Discrete-Time Optimal Control Problem

A discrete-time controlled dynamical system

\[ x_{k+1} = f(x_k, u_k), \quad k = 0, 1, \ldots, \text{ initial condition } x_0 \]
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A discrete-time controlled dynamical system

\[ x_{k+1} = f(x_k, u_k), \quad k = 0, 1, \ldots, \text{initial condition } x_0 \]

**Problem**: Given a time horizon \([0, N]\), find the optimal input sequence \(u = (u_0, \ldots, u_{N-1})\) that minimizes

\[ J(u) = \sum_{k=0}^{N-1} \ell(x_k, u_k) + \phi(x_N) \]

- Running cost \(\ell(x_k, u_k) \geq 0\)
- Terminal cost \(\phi(x_N) \geq 0\)

Extension to discrete-time hybrid system

\[ x_{k+1} = f(x_k, u_k, \sigma_k), \quad k = 0, 1, \ldots \]
Linear Quadratic Regulation (LQR) Problem

A discrete-time linear system with given initial condition \( x_0 \):

\[
x_{k+1} = Ax_k + Bu_k
\]

Problem: find optimal input sequence \( u = (u_0, \ldots, u_{N-1}) \) that minimizes

\[
J(u) = \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k) + x_N^T Q x_N
\]
Linear Quadratic Regulation (LQR) Problem

A discrete-time linear system with given initial condition $x_0$:

$$x_{k+1} = Ax_k + Bu_k$$

**Problem:** find optimal input sequence $u = (u_0, \ldots, u_{N-1})$ that minimizes

$$J(u) = \sum_{k=0}^{N-1} \left( x_k^T Q x_k + u_k^T R u_k \right) + x_N^T Q_f x_N$$

- State weight matrix $Q = Q^T \succeq 0$
- Control weight matrix $R = R^T > 0$ (no free control)
- Final state weight matrix $Q_f = Q_f^T \succeq 0$

LQR Problem: Motivation

A discrete-time linear system with given initial condition $x_0$:

$$x_{k+1} = Ax_k + Bu_k$$

**Problem:** find optimal input sequence $u = (u_0, \ldots, u_{N-1})$ that minimizes

$$J(u) = \sum_{k=0}^{N-1} \left( x_k^T Q x_k + u_k^T R u_k \right) + x_N^T Q_f x_N$$

Compromises between the conflicting goals:
- minimize overall control effort
- minimize overall state deviation from 0

Larger control input can drive the state to zero faster
LQR Problem: Special Cases

Energy efficient stabilization: $Q = Q_f = \alpha I$, $R = \beta I$

$$J(u) = \alpha \sum_{k=0}^{N} \| x_k \|^2 + \beta \sum_{k=0}^{N-1} \| u_k \|^2$$

- Weights $\alpha, \beta > 0$ determine the emphasis between two objectives:
  - (i) state stays close to 0;
  - (ii) use less control energy
LQR Problem: Special Cases

Problem: find the control sequence $u = (u_0, \ldots, u_{N-1})$ with the least energy that can steer the system state from $x_0$ to $x_N = 0$.

- Set $Q = 0$ since we do not care about deviation from 0 of states at times 0, 1, $\ldots$, $N-1$.
- Choose a very large $\alpha$ since the final state $x_N$ needs to be 0 in optimal solution.
LQR Problem: Special Cases

Problem: find the control sequence $u = (u_0, \ldots, u_{N-1})$ with the least energy that can steer the system state from $x_0$ to $x_N = 0$

- Set $Q = 0$ since we do not care about deviation from $0$ of states at times $0, 1, \ldots, N-1$
- Choose a very large $\alpha$ since the final state $x_N$ needs to be $0$ in optimal solution

Minimum energy steering to $0$: $Q = 0$, $Q_f = \alpha I$, $R = I$

$$J(u) = \alpha \|x_N\|^2 + \sum_{k=0}^{N-1} \|u_k\|^2$$

LQR Problem: Special Cases

A discrete-time linear system with output and given initial condition $x_0$:

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k$$

Problem: find optimal input sequence $u = (u_0, \ldots, u_{N-1})$ that minimizes

$$J(u) = \alpha \sum_{k=0}^{N} \|y_k\|^2 + \beta \sum_{k=0}^{N-1} \|u_k\|^2 \quad (\alpha > 0, \beta > 0)$$
**LQR Problem: Special Cases**

A discrete-time linear system with output and given initial condition $x_0$:

\[ x_{k+1} = Ax_k + Bu_k \]
\[ y_k = Cx_k \]

**Problem**: find optimal input sequence $u = (u_0, \ldots, u_{N-1})$ that minimizes

\[ J(u) = \alpha \sum_{k=0}^{N} ||y_k||^2 + \beta \sum_{k=0}^{N-1} ||u_k||^2 \quad (\alpha > 0, \beta > 0) \]

As an LQR problem

- State weight matrix $Q = \alpha C^T C \succeq 0$
- Control weight matrix $R = \beta I \succ 0$
- Final state weight matrix $Q_f = Q = \alpha C^T C \succeq 0$
LQR Problem: Special Cases

A discrete-time linear system with given initial condition $x_0$:

$$x_{k+1} = Ax_k + Bu_k$$

**Problem:** track a reference state trajectory $x_r^0, x_r^1, \ldots, x_r^N$ with efficient control:

$$J(u) = \alpha \sum_{k=0}^{N} \|x_k - x_r^k\|^2 + \beta \sum_{k=0}^{N-1} \|u_k\|^2$$

Tracking error penalty + control energy

Can be formulated as a (time-varying) LQR problem.
LQR Problem: Special Cases

A discrete-time linear system with given initial condition $x_0$:

$$x_{k+1} = Ax_k + Bu_k$$

Problem: track a reference state trajectory $x'_0, x'_1, \ldots, x'_N$ with efficient control:

$$J(u) = \alpha \sum_{k=0}^{N} \| x'_k - x'_k \|^2 + \beta \sum_{k=0}^{N-1} \| u_k \|^2$$

Can be formulated as a (time-varying) LQR problem

• Augment the state $x$ to $\tilde{x} = \begin{bmatrix} x \\ z \end{bmatrix}$ with $z \in \mathbb{R}$; let $\tilde{x}_0 = \begin{bmatrix} x_0 \\ 1 \end{bmatrix}$
LQR Problem: Special Cases

A discrete-time linear system with given initial condition $x_0$:

$$x_{k+1} = Ax_k + Bu_k$$

Problem: track a reference state trajectory $x_0^r, x_1^r, \ldots, x_N^r$ with efficient control:

$$J(u) = \alpha \sum_{k=0}^{N} \|x_k - x_k^r\|^2 + \beta \sum_{k=0}^{N-1} \|u_k\|^2$$

Can be formulated as a (time-varying) LQR problem

- Augment the state $x$ to $\bar{x} = \begin{bmatrix} x \\ z \end{bmatrix}$ with $z \in \mathbb{R}$; let $\bar{x}_0 = \begin{bmatrix} x_0 \\ 1 \end{bmatrix}$
- Augmented state dynamics to $\bar{x}_{k+1} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \bar{x}_k + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k$
- Choose $\bar{Q}_k = \alpha \begin{bmatrix} I & -(x_k^r)^T \\ -(x_k^r) & \|x_k^r\|^2 \end{bmatrix}$, $\bar{R} = \beta I$, $\bar{Q}_f = \bar{Q}_N$

Switched LQR Problem

A discrete-time switched linear system with given initial condition $x_0$:

$$x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k} u_k,$$

- continuous state: $x_k \in \mathbb{R}^n$
- discrete state (mode): $\sigma_k \in \Sigma = \{1, 2, \ldots, M\}$
Switched LQR Problem

A discrete-time switched linear system with given initial condition $x_0$:

$$x_{k+1} = A_{\sigma_k}x_k + B_{\sigma_k}u_k,$$

**Problem:** Find the optimal input sequence $(u_0, \ldots, u_{N-1})$ and mode sequence $(\sigma_0, \ldots, \sigma_{N-1})$ that minimize the cost function

$$\sum_{k=0}^{N-1} \left( x_k^T Q_{\sigma_k} x_k + u_k^T R_{\sigma_k} u_k \right) + x_N^T Q_f x_N.$$

- State weight and control weight matrices mode dependent

**Observations:**
- In different modes, both dynamics and running costs are different
- If mode sequence is given, becomes a time-varying LQR problem
- Main challenge is determining the mode sequence
Example

Building cooling system:
- Multiple building zones
- Air Handling Units (AHUs)

State variables:
- Zone temperatures, humidity

Controls:
- AHU damper open/close
- Fan powers

Objectives:
- Maintain comfort
- Reduce energy usage

Outline
- Solve LQR problem using dynamic programming method
- Extend the method to solve SLQR problem
- Complexity reduction techniques
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- Solve LQR problem using dynamic programming method
- Extend the method to solve SLQR problem
- Complexity reduction techniques

We first look at the LQR problem:

\[
\text{Minimize } \sum_{k=0}^{N-1} \left( x_k^T Q x_k + u_k^T R u_k \right) + x_N^T Q x_N \\
\text{subject to } x_{k+1} = A x_k + B u_k, \ k = 0, \ldots, N - 1 \\
x_0 \text{ fixed}
\]

Direct Approach: LQR via Least-squares

The state along the time horizon \([0, N]\) is a linear function of \(u\) and \(x_0\):

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_N
\end{bmatrix}
= 
\begin{bmatrix}
  B & 0 & \cdots & 0 \\
  AB & B & 0 & \cdots \\
  \vdots & \vdots & \ddots & \vdots \\
  A^{N-1}B & A^{N-2}B & \cdots & B
\end{bmatrix}
\begin{bmatrix}
  u_0 \\
  u_1 \\
  \vdots \\
  u_{N-1}
\end{bmatrix}
+ 
\begin{bmatrix}
  A \\
  A^2 \\
  \vdots \\
  A^N
\end{bmatrix}
\begin{bmatrix}
  x_0
\end{bmatrix}
\]
Direct Approach: LQR via Least-squares

The state along the time horizon \([0, N]\) is a linear function of \(u\) and \(x_0\):

\[
\begin{bmatrix}
x_1 \\ x_0 \\ \vdots \\ x_N
\end{bmatrix} =
\begin{bmatrix}
B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B
\end{bmatrix}
\begin{bmatrix}
u_0 \\ \vdots \\ \vdots \\ u_{N-1}
\end{bmatrix}
+
\begin{bmatrix}
A \\ \vdots \\ \vdots \\ A^N
\end{bmatrix} x_0
\]

Minimize the function:

\[
J(u) = x^T Q x + u^T R u
\]

\[
= (Gu + Hx_0)^T Q (Gu + Hx_0) + u^T R u
\]
Direct Approach: LQR via Least-squares

The state along the time horizon $[0, N]$ is a linear function of $u$ and $x_0$:

$$x = Gu + Hx_0$$

Minimize the function:

$$J(u) = x^TQx + u^TRu$$

$$= (Gu + Hx_0)^TQ(Gu + Hx_0) + u^TRu$$

$$= ||Q^{1/2}(Gu + Hx_0)||^2 + ||R^{1/2}u||^2$$

This is a least-squares problem.

The optimal control is

$$u^* = -(R + G^TQG)^{-1}G^TQHx_0$$
Direct Approach: LQR via Least-squares

Limitations of Direct Approach:
- Matrix inversion needed to find optimal control
- Problem (matrices) dimension increases with time horizon $N$
- Impractical for large $N$ let alone infinite horizon case
- Sensitivity of solutions to numerical errors

Observations:
- Problem easier to solve for shorter time horizon $N$
- $(N + 1)$-horizon solution related to $N$-horizon solution
- Exploit this relation to design an iterative solution procedure
Direct Approach: LQR via Least-squares

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Observations:
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- Exploit this relation to design an iterative solution procedure

Dynamic programming: an iterative approach that can
- Re-use results for smaller $N$ to solve for larger $N$ case
- In each iteration only need to deal with a problem of fixed size

Dynamic Programming Approach

Idea: Solve a sequence of optimal control problems over time horizons $[t, N]$, for decreasing $t = N, N - 1, \ldots, 0$
Dynamic Programming Approach

Idea: Solve a sequence of optimal control problems over time horizons \([t, N]\), for decreasing \(t = N, N - 1, \ldots, 0\).

- **Value function** at time \(t\) is the optimal cost over \([t, N]\):

\[
V_t(x) = \min_{u_t, u_{t+1}, \ldots, u_{N-1}} \sum_{k=t}^{N-1} \left( x_T^k Q x_k + u_T^k R u_k \right) + x_N^T Q x_N
\]

with the initial condition \(x_t = x\).
Dynamic Programming Approach

Idea: Solve a sequence of optimal control problems over time horizons \([t, N]\), for decreasing \(t = N, N-1, \ldots, 0\).

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\]

with the initial condition \(x_t = x\).

- Value function backward iteration:
  \(V_{t-1}(\cdot)\) can be computed based on \(V_t(\cdot)\).

- Optimal cost of original problem is \(V_0(x_0)\).

- Optimal input sequence can be recovered from value functions.

Motivating Example

- Start from point A
- Try to reach point B
- Each step only move right \((\rightarrow N = 6)\)
- Cost labeled on each edge

**Problem:** Path from A to B with the least cost?
Motivating Example

• Start from point A
• Try to reach point B
• Each step only move right ($\rightarrow N = 6$)
• Cost labeled on each edge

Problem: Path from A to B with the least cost?

• For $\ell$-by-$\ell$ grid, the total number of legal paths is $2^{2\ell}$, which grows fast with $\ell$. In our case $\ell = 3$, hence total number of legal path is 20.

Value Functions

Value function at $z$ is the least possible cost to reach $B$ from $z$
Value Functions

Value function at $z$ is the least possible cost to reach $B$ from $z$

**Principle of Optimality:** If a least-cost path from $A$ to $B$ is

$$x_0^* = A \rightarrow x_1^* \rightarrow x_2^* \rightarrow \cdots \rightarrow x_6^* = B,$$

then any truncation of it:

$$x_t^* \rightarrow x_{t+1}^* \rightarrow \cdots \rightarrow x_6^* = B$$

is also a least-cost path from $x_t^*$ to $B$.

Value function at any point $z$ reached at time $t$ satisfies

$$V_t(z) = \min\{w_u + V_{t+1}(z'_u), w_d + V_{t+1}(z'_d)\}$$

Optimal action when $x_t = z$ is the one providing the minimum argument and can be recovered from $V_{t+1}(\cdot)$. 
Value Function Iteration: Results
Value Function Iteration: Some Observations

Reduced computational complexity: for $\ell$-by-$\ell$ grid
- Only need to compute $\ell^2$ value functions
- No need to enumerate $\frac{(2\ell)!}{\ell!^2}$ paths
- Solve an optimization problem of fixed size in each iteration

Provide solutions to a family of optimal control problems
- Even if starting from a different initial position, there is no need for re-computation
- The input is determined as a function of the current state (state feedback static policy)
Value Function Iteration: Some Observations

Reduced computational complexity: for \( \ell \times \ell \) grid
- Only need to compute \( \ell^2 \) value functions
- No need to enumerate \( (2\ell!)^2 \) paths
- Solve an optimization problem of fixed size in each iteration

Provide solutions to a family of optimal control problems
- Even if starting from a different initial position, there is no need for re-computation
- The input is determined as a function of the current state (state feedback static policy)

Particularly suitable for multi-stage decision problems when the number of control choices is small at each stage

Value Functions of LQR Problem

The \textit{value function} at time \( t \in \{0,1,\ldots,N\} \) and state \( x \in \mathbb{R}^n \) is

\[
V_t(x) = \min_{u_0,u_1,\ldots,u_{N-1}} \sum_{k=0}^{N-1} \left( x_k^T Q x_k + u_k^T R u_k \right) + x_N^T Q f x_N
\]
Value Functions of LQR Problem

The value function at time $t \in \{0, 1, \ldots, N\}$ and state $x \in \mathbb{R}^n$ is

$$V_t(x) = \min_{u_t, u_{t+1}, \ldots, u_{N-1}} \sum_{k=t}^{N-1} (x_k^T Q x_k + u_k^T R u_k) + x_N^T Q f x_N$$

- $V_t(x)$ is the optimal cost of the LQR problem within a shorter time horizon (from time $t$ to $N$), starting from the initial condition $x_t = x$
- $V_0(x_0)$ is the optimal cost of the original LQR problem

LQR problem: Dynamic Programming Solution

Bellman equation:

$$V_t(x) = \min_{u_t, u_{t+1}, \ldots, u_{N-1}} \left[ x_t^T Q x_t + u_t^T R u_t + V_{t+1}(Ax + Bu) \right]$$

$$= x_t^T Q x_t + \min_{u_t, u_{t+1}, \ldots, u_{N-1}} \left[ V_{t+1}(Ax + Bu) \right]$$
LQR problem: Dynamic Programming Solution

Bellman equation:

\[ V_t(x) = \min_u \{ x^T Q x + x^T R x + V_{t+1}(Ax + Bu) \} \]

Optimal control:

\[ u^*_t(x) = \arg \min_v \{ v^T R v + V_{t+1}(Ax + Bu) \} \]

Value function at time \( N \) is quadratic:

\[ V_N(x) = x^T Q_x x \]
LQR problem: Dynamic Programming Solution

- Value function at time $N$ is quadratic: $V_N(x) = x^T Q x$
- Suppose $V_{t+1}(x) = x^T P_{t+1} x$ is quadratic, then

$$V_t(x) = \min_v \left[ x^T Q x + v^T R v + V_{t+1}(Ax + Bv) \right] = x^T P_t x$$
LQR problem: Dynamic Programming Solution

By setting $V_{t+1}(x) = x^T P_{t+1} x$ in the expression of $V_t(x)$, we get

$$V_t(x) = \min_v \left[ x^T Q x + v^T R v + V_{t+1}(Ax + Bu) \right]$$

$$= \min_v \left[ x^T Q x + v^T R v + (Ax + Bu)^T P_{t+1} (Ax + Bu) \right]$$

Minimizer is given by

$$u^*_t = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A x$$

If we plug it back into the expression of $V_t(x)$, we obtain

$$V_t(x) = x^T (Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A) x$$
LQR problem: Dynamic Programming Solution

- Value function at time $N$ is quadratic: $V_N(x) = x^T Q x$
- Suppose $V_{t+1}(x) = x^T P_{t+1} x$ is quadratic, then
  \[
  V_t(x) = \min_v \left[ x^T Q x + v^T R v + V_{t+1}(Ax + Bu) \right] = x^T P_t x
  \]
  is quadratic with $P_t$ obtained from $P_{t+1}$ by Riccati mapping:
  \[
  P_t := Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A
  \]
  which is achieved by the linear state feedback control
  \[
  u^*_t = - (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A x = -K_t x
  \]

- Value function at any time is quadratic (easy numeric representation)
- Optimal control is of linear state feedback form with time-varying gains
- Yields the optimal solutions for all initial conditions $x_0$ and all initial times $t_0 \in \{0, 1, \ldots, N\}$ simultaneously
- Easily extended to time-varying dynamics and costs cases
LQR Solution Algorithm

Set $P_N = Q$

for $t = N - 1, N - 2, \ldots, 0$ do

Compute the value functions backward in time:

$$P_t := Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$$

end for

Return $V_0(x_0)$ as the optimal cost

Set $x^*_0 = x_0$

for $t = 0, 1, \ldots, N - 1$ do

Recover the optimal control and state trajectory forward in time:

$$u^*_t = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A x^*_t$$

$$x^*_{t+1} = A x^*_t + B u^*_t$$

end for

Return $u^*_t$ and $x^*_t$ as the optimal control and state sequences

---

Example

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k, \quad y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k = C x_k, \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Cost function: $J(U) = \sum_{k=0}^{N-1} ||u_k||^2 + \rho \sum_{k=0}^{N-1} ||y_k||^2$ ($N = 20$)

Graphs showing control $u(t)$ and output $y(t)$ over time.
Example

Optimal control is of the form

\[ u^*_t = [a_t \ b_t] x^*_t, \ t = 0, 1, \ldots, 19 \]

The Kalman gains \( a_t \) and \( b_t \) rapidly converge to some constant values.

Convergence of Riccati Recursion

**Theorem**

*If \((A, B)\) is stabilizable, then Riccati recursion will converge to a solution \( P_m \) of the Algebraic Riccati Equation (ARE)*

\[ P_m = Q + A^T P_m A - A^T P_m B (R + B^T P_m B)^{-1} B^T P_m A \]

If further \( Q = C^T C \) for some \( C \) such that \((C, A)\) is detectable, then \( P_m \) is unique, and under the steady-state optimal control gain

\[ K_{ss} = (R + B^T P_m B)^{-1} B^T P_m A, \]

the closed-loop system \( A_{cl} = A - BK_{ss} \) is stable.

Important Properties of Riccati Mapping

The Riccati mapping \( P_t = \rho(P_{t+1}) \) defined by

\[
P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A
\]

is a mapping \( \rho : S_+ \to S_+ \) between set of positive semidefinite matrices.

**Proposition (Monotonicity)**

For \( P, P' \in S_+ \) with \( P \preceq P' \), we have \( \rho(P) \preceq \rho(P') \)

**Proposition (Concavity)**

For \( P, P' \in S_+ \) and \( \theta \in [0, 1] \), \( \rho(\theta P + (1 - \theta)P') \preceq \theta \rho(P) + (1 - \theta) \rho(P') \)


Back to Switched LQR Problem

A discrete-time switched linear system with given initial condition \( x_0 \):

\[
x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k} u_k,
\]

- continuous state: \( x_k \in \mathbb{R}^n \)
- discrete state (mode): \( \sigma_k \in \Sigma = \{1, 2, \ldots, M\} \)

**Problem:** Find the optimal input sequence \( (u_0, \ldots, u_{N-1}) \) and mode sequence \( (\sigma_0, \ldots, \sigma_{N-1}) \) that minimize the cost function

\[
\sum_{k=0}^{N-1} \left( x_k^T Q_{\sigma_k} x_k + u_k^T R_{\sigma_k} u_k \right) + x_N^T Q_f x_N
\]
Back to Switched LQR Problem

Find control sequence $u_0, \ldots, u_{N-1}$ and mode sequence $\sigma_0, \ldots, \sigma_{N-1}$ to minimize

$$\sum_{k=0}^{N-1} \left( x_k^T Q_{\sigma_k} x_k + u_k^T R_{\sigma_k} u_k \right) + x_N^T Q_f x_N $$

subject to

$$x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k} u_k, \ k = 0, \ldots, N-1$$

$x_0$ fixed

Value function at each $t = 0, 1, \ldots, N$ and $x$ is the optimal cost over horizon $[t, N]$ assuming $x_t = x$

$$V_t(x) = \min_{\sigma_t, \ldots, \sigma_{N-1}} \sum_{k=t}^{N-1} \left( x_k^T Q_{\sigma_k} x_k + u_k^T R_{\sigma_k} u_k \right) + x_N^T Q_f x_N$$

Back to Switched LQR Problem

Observations:

- Solution strategy: dynamic programming
- In each step, need to determine both the optimal $u$ and $\sigma$
- Value function no longer quadratic
- $V_0(x_0)$ is the optimal cost of the original problem
- Value function $V_t(x)$ does not depend on mode $\sigma_{t-1}$
- Hint: no switching cost

Robust optimal control: assume $\sigma$ is not controllable

$$\inf_{u} \sup_{\sigma} J(u, \sigma)$$

is a convex problem!
Bellman Equation of SLQR Problem

Value functions at different times are related by

\[ V_t(x) = \min_{\sigma_t, \nu_t} \left[ x^T Q_{\sigma} x + \nu^T R_{\sigma} \nu + V_{t+1}(A_{\sigma} x + B_{\sigma} \nu) \right] \]

Optimal control and mode are the ones achieving minimum above:
- Optimal state-dependent switching policy \( \sigma_t^*(x) \)
- Optimal state feedback controller \( u_t^*(x) \)

Bad news: value functions are in general not quadratic
- \( V_N(x) = x^T Q_f x \) is quadratic
- However, for \( t = N - 1, N - 2, \ldots \)

---

**Bellman equation for the LQR problem of \( \sigma \)-th subsystem**

\[ V_{t-1}(x) = \min_{\sigma_{t-1}} \left[ x^T Q_{\sigma_{t-1}} x + \nu^T R_{\sigma_{t-1}} \nu + V_{t}(A_{\sigma_{t-1}} x + B_{\sigma_{t-1}} \nu) \right] \]

\( \rho_{\sigma_{t-1}} \) is Riccati mapping of subsystem \((A_{\sigma_{t-1}}, B_{\sigma_{t-1}})\) with weights \( Q_{\sigma_{t-1}}, R_{\sigma_{t-1}} \)
\( t = N - 1 \) Case

\( V_{N-1}(x) \) is pointwise minimum of a number of quadratic functions
\[ \rightarrow \text{piecewise quadratic} \]

\[ V_{N-1}(x) = \min_{P \in P_{N-1}} x^T P x \]

where \( P_{N-1} = \{ \rho_1(Q), \ldots, \rho_M(Q) \} = \rho_M(Q) \)

- State space partitioned into cones (radially invariant minimizer)
- One optimal mode for each cone
- One optimal linear state feedback controller for each cone

\[ t = N - 2 \) Case

\[ V_{N-2}(x) = \min_{P \in P_{N-2}} x^T P x \]

Conclusion: value function \( V_{N-2}(x) \) is the pointwise minimum of \( M^2 \) quadratic functions:

\[ V_{N-2}(x) = \min_{P \in P_{N-2}} x^T P x \]

where \( P_{N-2} = \rho_{\sigma}(P_{N-1}) = \{ \rho_\sigma(P) : P \in P_{N-1}, \sigma \in \Sigma \} \).
**General Case**

If at $t+1$, $V_{t+1}(x) = \min_{P \in \mathcal{P}_{t+1}} x^T P x$ for a set $\mathcal{P}_{t+1}$ of p.s.d. matrices, then at time $t$, the value function is given by

$$V_t(x) = \min_{P \in \mathcal{P}_t} x^T P x$$

where $\mathcal{P}_t$ is obtained from $\mathcal{P}_{t+1}$ by switched Riccati recursion:

$$\mathcal{P}_t = \rho_\Sigma(\mathcal{P}_{t+1}) := \bigcup_{\sigma \in \Sigma} \rho_\sigma(\mathcal{P}_{t+1})$$

Size of $\mathcal{P}_t$ is bigger than $\mathcal{P}_{t+1}$: $|\mathcal{P}_t| = M \cdot |\mathcal{P}_{t+1}|$

---

**SLQR Solution Algorithm**

Set $\mathcal{P}_N = \{Q_t\}$

for $t = N - 1, N - 2, \ldots, 0$

- Compute the set of p.s.d. matrices:
  $$\mathcal{P}_t = \rho_\Sigma(\mathcal{P}_{t+1})$$

end for

Return $V_0(x_0) = \min_{P \in \mathcal{P}_0} x_0^T P x_0$ as the optimal cost

Set $x_0^* = x_0$

for $t = 0, 1, \ldots, N - 1$

- Recover the optimal mode $\sigma_t^*$ and the optimal control $u_t^*$ from
  $$\begin{align*}
  (\sigma_t^*, u_t^*) &= \arg \min_{\sigma_t, u_t} \left[ (x_t^*)^T Q_t x_t^* + u_t^T R_t u_t + V_{t+1}(A_{\sigma_t} x_t^* + B_{\sigma_t} u_t) \right] \\
  &\quad \text{Let } x_{t+1}^* = A_{\sigma_t} x_t^* + B_{\sigma_t} u_t^*
  \end{align*}$$

end for

Return $\sigma_t^*$ and $u_t^*$ as the optimal mode and control sequences
Complexity Reduction

Issue: Number of matrices in $P_t$ grows exponentially

In the set $P_t$ defining the value function $V_t(x) = \min_{P \in P_t} x^T P x$

- Matrix $P \in P_t$ is called effective if for at least one $x \neq 0$
  
  $$x^T P x < x^T P' x, \quad \forall P' \in P_t \setminus \{P\}$$

- Otherwise $P$ is called redundant
Complexity Reduction

**Issue:** Number of matrices in $P_t$ grows exponentially

In the set $P_t$ defining the value function $V_t(x) = \min_{P \in P_t} x^T P x$

- Matrix $P \in P_t$ is called **effective** if for at least one $x \neq 0$
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- Otherwise $P$ is called **redundant**

Redundant matrices can be discarded without affecting optimal solution because of the monotonicity of Riccati mapping.

- **Sufficient condition** for $P \in P_t$ to be redundant:
  \[ P \succeq \text{a convex combination of } P' \in P_t \setminus \{P\} \]
Complexity Reduction

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- Otherwise $P$ is called **redundant**

Redundant matrices can be discarded without affecting optimal solution because of the monotonicity of Riccati mapping

**Sufficient condition** for $P \in P_t$ to be redundant:

\[ P \succeq \text{a convex combination of } P' \in P_t \setminus \{P\} \]

Proof:

\[ x^T P x \geq \sum_{P_i \in P_t \setminus \{P\}} \alpha_i x^T P_i x \geq x^T P_j x, \quad \text{for some } P_j \in P_t \setminus \{P\} \]

LMI feasibility condition to test.
Example of Ineffective Matrices

A switched LQR problem specified by

\[
A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.5 & 1 \\ 0 & 1.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

\[Q_0 = 1, \ R_\sigma = 1, \ N = 20\]

16 matrices in \( P_{16} \)
evaluated along half of the unit circle because of the radial invariance.

Decision Tree Pruning
Further Reduction by Relaxation

Remove more matrices by relaxing Condition (1) to

\[ P \succeq \sum_{i \in I} \alpha_i P_i - \varepsilon I \]  

(2)

- \( \varepsilon > 0 \) is a small constant specifying approximation quality
- Even a small \( \varepsilon \) could result in significant reduction in complexity

Example

\[
A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

\[
Q_1 = Q_2 = I, \quad R_1 = R_2 = 1, \quad Q_f = I, \quad N = 100
\]
Example (cont.)

Optimal switching policy (Gray region: mode 1 optimal; Black region: mode 2 optimal)

Another Example

\[ A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]
\[ A_2 = \begin{bmatrix} 1.5 & 1 \\ 0 & 1.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
\[ Q_x = I, \quad R_x = 1, \quad N = 20 \]

- Without any reduction, complexity grows exponentially
- With reduction, complexity saturates at 360 matrices
- With relaxation \((c = 10^{-3})\), complexity saturates at 14 matrices
SLQR Problem with Switching Cost

Cost function to be minimized:

\[
\sum_{k=0}^{N-1} \left( x_k^T Q_{\sigma_k} x_k + u_k^T R_{\sigma_k} u_k + \underbrace{w_{\sigma_k,\sigma_{k+1}}(x_k)}_{\text{switching cost}} \right) + x_N^T Q_{\sigma_N} x_N
\]

Value function \( V_t(\sigma, x) \) is the optimal cost-to-go starting from \( x_t = x \) with previous mode being \( \sigma_{t-1} = \sigma \).

Bellman equation:

\[
V_t(\sigma, x) = \min_{\sigma' \in \mathcal{M}} \left[ x^T Q_{\sigma'} x + v^T R_{\sigma'} v + \underbrace{w_{(\sigma,\sigma')} (x)}_{\text{switching cost}} + V_{t+1}(\sigma', A_{\sigma'} x + B_{\sigma'} v) \right]
\]
**SLQR Problem with Switching Cost**

**Cost function to be minimized:**

\[
\sum_{k=0}^{N-1} \left( x_k^T Q x_k + u_k^T R u_k + w(\sigma_k, \sigma_{k+1}) (x_k) \right) + x_N^T Q_f x_N
\]

**Value function** \( V_t(\sigma, x) \) is the optimal cost-to-go starting from \( x_t = x \) with previous mode being \( \sigma_{t-1} = \sigma \)

**Bellman equation:**

\[
V_t(\sigma, x) = \min_{\sigma_t, u_t} \left[ x^T Q x + v^T R v + w(\sigma_t, \sigma_{t+1}) (x) + V_{t+1}(\sigma_{t+1}, A_{\sigma_t} x + B_{\sigma_t} v) \right]
\]

- Optimal switching policy \( \sigma^*_t(\sigma, x) \) and optimal control \( u^*_t(\sigma, x) \)
- Both depend on previous mode \( \sigma \) and current state \( x \)
- Same technique with piecewise quadratic value functions if \( w(\sigma, \sigma')() \)
  is quadratic, linear, or constant.

---

**Continuous-Time LQR Problem**

A continuous-time linear system with given initial condition \( x_0 \):

\[
\dot{x} = Ax + Bu
\]

**Problem:** find the optimal control input \( u(t) \) over the time horizon \([0, t_f]\) that minimizes

\[
J = \int_0^{t_f} \left( x^T Q x + u^T R u \right) dt + x(t_f)^T Q_f x(t_f)
\]

- State running weight \( Q = Q^T \geq 0 \)
- Control running weight \( R = R^T > 0 \)
- Final state weight \( Q_f = Q_f^T \geq 0 \)
Continuous-Time LQR Problem

Value function at time $t \in [0, t_f]$:

$$V_t(x) = \min_{u(s), s \in [t, t_f]} \int_t^{t_f} \left[ x(s)^T Q x(s) + u(s)^T R u(s) \right] ds + x(t_f)^T Q x(t_f)$$

- $V_t(x)$ is the optimal cost-to-go at time $t$ from state $x$
- optimal cost of the original LQR problem is given by $V_0(x_0)$
- at time $t_f$, the value function is quadratic $V_{t_f}(x) = x^T Q_f x$

As in the discrete-time case, the value function can be shown to be quadratic at any time:

$$V_t(x) = x^T P(t) x, \quad t \in [0, t_f]$$
A Heuristic Derivation of Value Functions

- Assume that the system starts from $x$ at time $t$
  \[ x(t) = x, \; t \in [0, t_f), \; x \in \mathbb{R}^n \]
- Assume that the control input is kept constant for a brief $\delta$-length time horizon
  \[ u(s) = w, \; s \in [t, t + \delta] \]
- Assume that the value function is quadratic at any time:
  \[ V_t(x) = x^T P(t) x, \; t \in [0, t_f] \]

Bellman equation:

\[
V_t(x) \approx \min_{u(t) = w} \left[ \delta(x^T Q x + w^T R w) + V_{t+\delta}(x + \delta(A x + B w)) \right]
\]

\[ \text{cost-to-go at time } t \]
\[ \text{cost during } [t, t + \delta] \]
\[ \text{cost-to-go from time } t + \delta \]
A Heuristic Derivation of Value Functions

Bellman equation:

\[ V_t(x) \simeq \min_{u} \left[ \delta(x^T Q x + w^T R w) + V_{t+1}(x + \delta(A x + B w)) \right] \]

\[ V_{t+1}(x + \delta(A x + B w)) \]
\[ = [x + \delta(A x + B w)]^T P(t + \delta) x + \delta(A x + B w) \]
\[ \simeq [x + \delta(A x + B w)]^T \left[ P(t) + \delta \dot{P}(t) [x + \delta(A x + B w)] \right] \]
\[ \simeq x^T P(t) x + \delta \left[ x^T P(t)(A x + B w) + (A x + B w)^T P(t) x + x^T \dot{P}(t) x \right] \]
\[ = V_t(x) + \delta \left[ x^T P(t)(A x + B w) + (A x + B w)^T P(t) x + x^T \dot{P}(t) x \right] \]
A Heuristic Derivation of Value Functions

Bellman equation:
\[ V_t(x) \simeq \min_w \left[ \delta(x^T Q x + w^T R w) + V_{t+\delta}(x + \delta(\delta x + B w)) \right] \]
\[ = V_t(x) + \delta \left[ x^T P(t)(A x + B w) + (A x + B w)^T P(t) x + x^T \dot{P}(t) x \right] \]

As \( \delta \to 0 \), Bellman equation becomes asymptotically:
\[ 0 = \min_w \left\{ x^T Q x + w^T R w + x^T P(t)(A x + B w) + (A x + B w)^T P(t) x \\
+ x^T \dot{P}(t) x \right\} \]

The optimal control law at time \( t \) is then:
\[ u^*(t) = \arg \min_w \left\{ x^T Q x + w^T R w + x^T P(t)(A x + B w) + (A x + B w)^T P(t) x + x^T \dot{P}(t) x \right\} \]
\[ = -R^{-1}B^T P(t) x \]
\[ K(t) \text{: Kalman gain} \]
A Heuristic Derivation of Value Functions

The optimal control law at time $t$ is then:

$$u^*(t) = \arg \min_w \{ x^T Q x + w^T R w + x^T P(t)(A x + B w) + (A x + B w)^T P(t) x + x^T \dot{P}(t) x \}$$

$$= -B R^{-1} P(t) x$$

$K(t)$: Kalman gain

Plug this back into the asymptotic version of the Bellman equation:

$$0 = \{ x^T Q x + w^T R w + x^T P(t)(A x + B w) + (A x + B w)^T P(t) x + x^T \dot{P}(t) x \}_{w = u^*}$$

$$\rightarrow 0 = x^T \{ Q + P(t) A + A^T P(t) - P(t) BR^{-1} B^T P(t) + \dot{P}(t) \} x, \ \forall x$$

$$\rightarrow -\dot{P}(t) = Q + P(t) A + A^T P(t) - P(t) BR^{-1} B^T P(t)$$

Initial condition: $P(t_f) = Q_f$ and integrated backward in time till time 0

LQR problem: Dynamic Programming Solution

The value functions are quadratic

$$V_t(x) = x^T P(t) x$$

with $P(t)$ satisfying the Riccati (matrix) differential equation:

$$-\dot{P}(t) = Q + P(t) A + A^T P(t) - P(t) BR^{-1} B^T P(t) + \dot{P}(t)$$

The optimal control is a linear state feedback controller:

$$u^*(t) = -R^{-1} B^T P(t)x$$
**LQR Solution Algorithm**

Set $P(t_f) = Q$

Solve the matrix Riccati equation backward in time:

$$-\dot{P}(t) = Q + P(t)A + A^TP(t) - P(t)BR^{-1}B^TP(t)$$

Return $V_0(x_0) = x_0^TP(0)x_0$ as the optimal cost

Set $x^*(0) = x_0$

Recover the optimal control and trajectory forward in time

$$\begin{cases}
\dot{x}^*(t) = Ax^*(t) + Bu^*(t), & t \in [0, t_f],

u^*(t) = K(t)x^*(t),
\end{cases}$$

where $K(t)$ is the Kalman gain computed by

$$K(t) = -R^{-1}B^TP(t)$$

---

**Switched LQR Problem**

A continuous-time switched linear system with given initial condition $x_0$:

$$\dot{x} = A_\sigma x + B_\sigma u$$

- continuous state: $x(t) \in \mathbb{R}^n$
- discrete state (mode): $\sigma(t) \in \Sigma = \{1, 2, \ldots, M\}$
Switched LQR Problem

A continuous-time switched linear system with given initial condition $x_0$:

$$\dot{x} = A_\sigma x + B_\sigma u$$

**Problem:** Find the optimal mode $\sigma(t) \in \Sigma$ and input $u(t)$ over the time horizon $[0,t_f]$ that minimize the cost function

$$\int_0^{t_f} \left( x^T Q_\sigma x + u^T R_\sigma u \right) dt + x(t_f)^T Q_f x(t_f)$$

- State running weight $Q_\sigma = Q_\sigma^T \succeq 0, \sigma \in \Sigma$
- Control running weight $R_\sigma = R_\sigma^T > 0, \sigma \in \Sigma$
- Final state weight $Q_f = Q_f^T \succeq 0$
- No switching cost

**Observations:**
- In different modes, both dynamics and running costs are different
- If mode sequence is given, becomes a time-varying LQR problem
- Main challenge is determining the mode sequence
Continuous-Time SLQR Problem

Value function at time $t \in [0, t_f]$:

$$V_t(x) = \min_{u(s), \sigma(s) \in \mathcal{U}, \sigma : [t, t_f]} \left\{ \int_t^{t_f} \left[ x(s)^T Q_{s(t)} x(s) + u(s)^T R_{s(t)} u(s) \right] ds + x(t_f)^T Q_{t_f} x(t_f) \right\}$$

- $V_t(x)$ is the optimal cost-to-go at time $t$ from state $x$.
- Value function independent of $\sigma$ due to the absence of switching cost.
- Optimal cost of the original SQR problem is given by $V_0(x_0)$.
- At time $t_f$, the value function is quadratic: $V_{t_f}(x) = x^T Q_{t_f} x$.

As in the discrete-time case, the value function at any time is the minimum of a (time-varying) set of quadratic functions:

$$V_t(x) = \inf_{P(t) \in \mathcal{P}(t)} x^T P_t x.$$
Derivation of Value Functions

To obtain a more tractable optimal control problem:

- embed the switched system in the larger family

\[
\dot{x} = A_\lambda x + B_\lambda u, \quad x(0) = x_0
\]

where \( A_\lambda = \sum_{i=1}^{M} \lambda_i A_i \) and \( B_\lambda = \sum_{i=1}^{M} \lambda_i B_i \) are parameterized by

\[
\lambda = (\lambda_1, \ldots, \lambda_M) \text{ with } \lambda_i \geq 0, i = 1, \ldots, M, \sum_{i=1}^{M} \lambda_i = 1
\]

\( \rightarrow \lambda \in S, \) \( S \) being a simplex.

When \( \lambda \) takes value in a vertex of \( S \), we get one of the dynamical systems among which switching occurs. For instance, if \( \lambda_i = 1 \), then,

\[
\dot{x} = A_i x + B_i u
\]
Derivation of Value Functions

To obtain a more tractable optimal control problem:

- reformulate the optimal control problem as follows:

  \[
  \text{Find } u(t) \text{ and } \lambda(t), \ t \in [0, t_f], \text{ to minimize}
  \int_0^{t_f} \left( x^T Q \dot{x} + u^T R \dot{u} \right) dt + x(t_f)^T Q x(t_f)
  \]

  subject to
  \[
  \dot{x} = A \lambda x + B \lambda u, \ t \in [0, t_f],
  \]

  \[
  x_0 \text{ fixed}
  \]

  where
  \[
  Q_i = \sum_{i=1}^{M} \lambda_i Q_i \quad \text{and} \quad R_i = \sum_{i=1}^{M} \lambda_i R_i
  \]

  If the optimal \( \lambda(t), t \in [0, t_f] \), takes values in the vertices of the simplex \( S \), then, the solution is also optimal for the original switched problem, otherwise only a suboptimal solution can be determined.

Derivation of Value Functions

- Assume that the system starts from \( x \) at time \( t \)

  \[
  x(t) = x, \ t \in [0, t_f], x \in \mathbb{R}^n
  \]

- Assume that the control input is kept constant for a brief \( \delta \)-length time horizon

  \[
  u(s) = w, \ s \in [t, t+\delta]
  \]

- Assume that the value function is the minimum of a (time-varying) set of quadratic functions:

  \[
  V_t(x) = \inf_{P \in \mathbb{P}(x)} x^T P x
  \]
Derivation of Value Functions

Bellman equation:

\[ V_t(x) \approx \min_{w, \lambda} \left[ \delta(x^T Q_{\lambda} x + w^T R_{\lambda} w) + V_{t+\delta}(x + \delta(A_{\lambda} x + B_{\lambda} w)) \right] \]

Note that since

\[ V_{t+\delta}(x) = \inf_{P(t+\delta)(P(t+\delta)))} x^T P(t+\delta)x \]

we get

\[ V_t(x) \approx \min_{w, \lambda, P(t+\delta)(P(t+\delta)))} \left[ \delta(x^T Q_{\lambda} x + w^T R_{\lambda} w) + \right. \]

\[ \left. (x + \delta(A_{\lambda} x + B_{\lambda} w))^T P(t+\delta)(x + \delta(A_{\lambda} x + B_{\lambda} w)) \right] \]
Derivation of Value Functions

Expand $P(t + \delta) \in \mathcal{P}(t + \delta)$ as $P(t) \simeq P(t) + \delta \dot{P}(t)$ for some $P(t) \in \mathcal{P}(t)$

Let $\delta \to 0$. The Bellman equation becomes asymptotically:

$$0 = \min_{w, \lambda, P(t) \in \mathcal{P}(t)} \left\{ x^T Q x + w^T R x + x^T P(t) (A \lambda x + B \lambda w) \\
+ (A x + B w)^T P(t) x + x^T \dot{P}(t) x \right\}$$
Derivation of Value Functions

Expand $P(t + \delta) \in P(t + \delta)$ as $P(t + \delta) \simeq P(t) + \delta \dot{P}(t)$ for some $P(t) \in P(t)$

Let $\delta \to 0$. The Bellman equation becomes asymptotically:

$$0 = \min_{w, \lambda, P(t) \in P(t)} \left\{ x^T Q \lambda + w^T R \lambda x + \lambda x^T P(t) (A \lambda x + B \lambda w) + (A \lambda x + B \lambda w)^T P(t) x + x^T \dot{P}(t) x \right\}$$

Using the optimal control, the value function is of the form

$$V_t(x) = \inf_{P(t) \in P(t)} x^T P x$$

where the set $P(t)$ satisfies

$$-\dot{P}(t) \in \{ Q + P(t) A + A^T P(t) - P(t) B R^{-1} B^T P(t) : \lambda \in S \}$$

$\forall P(t) \in P(t)$.

Value Functions of C.-T. SLQR Problem

The value function $V_t(x)$ is still of the form

$$V_t(x) = \inf_{P(t) \in P(t)} x^T P x$$

$P(t)$ can be computed from the Riccati differential inclusion

$$-\dot{P}(t) \in \{ Q + P(t) A + A^T P(t) - P(t) B R_i^{-1} B^T P(t) : \lambda \in S \}$$

where $A, B, Q, R$ is any convex combination of $A_i, B_i, Q_i, R_i$ for $i \in \Sigma$.
**Value Functions of C.-T. SLQR Problem**

The value function $V_t(x)$ is still of the form

$$V_t(x) = \inf_{P \in \mathbb{P}(t)} x^T P x$$

$P(t)$ can be computed from the Riccati differential inclusion

$$-\dot{P}(t) \in \{ Q + P(t) A_\lambda + A_\lambda^T P(t) - P(t) B_\lambda R_{\lambda}^{-1} B_\lambda^T P(t) : \lambda \in \mathcal{S} \}$$

where $A_\lambda, B_\lambda, Q_\lambda, R_\lambda$ is any convex combination of $A_i, B_i, Q_i, R_i$ for $i \in \Sigma$

In general, $P(t)$ is very difficult to compute analytically and numerically

- Discretize the C.-T. SLS into D.-T. SLS