Principles of Programming Languages (H)

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Overview

1. Introduction on purity and evaluation
2. Basic Haskell
3. More advanced concepts
We will consider now some basic concepts of Haskell, by implementing them in Scheme:

- What is a *pure* functional language?
- *Non-strict* evaluation strategies
- *Currying*
What is a **functional** language?

- In mathematics, **functions** do not have **side-effects**
- e.g. if $f : \mathbb{N} \rightarrow \mathbb{N}$, $f(5)$ is a fixed value in $\mathbb{N}$, and do not depend on time (also called **referential transparency**)
- this is clearly not true in conventional programming languages, Scheme included
- Scheme is *mainly* functional, as programs are **expressions**, and computation is evaluation of such expressions
- but some expressions have **side-effects**, e.g. `vector-set!`
- Haskell is **pure**, so we will see later how to manage inherently side-effectful computations (e.g. those with I/O)
Evaluation of functions

- We have already seen that, in **absence of side effects** (purely functional computations) from the point of view of the result the **order** in which functions are applied **does not matter** (almost).

- However, it matters in other aspects, consider e.g. this function:

```scheme
(define (sum-square x y)
  (+ (* x x)
    (* y y)))
```
A possible evaluation:

```
(sum-square (+ 1 2) (+ 2 3))
;; applying the first +
= (sum-square 3 (+ 2 3))
;; applying +
= (sum-square 3 5)
;; applying sum-square
= (+ (* 3 3)(* 5 5))
...  
= 34
```

is it that of Scheme?
(sum-square (+ 1 2) (+ 2 3))
;; applying sum-square
= (+ (* (+ 1 2) (+ 1 2)) (* (+ 2 3) (+ 2 3)))
;; evaluating the first (+ 1 2)
= (+ (* 3 (+ 1 2)) (* (+ 2 3) (+ 2 3)))
...
= (+ (* 3 3) (* 5 5))
...
= 34

- The two evaluations differ in the **order** in which function applications are evaluated.
- A function application ready to be performed is called a **reducible expression** (or **redex**).
Evaluation strategies: call-by-value

- in the first example of evaluation of mult, redexes are evaluated according to a (leftmost) **innermost strategy**
- i.e., when there is more than one redex, the leftmost one that does not contain other redexes is evaluated
  - e.g. in `(sum-square (+ 1 2) (+ 2 3))` there are 3 redexes: `(sum-square (+ 1 2) (+ 2 3))`, `(+ 1 2)` and `(+ 2 3)` the innermost that is also leftmost is `(+ 1 2)`, which is applied, giving expression `(sum-square 3 (+ 2 3))`
- in this strategy, **arguments** of functions are always evaluated **before** evaluating the function itself - this corresponds to passing arguments **by value**.
- note that Scheme does not require that we take the **leftmost**, but this is very common in mainstream languages
Evaluation strategies: call-by-name

- a dual evaluation strategy: redexes are evaluated in an *outermost* fashion
- we start with the redex that is *not contained in any other redex*, i.e. in the example above, with `(sum-square (+ 1 2) (+ 2 3))`, which yields `(+ (* (+ 1 2)(+ 1 2)) (* (+ 2 3)(+ 2 3)))`
- in the outermost strategy, functions are always applied before their arguments, this corresponds to passing arguments by name (like in Algol 60).
Termination and call-by-name

- e.g. first we define the following two simple functions:

```scheme
(define (infinity)
  (+ 1 (infinity)))

(define (fst x y) x)
```

- consider the expression (fst 3 (infinity)):
  - Call-by-value: (fst 3 (infinity)) = (fst 3 (+ 1 (infinity))) = (fst 3 (+ 1 (+ 1 (infinity)))) = ...
  - Call-by-name: (fst 3 (infinity)) = 3

- if there is an evaluation for an expression that terminates, call-by-name terminates, and produces the same result (Church-Rosser confluence)
Haskell is lazy: call-by-need

- In call-by-name, if the argument is not used, it is never evaluated; if the argument is used several times, it is re-evaluated each time.
- **Call-by-need** is a memoized version of call-by-name where, if the function argument is evaluated, that value is stored for subsequent uses.
- In a “pure” (effect-free) setting, this produces the same results as call-by-name, and it is usually faster.
we saw that macros are different from function, as they do not evaluate and are expanded at **compile time**

a possible idea to overcome the nontermination of \((\text{fst } 3 \ (\text{infinity})))\), could be to use **thunks** to prevent evaluation, and then **force** it with an explicit call

indeed, there is already an implementation in Racket based on **delay** and **force**

we’ll see how to implement them with macros and thunks
Delay is used to return a promise to execute a computation (implements call-by-name)

moreover, it caches the result (memoization) of the computation on its first evaluation and returns that value on subsequent calls (implements call-by-need)
(struct promise
  (proc ; thunk or value
       value? ; already evaluated?
   ) #:mutable)
(define-syntax delay
  (syntax-rules ()
    ((_ (expr ...))
      (promise (lambda ()
                 (expr ...)) ; a thunk
       #f)))) ; still to be evaluated
force is used to force the evaluation of a promise:

(define (force prom)
  (cond
    ; is it already a value?
    ((not (promise? prom)) prom)
    ; is it an evaluated promise?
    ((promise-value? prom) (promise-proc prom))
    (else
     (set-promise-proc! prom
                        ((promise-proc prom)))
     (set-promise-value?! prom #t)
     (promise-proc prom)))))
Examples

```
(define x (delay (+ 2 5))) ; a promise
(force x) ;; => 7

(define lazy-infinity (delay (infinity)))
(force (fst 3 lazy-infinity)) ; => 3
(fst 3 lazy-infinity) ; => 3
(force (delay (fst 3 lazy-infinity))) ; => 3
```

- here we have call-by-need only if we make every function call a promise
- in Haskell call-by-need is the default: if we need call-by-value, we need to `force` the evaluation (we’ll see how)
Currying

- In Haskell, functions have only **one** argument!
- This is not a limitation, because functions with more arguments are **curried**
- We see here in Scheme what it means. Consider the function:

```scheme
(define (sum-square x y)
 (+ (* x x)
  (* y y)))
```

- It has signature \(\text{sum-square} : \mathbb{C}^2 \to \mathbb{C}\), if we consider the most general kind of numbers in Scheme, i.e. the complex field
curried version:

```
(define (sum-square x)
  (lambda (y)
    (+ (* x x)
       (* y y)))))
```

;; shorter version:
```
(define ((sum-square x) y)
  (+ (* x x)
     (* y y)))
```

- it can be used almost as the usual version: `((sum-square 3) 5)`
- the curried version has signature `sum-square : \mathbb{C} \to (\mathbb{C} \to \mathbb{C})`
  i.e. $\mathbb{C} \to \mathbb{C} \to \mathbb{C}$ ($\to$ is right associative)
Currying in Haskell

- In Haskell, every function is automatically curried and consequently managed.
- The name *currying*, coined by Christopher Strachey in 1967, is a reference to logician Haskell Curry.
- The alternative name *Schönfinkelisation* has been proposed as a reference to Moses Schönfinkel but didn’t catch on.
- Born in 1990, designed by committee to be:
  - **purely** functional
  - **call-by-need** (sometimes called **lazy evaluation**)
  - strong **polymorphic** and **static** typing
- Standards: Haskell '98 and '10
- Motto: "Avoid success at all costs"
  - ex. usage: Google’s *Ganeti* cluster virtual server management tool
- I mainly follow
- Beware! There are many *bad* tutorials on Haskell and monads, in particular, available online
A taste of Haskell’s syntax

- more complex and "human" than Scheme: parentheses are optional!
- function call is similar, though: \( f \ x \ y \) stands for \( f(x,y) \)
- there are infix operators and are made of non-alphabetic characters (e.g. *, +, but also <++>)
- \( \text{elem} \) is \( \in \). If you want to use it infix, just use ‘elem’
- \(--\) this is a comment
- lambdas: \( (\lambda (x \ y) (+ \ 1 \ x \ y)) \) is written \( \backslash x \ y \rightarrow 1+x+y \)
- Haskell has **static** typing, i.e. the type of everything must be known at **compile time**
- there is **type inference**, so usually we do not need to explicitly declare types
- *has type* is written `::` instead of `:` (the latter is **cons**)
- e.g.
  - `5 :: Integer`
  - `'a' :: Char`
  - `inc :: Integer -> Integer`
  - `[1, 2, 3] :: [Integer] – equivalent to `1:(2:(3:[])))`
  - `('b', 4) :: (Char, Integer)`
- strings are **lists of characters**
functions are declared through a sequence of *equations*

e.g.

\[
\begin{align*}
\text{inc } n &= n + 1 \\
\text{length :: } [\text{Integer}] &\rightarrow \text{Integer} \\
\text{length } [] &= 0 \\
\text{length } (x:xs) &= 1 + \text{length } xs
\end{align*}
\]

this is also an example of **pattern matching**

arguments are matched with the right parts of equations, top to bottom

if the match succeeds, the function body is called
the previous definition of `length` could work with any kind of lists, not just those made of integers

indeed, if we omit its type declaration, it is inferred by Haskell as having type

```haskell
length :: [a] -> Integer
```

lower case letters are **type variables**, so `[a]` stands for a list of elements of type `a`, for any `a`
Main characteristics of Haskell’s type system

- every well-typed expression is guaranteed to have a **unique principal type**
  - it is (roughly) the *least general type that contains all the instances of the expression*
  - e.g. `length :: a -> Integer` is too general, while `length :: [Integer] -> a` is too specific

- Haskell adopts a variant of the **Hindley-Milner** type system
  (used also in ML variants, e.g. F#)

- and the principal type can be **inferred automatically**

User-defined types

- are based on **data declarations**

```haskell
-- a "sum" type (union in C)
data Bool = False | True
```

- **Bool** is the (nullary) **type constructor**, while **False** and **True** are **data constructors** (nullary as well)

- data and type constructors live in separate name-spaces, so it is possible (and common) to use the same name for both:

```haskell
-- a "product" type (struct in C)
data Pnt a = Pnt a a
```

- if we apply a data constructor we obtain a **value** (e.g. Pnt 2.3 5.7), while with a type constructor we obtain a **type** (e.g. Pnt Bool)
Recursive types

- classical recursive type example:

  ```haskell
data Tree a = Leaf a | Branch (Tree a) (Tree a)
```

- e.g. data constructor `Branch` has type:

  ```haskell
  Branch :: Tree a -> Tree a -> Tree a
  ```

- An example tree:

  ```haskell
  aTree = Branch (Leaf 'a')
            (Branch (Leaf 'b') (Leaf 'c'))
  ```

- in this case `aTree` has type `Tree Char`
Lists are recursive types

- Of course, also lists are recursive. Using Scheme jargon, they could be defined by:

```haskell
data List a = Null | Cons a (List a)
```

- but Haskell has special syntax for them; in "pseudo-Haskell":

```haskell
data [a] = [] | a : [a]
```

- `[]` is a data and type constructor, while `:` is an infix data constructor.
An example function on Trees

```haskell
fringe :: Tree a -> [a]
fringe (Leaf x) = [x]
fringe (Branch left right) = fringe left ++ fringe right
```

• (++) denotes list concatenation, what is its type?
Syntax for fields

- as we saw, *product types* (e.g. `data Point = Point Float Float`) are like `struct` in C or in Scheme (analogously, *sum types* are like `union`)
- the access is positional, for instance we may define accessors:
  ```haskell
  pointx Point x _ = x
  pointy Point _ y = y
  ```
- there is a C-like syntax to have *named fields*:
  ```haskell
  data Point = Point {pointx, pointy :: Float}
  ```
- this declaration automatically defines two field names `pointx`, `pointy`
- and their corresponding *selector functions*
Type synonyms

- are defined with the keyword `type`
- some examples

```haskell
  type String = [Char]

  type Assoc a b = [(a,b)]
```

- usually for readability or shortness
newtype is used when we want to define a type with the same representation and behavior of an existing type (like type)
but having a separate identity in the type system (e.g. we want to define a kind of string \( \neq \) [Char])
e.g.
newtype Str = Str [Char]

note: we need to define a data constructor, to distinguish it from String
its data constructor is not lazy (difference with data)
More on functions and currying

- Haskell has **map**, and it can be defined as:
  
  ```haskell
  map f [] = []
  map f (x:xs) = f x : map f xs
  ```

- We can partially apply also infix operators, by using parentheses:
  ```haskell
  (+ 1) or (1 +) or (+)
  ```

  ```haskell
  map (1 +) [1,2,3] -- => [2,3,4]
  ```
:t at the prompt is used for getting **type**, e.g.

```
Prelude> :t (+1)
(+1) :: Num a => a -> a
Prelude> :t +
<interactive>:1:1: parse error on input ‘+’
Prelude> :t (+)

(+) :: Num a => a -> a -> a
```

**Prelude** is the standard library

we’ll see later the exact meaning of **Num a =>** with **type classes**. Its meaning here is that **a** must be a **numerical type**
Function composition and $\cdot$

- $\cdot$ is used for composing functions (i.e. $(f \cdot g)(x)$ is $f(g(x))$)

```
Prelude > let dd = (*2) . (1+)
Prelude > dd 6
14
Prelude > :t (.)
(.) :: (b -> c) -> (a -> b) -> a -> c
```

- $\cdot$ syntax for avoiding parentheses, e.g. $(10\cdot)(5+3) = (10\cdot)\ 5+3$
Infinite computations

- call-by-need is very convenient for dealing with never-ending computations that provide data
- here are some simple example functions:

\[
\begin{align*}
\text{ones} &= 1 : \text{ones} \\
\text{numsFrom } n &= n : \text{numsFrom } (n+1) \\
\text{squares} &= \text{map } (\lambda x . x^2) (\text{numsFrom } 0)
\end{align*}
\]

- clearly, we cannot evaluate them (why?), but there is \text{take} to get finite slices from them
- e.g.

\[
\text{take 5 squares} = [0, 1, 4, 9, 16]
\]
Infinite lists

- Convenient syntax for creating infinite lists:
- e.g. `ones` before can be also written as `[1,1..]`
- `numsFrom 6` is the same as `[6..]`
- **zip** is a useful function having type
  \[ \text{zip} :: [a] \rightarrow [b] \rightarrow [(a,b)] \]
  
  ```
  zip [1,2,3] "ciao"
  -- => [(1,'c'),(2,'i'),(3,'a')]
  ```

- **List comprehensions**
  
  ```
  [(x,y) | x <- [1,2], y <- "ciao"]
  -- => [(1,'c'),(1,'i'),(1,'a'),(1,'o'),(2,'c'),(2,'i'),(2,'a'),(2,'o')]
  ```
Infinite lists (cont.)

- a list with all the Fibonacci numbers
  (note: tail is cdr, while head is car)

```
fib = 1 : 1 :
    [a+b | (a,b) <- zip fib (tail fib)]
```
- **bottom** (aka \(\bot\)) is defined as \(\text{bot} = \text{bot}\)
- all errors have value \(\text{bot}\), a value shared by all types
- **error :: String -> a** is strange because is polymorphic only in the output
- the reason is that it returns **bot** (in practice, an exception is raised)
the matching process proceeds top-down, left-to-right
patterns may have **boolean guards**

\[
\text{sign } x \mid x > 0 = 1 \\
| x == 0 = 0 \\
| x < 0 = -1
\]

_ stands for *don’t care*
e.g. definition of **take**

\[
\begin{align*}
\text{take } 0 \_ &= [] \\
\text{take } \_ [[ ] &= [] \\
\text{take } n (x:xs) &= x : \text{take } (n-1) \text{ } xs
\end{align*}
\]
the order of definitions **matters**:

Prelude> :t bot
bot :: t

Prelude> take 0 bot
[]

on the other hand, `take bot []` does not terminate

what does it change, if we swap the first two defining equations?
**take** with **case**:

```haskell
take m ys = case (m, ys) of
    (0, _)  -> []
    (_, []) -> []
    (n, x:xs) -> x : take (n-1) xs
```
### let and where

- **let** is like Scheme’s letrec*:
  
  ```
  let x = 3
  y = 12
  in x+y -- => 15
  ```

- **where** can be convenient to scope binding over equations, e.g.:
  
  ```
  powset set = powset’ set [[]] where
  powset’ [] out = out
  powset’ (e:set) out = powset’ set (out ++ [ e:x | x <- out ])
  ```

- layout is like in Python, with meaningful whitespaces, but we can also use a C-like syntax:
  
  ```
  let {x = 3 ; y = 12} in x+y
  ```
**fold-left** is efficient in Scheme, because its definition is naturally tail-recursive:

\[
\begin{align*}
\text{foldl } f \ z \ [\!] & = z \\
\text{foldl } f \ z \ (x:xs) & = \text{foldl } f \ (f \ z \ x) \ xs
\end{align*}
\]

*note: in Racket it is defined with \((f \times z)\)*

this is not as efficient in Haskell, because of call-by-need:

- foldl (+) 0 [1,2,3]
- foldl (+) (0 + 1) [2,3]
- foldl (+) (((0 + 1) + 2) + 3) []
- (((0 + 1) + 2) + 3) = 6
There are various ways to enforce \textbf{strictness} in Haskell (analogously there are classical approaches to introduce laziness in strict languages)

e.g. on data with \textbf{bang patterns} (a datum marked with ! is considered strict)

\begin{verbatim}
data Complex = Complex !Float !Float
\end{verbatim}

there are extensions for using ! also in function parameters
Forcing evaluation

- Canonical operator to **force evaluation** is `seq :: a -> t -> t`
- `seq x y` returns `y`, **only if** the evaluation of `x` **terminates** (i.e. it performs `x` then returns `y`)
- a strict version of **foldl** (available in `Data.List`)

\[
\begin{align*}
\text{foldl}' \; f \; z \; [] &= z \\
\text{foldl}' \; f \; z \; (x:xs) &= \text{let } z' = f \; z \; x \\
&\quad \text{in } seq \; z' \; (\text{foldl}' \; f \; z' \; xs)
\end{align*}
\]

- strict versions of standard functions are usually primed
There is a convenient *strict* variant of $ (function application) called $!

Here is its definition:

\[(\$!) :: (a \rightarrow b) \rightarrow a \rightarrow b\]
\[f \$! x = \text{seq } x (f \ x)\]
not much to be said: Haskell has a simple module system, with import, export and namespaces

a very simple example

module CartProd where  --- export everything
infixr 9 -*-
-- right associative
-- precedence goes from 0 to 9, the strongest
x -*- y = [(i,j) | i <- x, j <- y]
import/export

```haskell
module Tree ( Tree (Leaf , Branch ) , fringe ) where
  data Tree a = Leaf a | Branch ( Tree a ) ( Tree a )
  fringe :: Tree a -> [a] ...
```

```haskell
module Main ( main ) where
  import Tree ( Tree (Leaf , Branch ) )
  main = print ( Branch ( Leaf 'a' ) ( Leaf 'b' ) )
```
modules provide the only way to build abstract data types (ADT)

the characteristic feature of an ADT is that the representation type is hidden: all operations on the ADT are done at an abstract level which does not depend on the representation

e.g. a suitable ADT for binary trees might include the following operations:

```haskell
data Tree a -- just the type name
  leaf :: a -> Tree a
  branch :: Tree a -> Tree a -> Tree a
  cell :: Tree a -> a
  left, right :: Tree a -> Tree a
  isLeaf :: Tree a -> Bool
```
ADT implementation

module TreeADT (Tree, leaf, branch, cell, left, right, isLeaf) where

data Tree a = Leaf a | Branch (Tree a) (Tree a)

leaf = Leaf
branch = Branch
cell (Leaf a) = a
left (Branch l r) = l
right (Branch l r) = r
isLeaf (Leaf _) = True
isLeaf _ = False

- in the export list the type name Tree appears without its constructors
- so the only way to build or take apart trees outside of the module is by using the various (abstract) operations
- the advantage of this information hiding is that at a later time we could change the representation type without affecting users of the type
we already saw *parametric polymorphism* in Haskell (e.g. in `length`)

**type classes** are the mechanism provided by Haskell for *ad hoc* polymorphism (aka *overloading*)

the first, natural example is that of numbers: 6 can represent an integer, a rational, a floating point number...

- e.g.

```haskell
Prelude> 6 :: Float
6.0

Prelude> 6 :: Integer -- unlimited
6

Prelude> 6 :: Int -- fixed precision
6

Prelude> 6 :: Rational
6 % 1
```
• also numeric operators and equality work with different kinds of numbers
• let’s start with equality: it is natural to define equality for many types (but not every one, e.g. functions - it’s undecidable)
• we consider here only value equality, not pointer equality (like Java’s == or Scheme’s eq?), because pointer equality is clearly not referentially transparent
• let us consider elem

\[
\begin{align*}
\text{x ‘elem‘ []} &= \text{False} \\
\text{x ‘elem‘ (y:ys)} &= \text{x==y || (x ‘elem‘ ys)}
\end{align*}
\]

• its type should be: a -> [a] -> Bool. But this means that (==) :: a -> a -> Bool, even though equality is not defined for every type


- **type classes** are used for overloading: a class is a "container" of overloaded operations
- we can declare a type to be an **instance** of a type class, meaning that it implements its operations
- e.g. class Eq

```haskell
class Eq a where
  (==) :: a -> a -> Bool
```

- now the type of (==) is

```haskell
(==) :: (Eq a) => a -> a -> Bool
```

- Eq a is a **constraint** on type a, it means that a must be an instance of Eq
Defining instances

- e.g. `elem` has type `(Eq a) => a -> [a] -> Bool`
- we can define instances like this:

  ```haskell
  instance (Eq a) => Eq (Tree a) where
  -- type a must support equality as well
  Leaf a == Leaf b = a == b
  (Branch l1 r1) == (Branch l2 r2) = (l1==l2) && (r1==r2)
  _ == _ = False
  ```

- an implementation of `(==)` is called a **method**

- **CAVEAT** do not confuse all these concepts with the homonymous concepts in OO programming: there are similarities but also big differences
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- in Java, an Object is an *instance* of a Class
- in Haskell, a Type is an *instance* of a Class
Eq and Ord in the Prelude

- Eq offers also a standard definition of \( \neq \), derived from \( == \):

```haskell
class Eq a where
  (==), (/=) :: a -> a -> Bool
  x /= y = not (x == y)
```

- we can also extend Eq with comparison operations:

```haskell
class (Eq a) => Ord a where
  (<), (<=), (>=), (>) :: a -> a -> Bool
  max, min :: a -> a -> a
```

- Ord is also called a subclass of Eq

- it is possible to have multiple inheritance: class \((X \ a, Y \ a) \Rightarrow Z \ a\)
Another important class: Show

- It is used for **showing**: to have an instance we must implement `show`.
- E.g., functions do not have a standard representation:

  ```haskell
  Prelude> (+)
  <interactive>:2:1:
  No instance for (Show (a0 -> a0 -> a0))
  arising from a use of ‘print’
  Possible fix:
  add an instance declaration for (Show (a0 -> a0 -> a0))
  ```

- Well, we can just use a trivial one:

  ```haskell
  instance Show (a -> b) where
  show f = "<< a function >>"
  ```
we can also represent binary trees:

```haskell
instance Show a => Show (Tree a) where
    show (Leaf a) = show a
    show (Branch x y) = "<" ++ show x ++ " | " ++ show y ++ ">
```

e.g.

```
Branch
  (Branch
    (Leaf 'a') (Branch (Leaf 'b') (Leaf 'c')))
  (Branch
    (Leaf 'd') (Leaf 'e'))
```

is represented as

```
<<'a' | 'b' | 'c'>> | <<'d' | 'e'>>
```
usually it is not necessary to explicitly define instances of some classes, e.g. Eq and Show

Haskell can be quite smart and do it automatically, by using **deriving**

for example we may define binary trees using an infix syntax and automatic Eq, Show like this:

```
infixr 5 :^:
data Tr a = Lf a | Tr a :^: Tr a
  deriving (Show, Eq)
```

e.g.

```
*Main> let x = Lf 3 :^: Lf 5 :^: Lf 2
*Main> let y = (Lf 3 :^: Lf 5) :^: Lf 2
*Main> x == y
False
*Main> x
Lf 3 :^: (Lf 5 :^: Lf 2)
```

An example with class Ord

- Rock-paper-scissors in Haskell

\[
data \text{RPS} = \text{Rock} \mid \text{Paper} \mid \text{Scissors} \text{ deriving (Show, Eq)}
\]

instance Ord RPS where
  \[x \leq y \mid x = y\] = True
  Rock \leq\ Paper = True
  Paper \leq\ Scissors = True
  Scissors \leq\ Rock = True
  _ \leq\ _ = False

- note that we only needed to define (\leq\) to have the instance
An example with class Num

- a simple re-implementation of rational numbers

```haskell
data Rat = Rat !Integer !Integer deriving Eq

simplify (Rat x y) = let g = gcd x y
                   in Rat (x `div` g) (y `div` g)
makerat x y = simplify (Rat x y)

instance Num Rat where
   (Rat x y) + (Rat x' y') = makerat (x*y'+x'*y) (y*y')
   (Rat x y) - (Rat x' y') = makerat (x*y'-x'*y) (y*y')
   (Rat x y) * (Rat x' y') = makerat (x*x') (y*y')
   abs (Rat x y) = makerat (abs x) (abs y)
   signum (Rat x y) = makerat (signum x * signum y) 1
   fromInteger x = makerat x 1
```

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An example with class Num (cont.)

- Ord:
  
  ```haskell
  instance Ord Rat where
  (Rat x y) <= (Rat x’ y’) = x*y’ <= x’*y
  ```

- a better show:
  
  ```haskell
  instance Show Rat where
  show (Rat x y) = show x ++ "/" ++ show y
  ```

- note: Rationals are in the Prelude!

- moreover, there is class Fractional for / (not covered here)

- but we could define our version of division as follows:
  
  ```haskell
  x // (Rat x’ y’) = x * (Rat y’ x’)
  ```
what is the type of the standard function `getChar`, that gets a character from the user? 

```
getChar :: theUser -> Char
```

first of all, it is not **referentially transparent**: two different calls of `getChar` could return different characters

In general, IO computation is based on **state change** (e.g. of a file), hence if we perform a **sequence of operations**, they must be performed in **order** (and this is not easy with **call-by-need**).
getChar can be seen as a function $:: \text{Time} \rightarrow \text{Char}$.

indeed, it is an **IO action** (in this case for Input):

getChar $:: \text{IO Char}$

quite naturally, to print a character we use **putChar**, that has type:

putChar $:: \text{Char} \rightarrow \text{IO ()}$

**IO** is an instance of the **monad** class, and in Haskell it is considered as an **indelible stain of impurity**
A very simple example of an IO program

- **main** is the default entry point of the program (like in C)

```haskell
main = do {
    putStr "Please, tell me something">
    thing <- getLine;
    putStrLn $ "You told me " ++ thing ++ "\".";
}
```

- special syntax for working with IO: **do**, `<-`
- we will see its real semantics later, used to define an IO action as an **ordered sequence** of IO actions
- "<-" (note: not =) is used to obtain a value from an IO action
- types:
  ```haskell
  main :: IO ()
  putStrLn :: String -> IO ()
  getLine :: IO String
  ```
compile with e.g. `ghc readFile.hs`

```haskell
import System.IO
import System.Environment

readfile = do
  args <- getArgs; -- command line arguments
  handle <- openFile (head args) ReadMode;
  contents <- hGetContents handle; -- note: lazy
  putStr contents;
  hClose handle;
main = readFile
```

- `readfile stuff.txt` reads "stuff.txt" and shows it on the screen
- `hGetContents` reads lazily the contents of the file
Of course, purely functional Haskell code can raise exceptions: \texttt{head \[]}, \texttt{3 \textquoteleft div\textquoteleft 0}, ... 

but if we want to catch them, we need an IO action:

\texttt{handle :: Exception e => (e -> IO a) \rightarrow IO a \rightarrow IO a;}

the 1st argument is the \texttt{handler}

Example: we catch the errors of \texttt{readfile}

\begin{verbatim}
import Control.Exception
import System.IO.Error
...
main = handle handler readfile
  where handler e
      | isDoesNotExistError e =
      | putStrLn "This file does not exist."
      | otherwise =
      | putStrLn "Something is wrong."
\end{verbatim}
Other classical data structures

- What about usual, practical data structures (e.g. arrays, hash-tables)?
- Traditional versions are imperative! If really needed, there are libraries with imperative implementations living in the IO monad
- Idiomatic approach: use immutable arrays (Data.Array), and maps (Data.Map, implemented with balanced binary trees)
- find are respectively $O(1)$ and $O(\log n)$; update $O(n)$ for arrays, $O(\log n)$ for maps
- of course, the update operations copy the structure, do not change it
import Data.Map

exmap = let m = fromList [("nose", 11), ("emerald", 27)]
    n = insert "rug" 98 m
    o = insert "nose" 9 n
    in (m ! "emerald", n ! "rug", o ! "nose")

- exmap evaluates to (27, 98, 9)
Example code: Arrays

- `(//)` is used for update/insert

- `listArray`'s first argument is the **range** of indexing (in the following case, indexes are from 1 to 3)

```haskell
import Data.Array

exarr = let m = listArray (1,3) ["alpha","beta","gamma"]
    n = m // [(2,"Beta")]
    o = n // [(1,"Alpha"), (3,"Gamma")]
    in (m ! 1, n ! 2, o ! 1)

exarr evaluates to ("alpha","Beta","Alpha")
```
We saw that IO is a type constructor, instance of *Monad*

But we still do not know what a Monad is

Recent versions of GHC make the trip a bit longer, because we need first to introduce the following classes:

- Foldable (not required, but useful)
- Functor
- Applicative (Functor)
Foldable is a class used for folding, of course.
The main idea is the one we know from foldl and foldr for lists:
we have a container, and a binary operation $f$, and we want to apply $f$ to all the elements in the container.
a minimal implementation of Foldable requires foldr.
Example: foldable binary trees

Let’s go back to our binary trees

```haskell
data Tree a = Empty | Leaf a | Node (Tree a) (Tree a)
```

we can easily define a *foldr* for them

```haskell
tfoldr f z Empty = z
tfoldr f z (Leaf x) = f x z
tfoldr f z (Node l r) = tfoldr f (tfoldr f z r) l
```

instance Foldable Tree where

```haskell
foldr = tfoldr
```

```haskell
> foldr (+) 0 (Node (Node (Leaf 1) (Leaf 3)) (Leaf 5))
9
```
**Maybe** is used to represent computations that may fail: we either have *Just v*, if we are lucky, or *Nothing*.

It is basically a simple "conditional container"

```
data Maybe a = Nothing | Just a
```

It is adopted in many recent languages, to avoid NULL and limit exceptions usage.

Examples are Scala (basically the ML family approach): Option[T], with values None or Some(v); Swift, with Optional<T>.

It is quite simple, so we will use it in our examples with Functors & C.
Of course, Maybe is foldable

instance Foldable Maybe where
    foldr _ z Nothing = z
    foldr f z (Just x) = f x z
**Functor** is the class of all the types that offer a *map* operation

(so there is an analogy with Foldable vs folds)

the map operation of functors is called **fmap** and has type:

\[ \text{fmap} :: (a \to b) \to f a \to f b \]

it is quite natural to define map for a container, e.g.:

```haskell
instance Functor Maybe where
    fmap _ Nothing = Nothing
    fmap f (Just a) = Just (f a)
```
Functor laws

- Well-defined functors should obey the following laws:
- \( \text{fmap id} = \text{id} \) (where \( \text{id} \) is the identity function)
- \( \text{fmap (f . g)} = \text{fmap f . fmap g} \)
- You can try, as an exercise, to check if the functors we are defining obey the laws
Trees can be functors, too

- First, let us define a suitable \textit{map} for trees:

\[
\begin{align*}
tmap \ f \ \text{Empty} &= \ \text{Empty} \\
tmap \ f \ (\text{Leaf} \ x) &= \ \text{Leaf} \ \$ \ f \ x \\
tmap \ f \ (\text{Node} \ l \ r) &= \ \text{Node} \ (tmap \ f \ l) \ (tmap \ f \ r)
\end{align*}
\]

- That's all we need:

\[
\text{instance \ Functor \ Tree \ where} \\
\quad \text{fmap} = \ tmap
\]

\[
\text{-- example} \\
\text{> fmap} \ (\text{+1}) \ (\text{Node} \ (\text{Node} \ (\text{Leaf} \ 1) \ (\text{Leaf} \ 2)) \ (\text{Leaf} \ 3)) \\
\text{Node} \ (\text{Node} \ (\text{Leaf} \ 2) \ (\text{Leaf} \ 3)) \ (\text{Leaf} \ 4)
\]
Applicative Functors

- In our voyage toward monads, we must consider also an extended version of functors, i.e. **Applicative functors**

- The definition looks indeed exotic:

```haskell
class (Functor f) => Applicative f where
    pure :: a -> f a
    (<*>) :: f (a -> b) -> f a -> f b
```

- note that $f$ is a type constructor, and $f$ a is a Functor type

- moreover, $f$ must be parametric with one parameter

- if $f$ is a container, the idea is not too complex:
  - pure takes a value and returns an $f$ containing it
  - $<*>$ is like $\text{fmap}$, but instead of taking a function, takes an $f$ containing a function, to apply it to a suitable container of the same kind
Maybe is an Applicative Functor

Here is its definition:

```haskell
instance Applicative Maybe where
  pure = Just

  Just f  <*> m     = fmap f m
  Nothing <*> _     = Nothing
```
Lists

- Of course, lists are instances of Foldable and Functor. What about Applicative?
- For that, it is first useful to introduce **concat**
  
  ```haskell
  concat :: Foldable t => t [a] -> [a]
  ```
- So we start from a container of lists, and get a list with the *concatenation* of them:
  
  ```haskell
  concat [[1,2],[3],[4,5]] is [1,2,3,4,5]
  ```
- it can be defined as: `concat l = foldr (++) [] l`
- its composition with *map* is called **concatMap**
  
  ```haskell
  concatMap f l = concat $ map f l
  > concatMap (\x -> [x, x+1]) [1,2,3]
  [1,2,2,3,3,4]
  ```
Lists are instances of Applicative

- With `concatMap`, we get the standard implementation of `<*>` for lists:

  ```haskell
  instance Applicative [] where
      pure x = [x]
      fs <*> xs = concatMap (\f -> map f xs) fs
  ```

- What can we do with it? For instance we can apply list of operations to lists:

  ```haskell
  > [(+1),(*2)] <*> [1,2,3]
  [2,3,4,2,4,6]
  ```

- Note that we `map` the operations in sequence, then we `concatenate` the resulting lists.
Following the list approach, we can make our binary trees an instance of Applicative Functors.

First, we need to define what we mean by tree concatenation:

\[
\begin{align*}
t\text{conc } \text{Empty } t &= t \\
t\text{conc } t \text{ Empty } &= t \\
t\text{conc } t_1 t_2 &= \text{Node } t_1 t_2
\end{align*}
\]

Now, concat and concatMap (here t\text{concmap} for short) are like those of lists:

\[
\begin{align*}
t\text{concat } t &= \text{tfoldr } t\text{conc } \text{Empty } t \\
t\text{concmap } f \ t &= t\text{concat } \$ \ t\text{map } f \ t
\end{align*}
\]
Here is the natural definition (practically the same of lists):

\[
\text{instance Applicative Tree where}
\]
\[
\text{pure} = \text{Leaf}
\]
\[
\text{fs} \triangleright\triangleright \text{xs} = \text{tconcmap (} \lambda f \rightarrow \text{tmap } f \text{ } \text{xs}) \text{ fs}
\]

Let's try it:

\[
> (\text{Node (Leaf (+1)) (Leaf (*2))) \triangleright\triangleright \\
\text{Node (Node (Leaf 1) (Leaf 2)) (Leaf 3)})
\]

\[
\text{Node (Node (Node (Leaf 2) (Leaf 3))}
\]
\[
\text{(Leaf 4))}
\]
\[
\text{(Node (Node (Leaf 2) (Leaf 4))}
\]
\[
\text{(Leaf 6))}
\]
A peculiar type class: Monad

- introduced by Eugenio Moggi in 1991, a monad is a kind of **algebraic data type** used to represent computations (instead of data in the domain model) - we will often call these computations **actions**
- monads allow the programmer to **chain** actions together to build an **ordered sequence**, in which each action is **decorated with additional processing rules** provided by the monad and performed automatically
- monads are **flexible** and **abstract**. This makes some of their **applications** a bit hard to understand.
monads can also be used to make imperative programming easier in a pure functional language

in practice, through them it is possible to define an imperative sub-language on top of a purely functional one

there are many examples of monads and tutorials (many of them quite bad) available in the Internet
class Applicative m => Monad m where

  -- Sequentially compose two actions, passing any value produced
  -- by the first as an argument to the second.
  (>>=) :: m a -> (a -> m b) -> m b

  -- Sequentially compose two actions, discarding any value produced
  -- by the first, like sequencing operators (such as the semicolon)
  -- in imperative languages.
  (>>) :: m a -> m b -> m b
  m >> k = m >>= \_ -> k

  -- Inject a value into the monadic type.
  return :: a -> m a
  return = pure

  -- Fail with a message.
  fail :: String -> m a
  fail s = error s
Note that only >>= is required, all the other methods have standard definitions

- >>= and >> are called bind
- m a is a computation (or action) resulting in a value of type a
- return is by default pure, so it is used to create a single monadic action. E.g. return 5 is an action containing the value 5.
- bind operators are used to compose actions
  - x >>= y performs the computation x, takes the resulting value and passes it to y; then performs y.
  - x >> y is analogous, but "throws away" the value obtained by x
Maybe is a Monad

Its definition is straightforward

```haskell
instance Monad Maybe where
  (Just x) >>= k = k x
  Nothing <<= _ = Nothing
  fail _ = Nothing
```
Examples with **Maybe**

- The information managed automatically by the monad is the “bit” which encodes the **success** (i.e. *Just*) or failure (i.e. *Nothing*) of the action sequence
- e.g. `Just 4 >>= Just >> Nothing >> Just 6` evaluates to *Nothing*
- a variant: `Just 4 >>= Just >> Nothing >> Just 6`
- another: `Just 4 >>= Just 1 >>= Just` (what is the result in this case?)
for a monad to behave correctly, method definitions must obey the following laws:

1) \textit{return} is the \textbf{identity element}:

\begin{align*}
\text{return } x \triangleright\!\!\!\!	riangleright f & \iff f \ x \\
 m \triangleright\!\!\!\!	riangleright \text{return} & \iff m \\
\end{align*}

2) \textbf{associativity} for binds:

\begin{align*}
 (m \triangleright\!\!\!\!	riangleright f) \triangleright\!\!\!\!	riangleright g & \iff m \triangleright\!\!\!\!	riangleright (\lambda x \rightarrow (f \ x \triangleright\!\!\!\!	riangleright g)) \\
\end{align*}

(monads are analogous to \textbf{monoids}, with \text{return} = 1 and \triangleright\!\!\!\!	riangleright = \_\_\_.)
Example: monadic laws application with Maybe

- \( (\text{return } 4 :: \text{Maybe Integer}) >>= \ x \rightarrow \text{Just } (x+1) \)
  Just 5
- \( \text{Just } 5 >>= \text{return} \)
  Just 5
- \( (\text{return } 4 >>= \ x \rightarrow \text{Just } (x+1)) >>= \ x \rightarrow \text{Just } (x*2) \)
  Just 10
- \( \text{return } 4 >>= (\ y \rightarrow \ ((\ x \rightarrow \text{Just } (x+1)) \ y) >>= \ x \rightarrow \text{Just } (x*2)) \)
  Just 10
The **do** syntax is used to avoid the explicit use of >> and >>

The essential translation of **do** is captured by the following two rules:

\[
\text{do } e_1 \ ; \ e_2 \quad \Leftrightarrow \quad e_1 \ >> \ e_2 \\
\text{do } p \ <- \ e_1 \ ; \ e_2 \quad \Leftrightarrow \quad e_1 \ >>= \ \lambda p \to e_2
\]

Note that they can also be written as:

\[
\text{do } e_1 \\
\quad e_2 \\
\text{do } e_1
\]

\[
\text{do } p \ <- \ e_1 \\
\quad e_2
\]

or:

\[
\text{do } \{ e_1 \ ; \\
\quad e_2 \ \} \\
\text{do } \{ p \ <- \ e_1 \ ; \\
\quad e_2 \ \}
\]
Caveat: **return** does not return

- IO is a build-in monad in Haskell: indeed, we used the *do* notation for performing IO
- there are some catches, though – it looks like an imperative sub-language, but its semantics is based on bind and pure
- For example:

```haskell
esp :: IO Integer
esp = do x <- return 4
         return (x+1)

> esp
5
```
The List Monad

- **List**: monadic binding involves joining together a set of calculations for each value in the list
- In practice, *bind* is `concatMap`

```haskell
instance Monad [] where
    xs >>= f = concatMap f xs
    fail _ = []
```
Lists: do vs comprehensions

- list comprehensions can be expressed in *do* notation
- e.g. this comprehension
  \[
  [(x, y) \mid x \leftarrow [1,2,3], \ y \leftarrow [1,2,3]]
  \]
  is equivalent to:
  
  ```
  do x <- [1,2,3]
      y <- [1,2,3]
      return (x,y)
  ```
the List monad (cont.)

to understand our example of comprehension, i.e.

\[\text{testcomp} = \text{do } x \leftarrow [1,2,3] \]
\[\quad y \leftarrow [1,2,3] \]
\[\quad \text{return } (x,y)\]

we can rewrite it following the monad definition:

\[\text{testcomp'} = \]
\[\quad [1,2,3] >>= (\lambda x \rightarrow [1,2,3] >>= (\lambda y \rightarrow \text{return } (x,y)))\]
that is:

```haskell
testcomp'' =
    concatMap f0 [1,2,3]
where f0 x = concatMap f1 [1,2,3]
    where f1 y = [(x,y)]
```
We can now define our own monad with binary trees. Knowing about lists, it is not too hard:

```
instance Monad Tree where
    xs >>= f = tconcmap f xs
    fail _ = Empty
```
Now some examples

- Monads are abstract, so monadic code is very flexible, because it can work with any instance of Monad
- A simple monadic comprehension:

```
exmon :: (Monad m, Num r) => m r -> m r -> m r
exmon m1 m2 = do x <- m1
                y <- m2
                return $ x - y
```
Let’s apply it to lists and trees

First, we try with lists:

> exmon [10, 11] [1, 7]
[9,3,10,4]

on trees is not much different

> exmon (Node (Leaf 10) (Leaf 11)) (Node (Leaf 1) (Leaf 7))
Node (Node (Leaf 9) (Leaf 3))
    (Node (Leaf 10) (Leaf 4))
Not just simple containers

- Monads can be used to implement parsers, continuations, ...
- and, of course, IO
- Let’s try exmon with IO Int:
  
  -- read is like in Scheme, here is used to parse the number
  exmon (do putStr "?> "
        x <- getLine;
        return (read x :: Int))
    (return 10)

- What is the result, if we enter 12?