) by adding a few **"long distance" connections (small-world network)**.

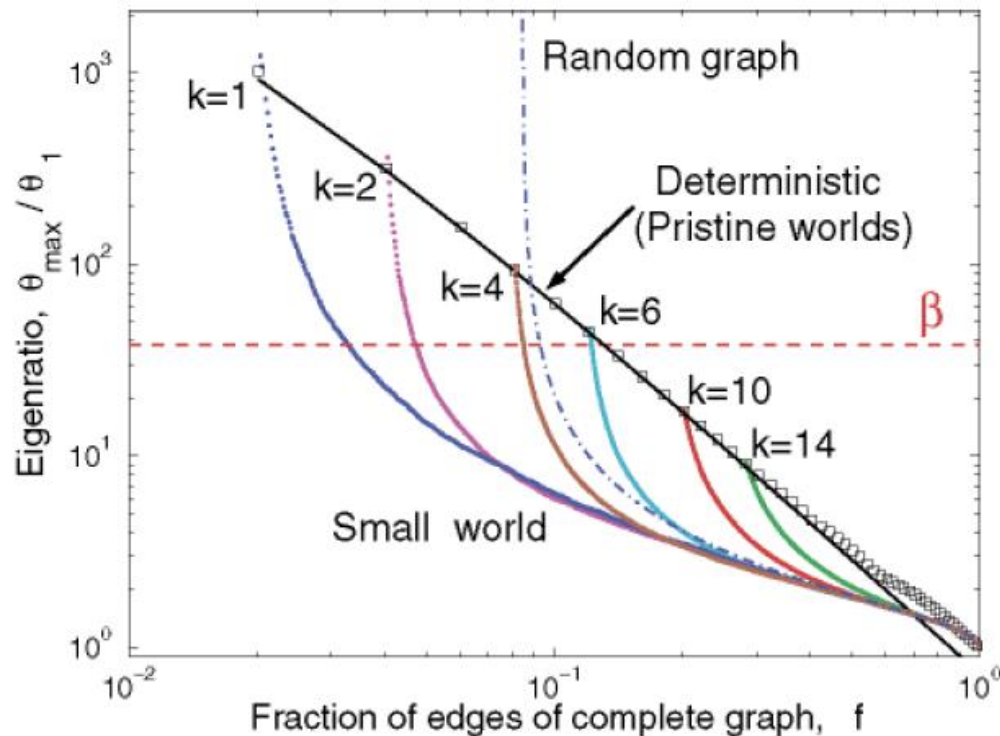
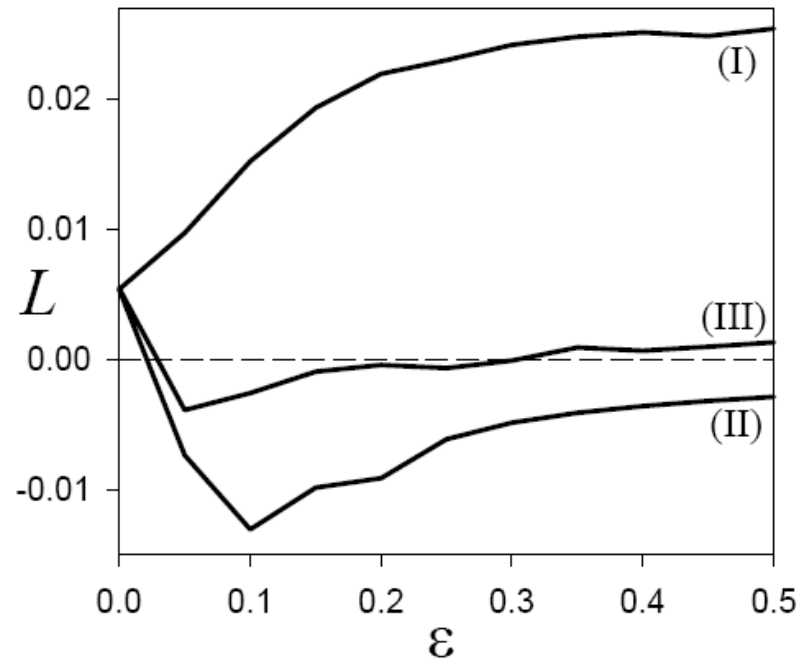


FIG. 2 (color online). Decay of the eigenratio in a $n = 100$ lattice as $f \binom{n}{2}$ edges are added following purely deterministic, semirandom (SW), and purely random schemes. Networks become synchronizable below the dashed line (β). The squares (numerical) and the solid line [analytic Eq. (4)] show the eigenratio decay of pristine worlds through the deterministic addition of short-range connections. The dot-dashed line corresponds to purely random graphs [Eq. (5)], which become *almost surely* disconnected and unsynchronizable at $f \approx 0.0843$. The semirandom SW approach (dots, shown for ranges $k = 1, 2, 4, 6, 10, 14$) is more efficient in producing synchronization when $k < \ln n$.

Example: three-trophic food chain



- (I) $H = H' = \text{diag}[1 \ 0 \ 0]$: synchronization is **impossible** for any network
- (II) $H = H'' = \text{diag}[0 \ 1 \ 0]$: synchronization is **possible** for any network, provided d be sufficiently large
- (III) $H = H''' = \text{diag}[0 \ 0 \ 1]$: synchronization is **possible** (with suitable d) **only for some networks**