Hybrid Systems Course
Switched systems & stability

OUTLINE

Switched Systems

Stability of Switched Systems
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Switched Systems

Stability of Switched Systems

SWITCHED SYSTEMS

- a family of systems

\[ \dot{x} = f_q(x), \quad q \in Q \]
SWITCHED SYSTEMS

- a family of systems

\[ \dot{x} = f_q(x), \quad q \in Q \]

- a signal that orchestrates the switching between them

Note: The value of \( x \) is preserved when a switching occurs

SWITCHED SYSTEMS AS HYBRID SYSTEMS

\[ \dot{x} = f_q(x), \quad q \in Q = \{1, 2\} \]
SWITCHED SYSTEMS vs. HYBRID AUTOMATA

• switched systems can be seen as a higher-level abstraction of hybrid automata (details of the discrete behavior neglected)
• simpler to describe but with more solutions than the original hybrid automata (conservative analysis results)

Switched systems are of interest in their own right
SWITCHING

• time-dependent versus state-dependent switching

• autonomous versus controlled switching
TIME-DEPENDENT SWITCHING

\[ \sigma : [0, \infty) \rightarrow Q \] (exogenous) switching signal

- piecewise constant function of time
- \( \sigma(t) \) specifies the system that is active at time \( t \)

SWITCHED LINEAR SYSTEMS

\[ \dot{x} = A_\sigma x \]

- family of systems
  \[ \dot{x} = A_q x, \ q \in Q = \{1, 2, \ldots, m\} \]
- time-dependent switching rule
  \[ \sigma : [0, \infty) \rightarrow Q \]
STATE-DEPENDENT SWITCHING

\[ \sigma : X \rightarrow Q \] (endogenous) switching signal

- the state space \( X \) is partitioned into operating regions, each one associated to a system
- \( \sigma(x) \) specifies the system that is active when the state is \( x \)
SWITCHING

• time-dependent versus state-dependent switching

• autonomous versus controlled switching

AUTONOMOUS SWITCHING

• Switching events are triggered by an external mechanism over which we do not have control

Examples:
  unpredictable environmental factors
  component failures
CONTROLLED SWITCHING

• Switching are imposed so as to achieve a desired behavior of the resulting system → switched control systems

Reasons for switching:
• large modeling uncertainty
• nature of the control problem (phase systems)
• sensor/actuator limitations
• …

SWITCHING CONTROL

The closed-loop system is a switched system

Applications: control of complex plants with multiple operating modes (e.g. flight control) systems affected by large modeling uncertainty
QUANTIZED CONTROL

OUTLINE

Switched Systems

Stability of Switched Systems
SWITCHED SYSTEMS: TIME-DEPENDENT SWITCHING

\[ \dot{x} = f_\sigma(x) \]

- family of systems
  \[ \dot{x} = f_q(x), \; q \in Q = \{1, 2, \ldots, m\} \]
  with \( f_q(0) = 0, \forall q \in Q \)

- piecewise constant switching signal
  \( \sigma : [0, \infty) \to Q \)

Stability of the equilibrium \( x_e = 0 \)?
Problem: find conditions that guarantee asymptotic stability under arbitrary switching
Problem: identify those switching signals that ensure asymptotic stability
• Stability for arbitrary switching

• Stability for constrained switching

• Stability for arbitrary switching

• Stability for constrained switching
GLOBAL UNIFORM ASYMPTOTIC STABILITY (GUAS)

\[ \dot{x} = f_\sigma(x) \]

\[ f_q(0) = 0, \ q \in Q = \{1, 2, \ldots, m\} \]

The equilibrium \( x_e=0 \) is GUAS if it is globally asymptotically stable, uniformly with respect to the switching signals \( \sigma \)

Assumption:

\[ \dot{x} = f_q(x), \ q \in Q = \{1, 2, \ldots, m\} \]

family of systems with GAS equilibrium in \( x=0 \)

Remark:

if the equilibrium \( x_e=0 \) is not GAS for one of the systems, then it cannot be GUAS for the switched system
COMMON LYAPUNOV FUNCTION

The family of systems
\[ \dot{x} = f_q(x), \; q \in Q = \{1, 2, \ldots, m\} \]
share a radially unbounded common Lyapunov function at \( x_e=0 \) if there exists a continuously differentiable function \( V \) such that

\[ V(x) > 0, \; \forall x \neq 0 \quad V(0) = 0 \]
\[ \|x\| \to \infty \implies V(x) \to \infty \]
\[ \frac{\partial V}{\partial x}(x)f_q(x) < 0, \; \forall x \neq 0, \; \forall q \in Q \]

COMMON LYAPUNOV FUNCTION

\[ \dot{x} = f_\sigma(x) \]

If all systems in the family
\[ \dot{x} = f_q(x), \; q \in \{1, 2, \ldots, m\} \]
share a radially unbounded common Lyapunov function at \( x_e=0 \), then, the equilibrium \( x_e=0 \) is GUAS.

Proof.
Same reasoning as for more general hybrid systems
GLOBALLY QUADRATIC LYAPUNOV FUNCTION

\[ \dot{x} = A_G x \]

If there exists \( P = P^T > 0 \) such that
\[ PA_q + A_q^T P < 0, \forall q \in Q = \{1, 2, \ldots, m\} \]
then, the equilibrium \( x_e = 0 \) is GUAS.

Proof.
\[ V(x) = x^T P x \] is a radially unbounded common Lyapunov function at \( x_e = 0 \).

GLOBALLY QUADRATIC LYAPUNOV FUNCTION

The existence of a globally quadratic Lyapunov function is not necessary for \( x_e = 0 \) to be GUAS

Example:
\[
A_1 = \begin{bmatrix}
-1 & -1 \\
1 & -1
\end{bmatrix} \quad A_2 = \begin{bmatrix}
-1 & -10 \\
0.1 & -1
\end{bmatrix}
\]

\( x_e = 0 \) is GUAS but there is no common quadratic Lyapunov function
SWITCHED SYSTEMS WITH A SPECIAL STRUCTURE

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, q \in Q = \{1, 2, \ldots, m\} \)
- commute
- are upper (or lower) triangular

COMMUTING HURWITZ MATRICES \(\Rightarrow\) GUAS

\[ \dot{x} = A_\sigma x \]

\( Q = \{1, 2\} \ A_1 A_2 = A_2 A_1 \)

\[
\begin{align*}
\sigma = 1 & \quad \sigma = 2 \quad \sigma = 1 \quad \sigma = 2 \quad \cdots \\
\downarrow s_1 & \quad \downarrow t_1 & \quad \downarrow s_2 & \quad \downarrow t_2 & \quad \cdots & \quad \downarrow t \\
\end{align*}
\]

\[ x(t) = e^{A_2 t_k} e^{A_1 t_k} \cdots e^{A_2 t_1} e^{A_1 t_1} x(0) \]

\[ = e^{A_2 (t_k + \cdots + t_1)} e^{A_1 (s_k + \cdots + s_1)} x(0) \rightarrow 0 \]
COMMUTING HURWITZ MATRICES $\Rightarrow$ GUAS

$$\dot{x} = A_\sigma x$$

$$Q = \{1, 2\} \quad A_1A_2 = A_2A_1$$

$\exists$ quadratic common Lyapunov function: $V(x) = x^T P_2 x$

$$P_1 A_1 + A_1^T P_1 = -I$$

$$P_2 A_2 + A_2^T P_2 = -P_1$$

\[ P_2 = \int_0^\infty e^{A_2^T \tau} P_1 e^{A_2 \tau} d\tau \quad P_1 = \int_0^\infty e^{A_1^T \tau} e^{A_1 \tau} d\tau \quad P_2 A_1 + A_1^T P_2 = -Q \]
\[
\begin{align*}
\dot{x} &= A_\sigma x \\
Q &= \{1, 2\}, \; X = \mathbb{R}^2 \\
\dot{x}_1 &= \lambda_{1,\sigma} x_1 + b_\sigma x_2 \\
\dot{x}_2 &= \lambda_{2,\sigma} x_2
\end{align*}
\]
TRIANGULAR HURWITZ MATRICES $\Rightarrow$ GUAS

\[
\dot{x} = A_\sigma x
\]

\[Q = \{1, 2\}, \ X = \mathbb{R}^2\]

\[
\dot{x}_1 = \lambda_{1, \sigma} x_1 + b_\sigma x_2 \\
\dot{x}_2 = \lambda_{2, \sigma} x_2 \\
\dot{x}_2 = \lambda_{2, \sigma} x_2 \rightarrow |x_2(t)| \leq e^{\text{max}_{\lambda_{2, \sigma}} t} |x_2(0)| \\
\dot{x}_1 = \lambda_{1, \sigma} x_1 + b_\sigma x_2
\]

exponentially stable system  exponentially decaying perturbation

TRIANGULAR HURWITZ MATRICES $\Rightarrow$ GUAS

\[
\dot{x} = A_\sigma x
\]

\[Q = \{1, 2\}, \ X = \mathbb{R}^2\]

\[
\dot{x}_1 = \lambda_{1, \sigma} x_1 + b_\sigma x_2 \\
\dot{x}_2 = \lambda_{2, \sigma} x_2 \\
\exists \text{ quadratic common Lyapunov function} \\
V(x) = x^T P x \\
\text{with } P \text{ diagonal}
SWITCHED SYSTEMS WITH A SPECIAL STRUCTURE

\[ \dot{x} = A_{\sigma}x \]

Hurwitz matrices \( A_q, \ q \in Q = \{1, 2, \ldots, m\} \)

- commute
- are upper (or lower) triangular
- can be transformed to upper (or lower) triangular form by a common similarity transformation

- Stability for arbitrary switching
- Stability for constrained switching
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, q \in Q = \{1, 2\} \)

\[
\begin{align*}
\sigma = 1 & \quad \sigma = 2 \\
\sigma = 1 & \quad \sigma = 2 \\
[1] & \quad [1] \\
t & \quad t \\
\end{align*}
\]

The switching intervals satisfy \( t_i, s_i \geq \tau_D \) dwell time

\[
x(t) = e^{A_2 t_k} e^{A_1 s_k} \ldots e^{A_2 t_i} e^{A_1 s_1} x(0)
\]
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_{\sigma} x \]

Hurwitz matrices \( A_q, \; q \in Q = \{1, 2\} \)

\[ \sigma = 1 \quad t \quad \sigma = 1 \quad t \quad \sigma = 2 \quad t \quad \sigma = 2 \quad t \quad \cdots \]

The switching intervals satisfy \( t_i, s_i \geq \tau_D \)

\[ x(t) = e^{A_2 t_k} e^{A_1 s_k} \cdots e^{A_2 t_1} e^{A_1 s_1} x(0) \]

\[ \| e^{A_i \Delta t} \| \leq \mu e^{-\lambda_0 \tau_D} \leq \mu e^{-\lambda_0} < 1 \]

slowest decay rate so that the inequality holds \( \forall \; i \)

\[ \mu < e^{\tau D (\lambda_0 - \lambda)} \quad \lambda \in (0, \lambda_0) \]
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, q \in Q = \{1, 2\} \)

\[ \sigma = 1 \quad \sigma = 2 \quad \sigma = 1 \quad \sigma = 2 \quad \cdots \quad t \]

The switching intervals satisfy \( t_i, s_i \geq \tau_D \)

\[ x(t) = e^{A_2 t_k} e^{A_1 s_k} \cdots e^{A_2 t_1} e^{A_1 s_1} x(0) \]

\[ \|e^{A_\tau \Delta t}\| \leq \mu e^{-\lambda_0 \Delta t} \leq e^{-\lambda \Delta t} < 1 \]

\[ \tau_D \geq \frac{\log \frac{\mu}{\lambda_0 - \lambda}}{\lambda_0 - \lambda} \]

\( \lambda \in (0, \lambda_0) \)
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, q \in Q = \{1, 2\} \)

\[
\begin{array}{c c c c c}
\sigma = 1 & \sigma = 2 & \sigma = 1 & \sigma = 2 & \ldots \\
\sigma_1 & \sigma_2 & \sigma_1 & \sigma_2 & \ldots \\
s_1 & t_1 & s_2 & t_2 & \ldots \\
& \vdots & & \vdots & \\
\end{array}
\]

\[ x(t) = e^{A_2 t_k} e^{A_1 s_k} \ldots e^{A_2 t_1} e^{A_1 s_1} x(0) \]

\[ \|e^{A_1 \Delta t}\| \leq e^{-\lambda \Delta t} \quad \Rightarrow \quad \|x(t)\| \leq e^{-\lambda t}\|x(0)\| \]

DWELL TIME: EXTENSIONS

- adaptive version: the dwell time is selected based on matrix \( A_i \) so as to make the dynamics of the system contract by some \( \nu \in (0,1) \) during the dwell time

- average dwell time

\[ N_\sigma(T, t) \leq N_0 + \frac{T - t}{\tau_{AD}} \]

\[ \text{average dwell time} \]

\[ \text{# of switches on} \quad (t, T) \]

\begin{align*}
N_0 = 0 & \quad \text{no switching: cannot switch if} \quad T - t < \tau_{AD} \\
N_0 = 1 & \quad \text{dwell time: cannot switch twice if} \quad T - t < \tau_{AD}
\end{align*}

Same bound on \( \tau_{AD} \) as in the dwell time case.
Larger values of \( x(t) \) in finite time because of \( N_0 \)
STABILITY UNDER STATE-DEPENDENT SWITCHING

\[ \sigma: X \to Q : \sigma(x) = i \text{ if } x \in X_i \]

STATE-DEPENDENT COMMON LYAPUNOV FUNCTIONS

If \( V: \mathbb{R}^n \to \mathbb{R} \) is a \( C^1 \) radially unbounded function such that

\[
\begin{align*}
V(0) &= 0 \\
V(x) &> 0, \forall x \in \mathbb{R}^n \setminus \{0\} \\
\frac{\partial V}{\partial x}(x)A_{\sigma(x)}(x) &< 0, \forall x \in \mathbb{R}^n
\end{align*}
\]

then, \( x_e = 0 \) is GAS for \( \dot{x} = A_{\sigma(x)}x \)

Remarks:
need that \( \frac{\partial V}{\partial x}(x)A_q(x) < 0 \) only when \( \sigma \) is equal to \( q \), i.e. on \( X_q \)
matrices \( A_q \) are not required to be Hurwitz
STABILIZATION BY SWITCHING

\[
\dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{both unstable}
\]

Assume: \( A = \alpha A_1 + (1 - \alpha) A_2 \) stable for some \( \alpha \in (0,1) \)

\[
A^T P + PA < 0
\]
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \] both unstable

Assume: \( A = \alpha A_1 + (1 - \alpha) A_2 \) stable for some \( \alpha \in (0,1) \)

\[
\alpha (A_1^T P + PA_1) + (1 - \alpha) (A_2^T P + PA_2) < 0
\]

So for each \( x \neq 0 \):

either \( x^T (A_1^T P + PA_1) x < 0 \) or \( x^T (A_2^T P + PA_2) x < 0 \)
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{both unstable} \]

Assume: \( A = \alpha A_1 + (1 - \alpha) A_2 \) stable for some \( \alpha \in (0,1) \)

\[ \alpha (A_1^T P + PA_1) + (1 - \alpha) (A_2^T P + PA_2) < 0 \]

So for each \( x \neq 0 \):
either \( x^T (A_1^T P + PA_1) x < 0 \) or \( x^T (A_2^T P + PA_2) x < 0 \)

Region where system 1 is active Region where system 2 is active

\[ V(x) = x^T P x \quad \text{is a Lyapunov function at } x_e = 0 \]
for the system \( \dot{x} = A_{\sigma(x)} x \Rightarrow \text{GAS} \)
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{both unstable} \]

Assume: \( A = \alpha A_1 + (1 - \alpha) A_2 \) stable for some \( \alpha \in (0,1) \)

\[
\alpha (A_1^T P + PA_1) + (1 - \alpha) (A_2^T P + PA_2) < 0
\]

So for each \( x \neq 0 \):

either \( x^T (A_1^T P + PA_1) x < 0 \) or \( x^T (A_2^T P + PA_2) x < 0 \)

STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{both unstable} \]

**Theorem:**
If the matrices \( A_1 \) and \( A_2 \) have a Hurwitz combination, then, there exists a state dependent switching strategy such that the switching system \( \dot{x} = A_\sigma x \) is GAS
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{both unstable} \]

**Theorem:**
If the matrices \( A_1 \) and \( A_2 \) have a Hurwitz combination, then, there exists a state dependent switching strategy such that the switching system \( \dot{x} = A_\sigma x \) is GAS

**Extensions to the m>2 matrices case:**
- two matrices \( A_i \) and \( A_j \) have a Hurwitz combination
- more than 2 matrices have a Hurwitz combination

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Main source:

*Switching in Systems and Control*