Principles of Programming Languages (III)

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January 2016
Introduction on purity and evaluation

Haskell
We will consider now some basic concepts of Haskell, by implementing them in Scheme:

1. What is a *pure* functional language?
2. *Non-strict* evaluation strategies
3. *Currying*
What is a functional language?

1. In mathematics, **functions** do not have **side-effects**
2. e.g. if \( f : \mathbb{N} \rightarrow \mathbb{N} \), \( f(5) \) is a fixed value in \( \mathbb{N} \), and do not depend on \( time \) (also called **referential transparency**)
3. this is clearly not true in conventional programming languages, Scheme included
4. Scheme is *mainly* functional, as programs are **expressions**, and computation is evaluation of such expressions
5. but some expressions have **side-effects**, e.g. `vector-set!`
6. Haskell is **pure**, so we will see later how to manage inherently side-effectful computations (e.g. those with I/O)
We have already seen that, in absence of side effects (purely functional computations) from the point of view of the result the order in which functions are applied does not matter (almost).

However, it matters in other aspects, consider e.g. this function:

```lisp
(define (sum-square x y)
  (+ (* x x)
      (* y y)))
```
Evaluation of functions (Scheme)

1. A possible evaluation:

```
(sum-square (+ 1 2) (+ 2 3))
;; applying the first +
= (sum-square 3 (+ 2 3))
;; applying +
= (sum-square 3 5)
;; applying sum-square
= (+ (* 3 3)(* 5 5))
...
= 34
```

2. Is it that of Scheme?
Evaluation of functions (alia modo)

(sum-square (+ 1 2) (+ 2 3))
;; applying sum-square
= (+ (* (+ 1 2) (+ 1 2)) (* (+ 2 3) (+ 2 3)))
;; evaluating the first (+ 1 2)
= (+ (* 3 (+ 1 2)) (* (+ 2 3) (+ 2 3)))
... 
= (+ (* 3 3) (* 5 5))
... 
= 34

1. The two evaluations differ in the order in which function applications are evaluated.
2. A function application ready to be performed is called a reducible expression (or redex)
in the first example of evaluation of mult, redexes are evaluated according to a (leftmost) **innermost strategy**

i.e., when there is more than one redex, the leftmost one that does not contain other redexes is evaluated

e.g. in \((\text{sum-square} \ (+ \ 1 \ 2) \ (+ \ 2 \ 3))\) there are 3 redexes: \((\text{sum-square} \ (+ \ 1 \ 2) \ (+ \ 2 \ 3))\), \((+ \ 1 \ 2)\) and \((+ \ 2 \ 3)\) the innermost that is also leftmost is \((+ \ 1 \ 2)\), which is applied, giving expression \((\text{sum-square} \ 3 \ (+ \ 2 \ 3))\)

in this strategy, **arguments** of functions are always evaluated **before** evaluating the function itself - this corresponds to passing arguments **by value**.

note that Scheme does not require that we take the **leftmost**, but this is very common in mainstream languages
a dual evaluation strategy: redexes are evaluated in an **outermost** fashion

we start with the redex that is **not contained in any other redex**, i.e. in the example above, with \( \text{sum-square} \ (+\ 1\ 2\) (+\ 2\ 3)) \), which yields \(+ (\star (+\ 1\ 2)(+\ 1\ 2))(\star (+\ 2\ 3)(+\ 2\ 3))\)

in the outermost strategy, functions are always **applied before their arguments**, this corresponds to passing arguments **by name** (like in Algol 60).
e.g. first we define the following two simple functions:

\[
\begin{align*}
&\text{(define (infinity)} \\
&\quad (+ 1 \text{(infinity)})) \\
&\text{(define (fst x y) x)}
\end{align*}
\]

consider the expression (fst 3 (infinity)):

1. Call-by-value: \((\text{fst 3 (infinity)}) = (\text{fst 3 (+ 1 (infinity)})) = (\text{fst 3 (+ 1 (+ 1 (infinity)})))) = \ldots \)
2. Call-by-name: \((\text{fst 3 (infinity)}) = 3 \)

if there is an evaluation for an expression that terminates, \textbf{call-by-name terminates}, and produces the same result (Church-Rosser confluence)
Haskell is lazy: call-by-need

1. In call-by-name, if the argument is not used, it is never evaluated; if the argument is used several times, it is re-evaluated each time.

2. Call-by-need is a memoized version of call-by-name where, if the function argument is evaluated, that value is stored for subsequent uses.

3. In a “pure” (effect-free) setting, this produces the same results as call-by-name, and it is usually faster.
1. we saw that macros are different from function, as they do not evaluate and are expanded at **compile time**

2. a possible idea to overcome the nontermination of \((\text{fst} 3 \ (\text{infinity}))\), could be to use **thunks** to prevent evaluation, and then **force** it with an explicit call

3. indeed, there is already an implementation in Racket based on **delay** and **force**

4. we’ll see how to implement them with macros and thunks
Delay is used to return a promise to execute a computation (implements call-by-name)

moreover, it caches the result (memoization) of the computation on its first evaluation and returns that value on subsequent calls (implements call-by-need)
(struct promise
   (proc ; thunk or value
    value? ; already evaluated?
   ) #:mutable)
(define-syntax delay
  (syntax-rules ()
    ((_ (expr ...))
      (promise (lambda ()
              (expr ...)) ; a thunk
        #f))) ; still to be evaluated)
**Force (code)**

1. **force** is used to force the evaluation of a promise:

```scheme
(define (force prom)
  (cond
    ;; is it already a value?
    ;; ((not (promise? prom)) prom)
    ;; is it an evaluated promise?
    (promise-value? prom) (promise-proc prom))
  (else
    (set-promise-proc! prom
      ((promise-proc prom)))
    (set-promise-value?! prom #t)
    (promise-proc prom))))
```
(define x (delay (+ 2 5))) ; a promise 
(force x) ;; => 7

(define lazy-infinity (delay (infinity)))
(force (fst 3 lazy-infinity)) ; => 3 
(fst 3 lazy-infinity) ; => 3 
(force (delay (fst 3 lazy-infinity))) ; => 3

1. here we have call-by-need only if we make every function call a promise
2. in Haskell call-by-need is the default: if we need call-by-value, we need to force the evaluation (we’ll see how)
in Haskell, functions have only **one** argument!

this is not a limitation, because functions with more arguments are **curried**

we see here in Scheme what it means. Consider the function:

```scheme
(define (sum-square x y)
  (+ (* x x)
      (* y y)))
```

it has signature \( \text{sum-square} : \mathbb{C}^2 \rightarrow \mathbb{C} \), if we consider the most general kind of numbers in Scheme, i.e. the complex field
Currying (cont.)

1 curried version:

```
(define (sum-square x)
  (lambda (y)
    (+ (* x x)
     (* y y))))

;; shorter version:
(define ((sum-square x) y)
  (+ (* x x)
     (* y y)))
```

2 it can be used *almost* as the usual version: ((sum-square 3) 5)

3 the curried version has signature \(\text{sum-square} : \mathbb{C} \rightarrow (\mathbb{C} \rightarrow \mathbb{C})\) i.e. \(\mathbb{C} \rightarrow \mathbb{C} \rightarrow \mathbb{C}\) (\(\rightarrow\) is right associative)
in Haskell every function is automatically curried and consequently managed

the name *currying*, coined by Christopher Strachey in 1967, is a reference to

logician Haskell Curry

the alternative name *Schönfinkelisation* has been proposed as a reference to

Moses Schönfinkel but didn’t catch on
Haskell

1. Born in 1990, designed by committee to be:
   1. purely functional
   2. call-by-need (sometimes called lazy evaluation)
   3. strong polymorphic and static typing

2. Standards: Haskell '98 and '10

3. Motto: "Avoid success at all costs"
   
   1. ex. usage: Google’s Ganeti cluster virtual server management tool


5. Beware! There are many bad tutorials on Haskell and monads, in particular, available online
More complex and "human" than Scheme: parentheses are optional!

Function call is similar, though: \( f \ x \ y \) stands for \( f(x, y) \)

There are infix operators and are made of non-alphabetic characters (e.g. *, +, but also <++>)

elem is \( \in \). If you want to use it infix, just use ‘elem‘

-- this is a comment

Lambdas: (\( \lambda \ (x \ y) \ (+ \ 1 \ x \ y) \)) is written \( \lambda x \ y \rightarrow 1+x+y \)
Haskell has **static** typing, i.e. the type of everything must be known at **compile time**

there is **type inference**, so usually we do not need to explicitly declare types

*has type* is written `::` instead of `:` (the latter is **cons**)

e.g.

1. `5 :: Integer`
2. `'a' :: Char`
3. `inc :: Integer -> Integer`
4. `[1, 2, 3] :: [Integer]` – equivalent to `1:(2:(3:[]))`
5. `('b', 4) :: (Char, Integer)`

strings are **lists of characters**
functions are declared through a sequence of equations

\[ \text{inc } n = n + 1 \]
\[ \text{length } :: [\text{Integer}] \rightarrow \text{Integer} \]
\[ \text{length } [] = 0 \]
\[ \text{length } (x:xs) = 1 + \text{length } xs \]

this is also an example of \textbf{pattern matching}

arguments are matched with the right parts of equations, top to bottom

if the match succeeds, the function body is called
the previous definition of \texttt{length} could work with any kind of lists, not just those made of integers

indeed, if we omit its type declaration, it is inferred by Haskell as having type

\begin{verbatim}
length :: [a] -> Integer
\end{verbatim}

lower case letters are \textbf{type variables}, so \texttt{[a]} stands for \textit{a list of elements of type \texttt{a}, for any \texttt{a}}
Main characteristics of Haskell’s type system

1. every well-typed expression is guaranteed to have a unique principal type
   1. it is (roughly) the least general type that contains all the instances of the expression
   2. e.g. length :: a -> Integer is too general, while length :: [Integer] -> a is too specific

2. Haskell adopts a variant of the Hindley-Milner type system (used also in ML variants, e.g. F#)

3. and the principal type can be inferred automatically

User-defined types

1. are based on **data declarations**

   ```haskell
   -- a "sum" type (union in C)
   data Bool = False | True
   ```

2. **Bool** is the (nullary) **type constructor**, while **False** and **True** are **data constructors** (nullary as well)

3. data and type constructors live in separate name-spaces, so it is possible (and common) to use the same name for both:

   ```haskell
   -- a "product" type (struct in C)
   data Pnt a = Pnt a a
   ```

4. if we apply a data constructor we obtain a **value** (e.g. `Pnt 2.3 5.7`), while with a type constructor we obtain a **type** (e.g. `Pnt Bool`)
Recursive types

1. classical recursive type example:

\[
data \text{Tree } a = \text{Leaf } a \mid \text{Branch } (\text{Tree } a) (\text{Tree } a)
\]

2. e.g. data constructor Branch has type:

\[
\text{Branch} :: \text{Tree } a \to \text{Tree } a \to \text{Tree } a
\]

3. An example tree:

\[
\text{aTree} = \text{Branch } (\text{Leaf } 'a') \\
(\text{Branch } (\text{Leaf } 'b') (\text{Leaf } 'c'))
\]

4. in this case aTree has type Tree Char
Lists are recursive types

1. Of course, also lists are recursive. Using Scheme jargon, they could be defined by:

```
data List a = Null | Cons a (List a)
```

2. but Haskell has special syntax for them; in "pseudo-Haskell":

```
data [a] = [] | a : [a]
```

3. `[]` is a data and type constructor, while `:` is an infix data constructor
An example function on Trees

fringe :: Tree a -> [a]
fringe (Leaf x) = [x]
fringe (Branch left right) = fringe left ++ fringe right

1 (++) denotes list concatenation, what is its type?
as we saw, *product types* (e.g. `data Point = Point Float Float`) are like `struct` in C or in Scheme (analogously, *sum types* are like `union`)

the access is positional, for instance we may define accessors:

```haskell
pointx Point x _ = x
pointy Point _ y = y
```

there is a C-like syntax to have *named fields*:

```haskell
data Point = Point {pointx, pointy :: Float}
```

this declaration automatically defines two field names `pointx`, `pointy`

and their corresponding *selector functions*
Type synonyms

1. are defined with the keyword `type`
2. some examples

```haskell
type String = [Char]
type Assoc a b = [(a,b)]
```
3. usually for readability or shortness
newtype is used when we want to define a type with the same representation and behavior of an existing type (like type) but having a separate identity in the type system (e.g. we want to define a kind of string ≠ [Char])

e.g.

newtype Str = Str [Char]

note: we need to define a data constructor, to distinguish it from String

its data constructor is not lazy (difference with data)
Haskell has **map**, and it can be defined as:

```haskell
code
map f [] = []
map f (x:xs) = f x : map f xs
```

we can partially apply also infix operators, by using parentheses: `(+ 1)` or `(1 +)` or `(+)`

```haskell
code
map (1 +) [1,2,3]  -- => [2,3,4]
```
:t at the prompt is used for getting type, e.g.

```
Prelude> :t (+1)
(+1) :: Num a => a -> a
Prelude> :t +
<interactive>:1:1: parse error on input ‘+’
Prelude> :t (+)

(+) :: Num a => a -> a -> a
```

2 **Prelude** is the standard library

3 we’ll see later the exact meaning of `Num a =>` with type classes. Its meaning here is that `a` must be a numerical type
Function composition and $ \cdot $

1. $(\cdot)$ is used for composing functions (i.e. $(f \cdot g)(x)$ is $f(g(x)))$

PRELUDE> let dd = (*2) . (1+)
PRELUDE> dd 6
14
PRELUDE> :t (\cdot)
(\cdot) :: (b -> c) -> (a -> b) -> a -> c

2. $\$\text{ syntax for avoiding parentheses, e.g. } (10*) (5+3) = (10*) \$ 5+3$
call-by-need is very convenient for dealing with never-ending computations that provide data

Here are some simple example functions:

```haskell
ones = 1 : ones

numsFrom n = n : numsFrom (n+1)

squares = map (^2) (numsFrom 0)
```

clearly, we cannot evaluate them (why?), but there is take to get finite slices from them

e.g.

```haskell
take 5 squares = [0,1,4,9,16]
```
Infinite lists

1. Convenient syntax for creating infinite lists:
   - e.g. `ones` before can be also written as `[1,1..]`
   - `numsFrom 6` is the same as `[6..]`

2. `zip` is a useful function having type `zip :: [a] -> [b] -> [(a, b)]`

   ```
   zip [1,2,3] "ciao"
   -- => [(1,’c’),(2,’i’),(3,’a’)]
   ```

3. list comprehensions

   ```
   [(x,y) | x <- [1,2], y <- "ciao"]
   -- => [(1,’c’),(1,’i’),(1,’a’),(1,’o’),
    (2,’c’),(2,’i’),(2,’a’),(2,’o’)]
   ```
Infinite lists (cont.)

1. a list with all the Fibonacci numbers (note: tail is cdr, while head is car)

```haskell
fib = 1 : 1 :
    [a+b | (a,b) <- zip fib (tail fib)]
```
1. **bottom** (aka `⊥`) is defined as `bot = bot`

2. all errors have value `bot`, a value shared by all types

3. `error :: String -> a` is strange because it is polymorphic only in the output

4. the reason is that it returns **bot** (in practice, an exception is raised)
Pattern matching

1. the matching process proceeds top-down, left-to-right
2. patterns may have **boolean guards**

\[
\text{sign } x \mid x > 0 = 1 \\
| x == 0 = 0 \\
| x < 0 = -1
\]

3. _ stands for *don’t care*
4. e.g. definition of **take**

\[
\text{take 0 } _ = [] \\
\text{take } _ [ ] = [] \\
\text{take } n (x:xs) = x : \text{take (n-1) } xs
\]
the order of definitions **matters**:  

Prelude> :t bot  
bot :: t  
Prelude> take 0 bot  
[]

on the other hand, `take bot []` does not terminate

what does it change, if we swap the first two defining equations?
take with case:

\[
\text{take } m \ ys = \ \text{case } (m, ys) \ \text{of}
\]
\[
(0, \_ ) \rightarrow []
\]
\[
(\_, [] ) \rightarrow []
\]
\[
(n, x:xs) \rightarrow x : \ \text{take } (n-1) \ xs
\]
**let and where**

1. **let** is like Scheme’s `letrec*`:

   let x = 3
   y = 12
in x+y -- => 15

2. **where** can be convenient to scope binding over equations, e.g.:

   powset set = powset’ set [[]] where
   powset’ [] out = out
   powset’ (e:set) out = powset’ set (out ++
   [ e:x | x <- out ])

3. layout is like in Python, with meaningful whitespaces, but we can also use a C-like syntax:

   let {x = 3 ; y = 12} in x+y
fold-left is efficient in Scheme, because its definition is naturally tail-recursive:

\[
\begin{align*}
\text{foldl } f \ z \ [] & = z \\
\text{foldl } f \ z \ (x:xs) & = \text{foldl } f \ (f \ z \ x) \ xs
\end{align*}
\]

\textbf{Note: in Racket it is defined with } (f \ x \ z)\textbf{.}

This is not as efficient in Haskell, because of call-by-need:

\begin{enumerate}
\item \text{foldl } (+) \ 0 \ [1,2,3]
\item \text{foldl } (+) \ (0 + 1) \ [2,3]
\item \text{foldl } (+) \ ((0 + 1) + 2) \ [3]
\item \text{foldl } (+) \ (((0 + 1) + 2) + 3) \ []
\item (((0 + 1) + 2) + 3) = 6
\end{enumerate}
There are various ways to enforce **strictness** in Haskell (analogously there are classical approaches to introduce laziness in strict languages)

- e.g. on data with **bang patterns** (a datum marked with `!` is considered strict)

```
data Complex = Complex !Float !Float
```

- there are extensions for using `!` also in function parameters
Canonical operator to **force evaluation** is `seq :: a -> t -> t`.

`seq x y` returns `y`, **only if** the evaluation of `x` **terminates** (i.e. it performs `x` then returns `y`).

A strict version of `foldl` (available in `Data.List`):

```haskell
foldl' f z [] = z
foldl' f z (x:xs) = let z' = f z x
                    in seq z' (foldl' f z' xs)
```

Strict versions of standard functions are usually primed.
Special syntax for \texttt{seq}

1. There is a convenient \textit{strict} variant of $ (function application) called $!$

2. Here is its definition:

\[
\begin{align*}
($!$) & :: (a \to b) \to a \to b \\
\text{f$\;$$!\;\textit{x} &= \text{seq}\;\textit{x}\;\textit{f}\;\textit{x})
\end{align*}
\]
not much to be said: Haskell has a simple module system, with `import`, `export` and namespaces

a very simple example

```haskell
module CartProd where  --- export everything
infixr 9 -*-
-- right associative
-- precedence goes from 0 to 9, the strongest
x -*- y = [(i,j) | i <- x, j <- y]
```
1 import/export

module Tree ( Tree(Leaf,Bran,fringe) ) where
data Tree a = Leaf a | Branch (Tree a) (Tree a)
fringe :: Tree a -> [a] ...

module Main (main) where
import Tree ( Tree(Leaf,Bran) )
main = print (Branch (Leaf 'a') (Leaf 'b'))
modules provide the only way to build abstract data types (ADT)

the characteristic feature of an ADT is that the representation type is hidden: all operations on the ADT are done at an abstract level which does not depend on the representation

e.g. a suitable ADT for binary trees might include the following operations:

```haskell
data Tree a  -- just the type name
leaf        :: a -> Tree a
branch      :: Tree a -> Tree a -> Tree a
cell        :: Tree a -> a
left, right :: Tree a -> Tree a
isLeaf      :: Tree a -> Bool
```
module TreeADT (Tree, leaf, branch, cell, left, right, isLeaf) where

data Tree a = Leaf a | Branch (Tree a) (Tree a)
leaf = Leaf
branch = Branch
cell (Leaf a) = a
left (Branch l r) = l
right (Branch l r) = r
isLeaf (Leaf _) = True
isLeaf _ = False

1 in the export list the type name Tree appears without its constructors

2 so the only way to build or take apart trees outside of the module is by using the various (abstract) operations

2 the advantage of this information hiding is that at a later time we could change the representation type without affecting users of the type
Type classes and overloading

1. We already saw *parametric polymorphism* in Haskell (e.g. in `length`).
2. **Type classes** are the mechanism provided by Haskell for *ad hoc* polymorphism (aka *overloading*).
3. The first, natural example is that of numbers: 6 can represent an integer, a rational, a floating point number...
4. E.g.

   ```haskell
   Prelude > 6 :: Float
   6.0
   Prelude > 6 :: Integer -- unlimited
   6
   Prelude > 6 :: Int -- fixed precision
   6
   Prelude > 6 :: Rational
   6 % 1
   ```
also numeric operators and equality work with different kinds of numbers

let’s start with equality: it is natural to define equality for many types (but not every one, e.g. functions - it’s undecidable)

we consider here only **value equality**, not **pointer equality** (like Java's `==` or Scheme’s `eq?`), because pointer equality is clearly *not referentially transparent*

let us consider `elem`

```haskell
x `elem` []       = False
x `elem` (y:ys)   = x==y || (x `elem` ys)
```

its type should be: `a -> [a] -> Bool`. But this means that `(==) :: a -> a -> Bool`, even though equality is not defined for every type
class Eq

type classes are used for overloading: a class is a "container" of overloaded operations

we can declare a type to be an instance of a type class, meaning that it implements its operations

e.g. class Eq

class Eq a where
    (==) :: a -> a -> Bool

now the type of (==) is

(==) :: (Eq a) => a -> a -> Bool

Eq a is a constraint on type a, it means that a must be an instance of Eq
1. e.g. elem has type (Eq a) => a -> [a] -> Bool
2. we can define instances like this:

   instance (Eq a) => Eq (Tree a) where
   -- type a must support equality as well
   Leaf a == Leaf b = a == b
   (Branch l1 r1) == (Branch l2 r2) = (l1==l2) && (r1==r2)
   _ == _ = False

3. an implementation of (==) is called a **method**
4. **CAVEAT** do not confuse all these concepts with the homonymous concepts in OO programming: there are similarities but also big differences
## Haskell vs Java concepts

<table>
<thead>
<tr>
<th>Haskell</th>
<th>Java</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class</td>
<td>Interface</td>
</tr>
<tr>
<td>Type</td>
<td>Class</td>
</tr>
<tr>
<td>Value</td>
<td>Object</td>
</tr>
<tr>
<td>Method</td>
<td>Method</td>
</tr>
</tbody>
</table>

1. **in Java, an Object is an instance of a Class**
2. **in Haskell, a Type is an instance of a Class**
Eq offers also a standard definition of \( \neq \), derived from (==):

```hs
class Eq a where
  (==), (/=) :: a -> a -> Bool
  x /= y = not (x == y)
```

we can also extend Eq with comparison operations:

```hs
class (Eq a) => Ord a where
  (<), (<=), (>=), (>) :: a -> a -> Bool
  max, min :: a -> a -> a
```

Ord is also called a **subclass** of Eq

it is possible to have **multiple inheritance**: class (X a, Y a) => Z a
Another important class: Show

1. it is used for **showing**: to have an instance we must implement `show`

2. e.g., functions do not have a standard representation:

   Prelude> (+)
   <interactive>:2:1:
    No instance for (Show (a0 -> a0 -> a0))
    arising from a use of ‘print’
    Possible fix:
     add an instance declaration for (Show (a0 -> a0 -> a0))

3. well, we can just use a trivial one:

   instance Show (a -> b) where
    show f = "<< a function >>"
we can also represent binary trees:

```haskell
instance Show a => Show (Tree a) where
  show (Leaf a) = show a
  show (Branch x y) = "<" ++ show x ++ " | " ++ show y ++ ">
```

e.g.

```
Branch
  (Branch
    (Leaf 'a') (Branch (Leaf 'b') (Leaf 'c')))
  (Branch
    (Leaf 'd') (Leaf 'e'))
```

is represented as

```
<<'a' | <<'b' | 'c'>> | <<'d' | 'e'>>
```
usually it is not necessary to explicitly define instances of some classes, e.g. Eq and Show

Haskell can be quite smart and do it automatically, by using `deriving`

for example we may define binary trees using an infix syntax and automatic Eq, Show like this:

```haskell
infixr 5 :^:
data Tr a = Lf a | Tr a :^: Tr a

deriving (Show, Eq)
```

e.g.

```haskell
*Main> let x = Lf 3 :^: Lf 5 :^: Lf 2
*Main> let y = (Lf 3 :^: Lf 5) :^: Lf 2
*Main> x == y
False
*Main> x
Lf 3 :^: (Lf 5 :^: Lf 2)
```
An example with class Ord

1. Rock-paper-scissors in Haskell

   ```haskell
data RPS = Rock | Paper | Scissors deriving (Show, Eq)

   instance Ord RPS where
     x <= y | x == y = True
     Rock <= Paper = True
     Paper <= Scissors = True
     Scissors <= Rock = True
     _ <= _ = False
```

2. note that we only needed to define \((\leq\)) to have the instance
A simple re-implementation of rational numbers

data Rat = Rat !Integer !Integer deriving Eq

simplify (Rat x y) = let g = gcd x y
                in Rat (x ‘div‘ g) (y ‘div‘ g)

makeRat x y = simplify (Rat x y)

instance Num Rat where

    (Rat x y) + (Rat x’ y’) = makeRat (x*y’+x’*y) (y*y’)

    (Rat x y) - (Rat x’ y’) = makeRat (x*y’-x’*y) (y*y’)

    (Rat x y) * (Rat x’ y’) = makeRat (x*x’) (y*y’)

    abs (Rat x y) = makeRat (abs x) (abs y)

    signum (Rat x y) = makeRat (signum x * signum y) 1

    fromInteger x = makeRat x 1
An example with class Num (cont.)

1. **Ord:**
   
   ```haskell
   instance Ord Rat where
     (Rat x y) <= (Rat x' y') = x*y' <= x'*y
   ```

2. A better show:
   
   ```haskell
   instance Show Rat where
     show (Rat x y) = show x ++ "/" ++ show y
   ```

3. Note: Rationals are in the Prelude!

4. Moreover, there is class Fractional for / (not covered here)

5. But we could define our version of division as follows:
   
   ```haskell
   x // (Rat x' y') = x * (Rat y' x')
   ```
Input/Output is dysfunctional

1. what is the type of the standard function `getChar`, that gets a character from the user? `getChar :: theUser -> Char`?

2. first of all, it is not **referentially transparent**: two different calls of `getChar` could return different characters

3. In general, IO computation is based on **state change** (e.g. of a file), hence if we perform a **sequence of operations**, they must be performed in **order** (and this is not easy with **call-by-need**).
getChar can be seen as a function $\mathbf{::} \text{Time} \rightarrow \text{Char}$.

indeed, it is an **IO action** (in this case for Input): getChar $\mathbf{::} \text{IO} \ \text{Char}$

quite naturally, to print a character we use **putChar**, that has type:

$\text{putChar} \mathbf{::} \text{Char} \rightarrow \text{IO} ()$

**IO** is an instance of the **monad** class, and in Haskell it is considered as an **indelible stain of impurity**
A very simple example of an IO program

1. **main** is the default entry point of the program (like in C)
   
   ```haskell
   main = do {
     putStr "Please, tell me something">
     thing <- getLine;
     putStrLn $ "You told me " ++ thing ++ ".";
   }
   ```

2. special syntax for working with IO: **do**, `<-

3. we will see its real semantics later, used to define an IO action as an ordered sequence of IO actions

4. "<" (note: not \(\neq\)) is used to obtain a value from an IO action

5. types:
   
   ```haskell
   main    :: IO ()
   putStrLn :: String -> IO ()
   getline :: IO String
   ```
Command line arguments and IO with files

1. compile with e.g. **ghc readfile.hs**
   
   ```haskell
   import System.IO
   import System.Environment

   readfile = do 
     args <- getArgs; -- command line arguments
     handle <- openFile (head args) ReadMode;
     contents <- hGetContents handle; -- note: lazy
     putStr contents;
     hClose handle;
   }
   main = readFile
   ```

2. **readfile stuff.txt** reads "stuff.txt" and shows it on the screen

3. **hGetContents** reads lazily the contents of the file
Of course, purely functional Haskell code can raise exceptions: `head []`, `3 ‘div‘ 0`, ...

but if we want to catch them, we need an IO action:

```
handle :: Exception e => (e -> IO a) -> IO a -> IO a; the 1st
argument is the handler
```

Example: we catch the errors of `readfile`

```
import Control.Exception
import System.IO.Error
...
main = handle handler readFile
    where handler e
            | isDoesNotExistError e = putStrLn "This file does not exist."
            | otherwise = putStrLn "Got a problem."
```
Another typical issue: \textbf{state}

1. the typical way of managing \textit{state} in a purely functional language is by passing it around
2. i.e. if a function works on state, it will receive it as a parameter, and will return it among its outputs
3. we can easily see it with an example: let us consider again \textit{binary trees}

\begin{verbatim}
data Tree a = Leaf a | Branch (Tree a) (Tree a)
deriving (Show, Eq)
\end{verbatim}
it is easy to write a map function working on tree nodes:

```haskell
mapTree :: (a -> b) -> Tree a -> Tree b
mapTree f (Leaf a) = Leaf (f a)
mapTree f (Branch lhs rhs) = Branch (mapTree f lhs) (mapTree f rhs)
```

there is a class, called **Functor**, for containers (and more) which offer a map operation (called **fmap**)

```haskell
instance Functor Tree where
  fmap = mapTree
```
If we want to number every node going from left to right, the previous function does not work.

Indeed, we have to maintain a **state** in our visit of the tree:
Stateful `mapTree`

1. adding state makes the function more complicated:

   ```haskell
   mapTreeState :: (a -> state -> (state, b)) ->
                Tree a -> state -> (state, Tree b)
   mapTreeState f (Leaf a) state =
      let (state', b) = f a state
          in (state', Leaf b)
   mapTreeState f (Branch lhs rhs) state =
      let (state', lhs') = mapTreeState f lhs state
          (state'', rhs') = mapTreeState f rhs state'
          in (state'', Branch lhs' rhs')
   ```

2. note also that its type is getting hard to understand
Let’s try it:

```
testMTS x st = (st+1, (x, st))
```

```
testTree = Branch
    (Branch (Leaf 'a') (Branch (Leaf 'b') (Leaf 'c')))
    (Branch (Leaf 'd') (Leaf 'e'))
```

```
mapTreeState testMTS testTree 0
```

we get:

```
(5, Branch (Branch (Leaf ('a',0))
    (Branch (Leaf ('b',1)) (Leaf ('c',2))))
    (Branch (Leaf ('d',3)) (Leaf ('e',4))))
```

monads were introduced for managing more easily all these issues - but what is a monad?
introduced by Eugenio Moggi in 1991, a monad is a kind of algebraic data
type used to represent computations (instead of data in the domain model) -
we will often call these computations actions

monads allow the programmer to chain actions together to build an ordered
sequence, in which each action is decorated with additional processing
rules provided by the monad and performed automatically
monads also can be used to make imperative programming easier in a pure functional language.

In practice, through them it is possible to define an imperative sub-language on top of a purely functional one.

There are many examples of monads and tutorials (many of them quite bad) available in the Internet.

CAVEAT: don't panic! There is always a "scary theoretical taste" to many monadic concepts, taken from category theory, but in practice using and defining monads do not require a deep theoretical background.
first of all, **Monad** is a just type class:

```haskell
class Monad m where
    return :: a -> m a
    (>>=) :: m a -> (a -> m b) -> m b

    (>>) :: m a -> m b -> m b
    m >> k = m >>= \_ -> k -- default implementation

    fail :: String -> m a
    fail s = error s -- default implementation
```

2. `>>=` and `>>` are called **bind**

3. note that `m` is a type constructor, and `m a` is a type in the monad

4. moreover, `m` must be parametric with one parameter (see `m a` or `m b` in its definition).
1. \( m \ a \) is a *computation* (or action) resulting in a value of type \( a \).

2. `return` is used to create a single monadic action that actually does nothing. E.g. `return 5` does nothing (as an action) and produces the value 5.

3. While `bind` operators are used to compose actions:
   1. \( x >>= y \) performs the computation \( x \), takes the resulting value and passes it to \( y \); then performs \( y \).
   2. \( x >> y \) is analogous, but "throws away" the value obtained by \( x \).
An example of standard monad: **Maybe**

1. **Maybe** is used to represent computations that may fail: we either have *Just v*, if we are lucky, or *Nothing*.

   ```haskell
data Maybe a = Nothing | Just a
```

   ```haskell
instance Monad Maybe where
    return = Just
    fail s = Nothing
    Nothing >>= f = Nothing
    (Just x) >>= f = f x
```

2. It is adopted in many recent languages, to avoid e.g. exceptions. Examples are Scala (basically the ML family approach): Option[T], with values None or Some(v); Swift, with Optional<T>.
Examples with *Maybe*

1. The information managed automatically by the monad is the “bit” which encodes the **success** (i.e. *Just*) or failure (i.e. *Nothing*) of the action sequence.

2. e.g. `Just 4 >>= Just >> Nothing >> Just 6` evaluates to *Nothing*.

3. a variant: `Just 4 >>= Just >> Nothing >> Just 6`.

4. another: `Just 4 >>= Just 1 >> Nothing >> Just 6` (what is the result in this case?)
The monadic laws

for a monad to behave correctly, method definitions must obey the following laws:

1) **return** is the **identity element**:
   
   \[(\text{return } x) >>= f \iff f x\]
   
   \[m >>= \text{return} \iff m\]

2) **associativity** for binds:
   
   \[(m >>= f) >>= g \iff m >>= (\lambda x \rightarrow (f x >>= g))\]

3) (monads are analogous to **monoids**, with \textsf{return} = 1 and \textsf{>>=} = \cdot.)
Example: monadic laws application with Maybe

1. \( (\text{return } 4 :: \text{Maybe Integer}) >>= \lambda x \rightarrow \text{Just}(x+1) \)
   
   Just 5

2. \( (\text{return } 4 >>= \lambda x \rightarrow \text{Just}(x+1)) >>= \lambda x \rightarrow \text{Just}(x*2) \)

   Just 10

3. \( \text{return } 4 >>= (\lambda y \rightarrow \big((\lambda x \rightarrow \text{Just}(x+1)) y\big) >>= \lambda x \rightarrow \text{Just}(x*2)) \)

   Just 10
The **do** syntax is used to avoid the explicit use of `>>=` and `>>

The essential translation of **do** is captured by the following two rules:

\[
\begin{align*}
do \ e_1 ; \ e_2 & \iff e_1 >> e_2 \\
do p \gets e_1 ; \ e_2 & \iff e_1 >>= \lambda p \to e_2
\end{align*}
\]

note that they can also be written as:

\[
\begin{align*}
do \ e_1 e_2 & \\
do p \gets e_1 e_2
\end{align*}
\]

or:

\[
\begin{align*}
do \{ \ e_1 ; e_2 \} & \\
do \{ p \gets e_1 ; e_2 \}
\end{align*}
\]
Caveat: **return** does not return

For example:

```haskell
esp :: IO Integer
esp = do x <- return 4
             return (x+1)

*Main> esp
5
```
Examples of built-in monads

1. **List**: monadic binding involves joining together a set of calculations for each value in the list.

2. Given a list of $a$ and a function $f :: a \rightarrow b$, >>= applies $f$ to each of the $a$ in the input, and returns all of the generated $b$ put into a list.

3. i.e. it is **map**, or, more precisely, **concatMap**

   ```
   concatMap :: (a -> [b]) -> [a] -> [b]
   ```

   ```haskell
   *Main> concatMap (\x -> [x, x+1]) [1,2,3]
   [1,2,2,3,3,4]
   ```

4. **return** creates a singleton list.
Lists: do vs comprehensions

1 list comprehensions are just syntax:

2 i.e. the following code

   [(x,y) | x <- [1,2,3], y <- [1,2,3]]

3 is translated into:

   do x <- [1,2,3]
       y <- [1,2,3]
       return (x,y)
the List monad

definition:

instance Monad [] where
    return x = [x]
    l >>= f = concatMap f l
    fail _ = []
to understand our example of comprehension, i.e.

testcomp = do x <- [1,2,3]
                     y <- [1,2,3]
                     return (x,y)

we can rewrite it following the monad definition:

testcomp' =
    [1,2,3] >>= (\x -> [1,2,3] >>=
                     (\y ->
                         return (x,y)))
that is:

\[
\text{testcomp'' } = \\
\text{concatMap } f_0 \ [1,2,3] \\
\text{where } f_0 \ x = \text{concatMap } f_1 \ [1,2,3] \\
\text{where } f_1 \ y = [(x,y)]
\]
What about usual, practical data structures (e.g. arrays, hash-tables)?

Traditional versions are imperative! If really needed, there are libraries with imperative implementations living in the IO monad.

Idiomatic approach: use immutable arrays (Data.Array), and maps (Data.Map, implemented with balanced binary trees).

Find are respectively $O(1)$ and $O(\log n)$; update $O(n)$ for arrays, $O(\log n)$ for maps.

Of course, the update operations copy the structure, do not change it.
import Data.Map

exmap = let m = fromList [("nose", 11), ("emerald", 27)]
         n = insert "rug" 98 m
         o = insert "nose" 9 n
     in (m ! "emerald", n ! "rug", o ! "nose")

1 exmap evaluates to (27,98,9)
Example code: Arrays

1. (//) is used for update/insert

2. listArray’s first argument is the **range** of indexing (in the following case, indexes are from 1 to 3)

   ```
import Data.Array

exarr = let m = listArray (1,3) ["alpha","beta","gamma"]
        n = m // [(2,"Beta")]
        o = n // [(1,"Alpha"), (3,"Gamma")]
        in (m ! 1, n ! 2, o ! 1)
```

3. exarr evaluates to ("alpha","Beta","Alpha")
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