Hybrid Systems Course
Observer design for hybrid systems

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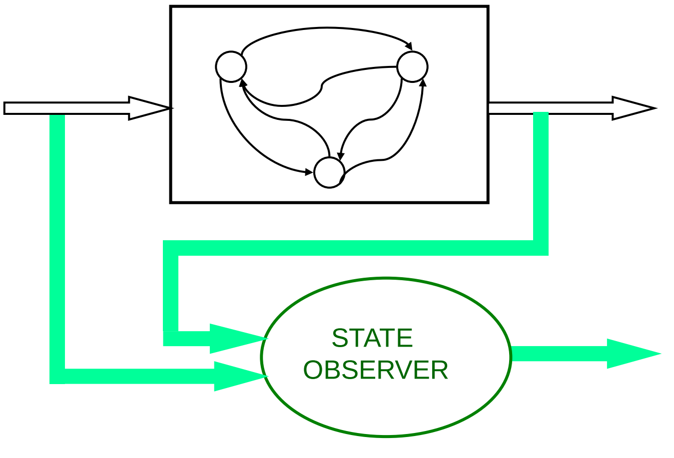
OBSERVER DESIGN PROBLEM

Goal: recover the state of a system from its input and output



MOTIVATION FOR OBSERVER DESIGN

- Need to monitor the evolution of the system
- Control algorithms require full state feedback

… but measuring the complete state of the system may be not economically feasible or even possible

OUTLINE

- observer design for continuous time linear systems
- observer design for switched linear systems with known switchings
- observer design for hybrid systems

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CONTINUOUS TIME LINEAR SYSTEMS

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

$u(t) \in \Re^m \equiv$ input

$y(t) \in \Re^p \equiv$ output

$x(t) \in \Re^n \equiv$ state

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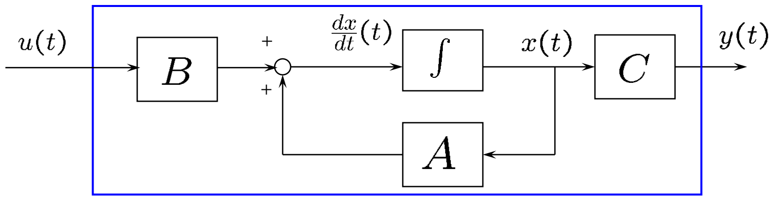

OBSERVABILITY NOTION

"possibility of reconstructing the state from past input and output measurements"

Definition [indistinguishable states]

x_1 and x_2 are indistinguishable if the output associated with $x(0) = x_1$ and $x(0) = x_2$ is identical for any input u(t), t$\geq$ 0

$$y_1(t) = Ce^{At}x_1 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau$$
$$y_2(t) = Ce^{At}x_2 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau$$
$$y_1(t) - y_2(t) = Ce^{At}(x_1 - x_2)$$

OBSERVABILITY NOTION

"possibility of reconstructing the state from past input and output measurements"

Definition [unobservable state]

x is unobservable if it is indistinguishable from the origin

$$Ce^{At}x = 0, \quad t \geq 0$$

OBSERVABILITY NOTION

"possibility of reconstructing the state from past input and output measurements"

Definition [unobservable state]

x is unobservable if it is indistinguishable from the origin

$$Ce^{At}x = Cx + CAxt + CA^2x\frac{t^2}{2!} + CA^3x\frac{t^3}{3!} + \cdots = 0, \quad t \geq 0$$

$$\begin{cases} Cx = 0 \\ CAx = 0 \\ CA^2x = 0 \\ \vdots \\ CA^{n-1}x = 0 \end{cases} \Longleftrightarrow \boxed{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}} x = 0$$

observability matrix O_n

OBSERVABILITY NOTION AND CHARACTERIZATION

"possibility of reconstructing the state from past input and output measurements"

Definition [unobservable state]

x is unobservable if it is indistinguishable from the origin

$$\Updownarrow$$

x belongs to the null space of the observability matrix O_n

OBSERVABILITY NOTION AND CHARACTERIZATION

"possibility of reconstructing the state from past input and output measurements"

Definition [unobservable state]

x is unobservable if it is indistinguishable from the origin

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x belongs to the null space of the observability matrix O_n

Further definitions:

x is observable if it is not unobservable

the set of all the unobservable states is the unobservable subspace, the orthogonal subspace is called observable subspace

OBSERVABILITY NOTION AND CHARACTERIZATION

"possibility of reconstructing the state from past input and output measurements"

Definition [observable system]

A system is observable if all states $x \neq 0$ are observable

$$\Updownarrow$$

the observability matrix O_n has maximum rank (n)

Remark:

if the observability matrix O_n has maximum rank (n), then, the pair (A,C) is called observable

OBSERVABILITY NOTION AND CHARACTERIZATION

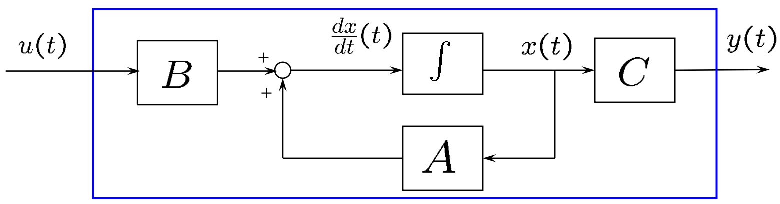
Key property:

(A,C) is observable

$$\Updownarrow$$

one can select matrix L such that A-LC has arbitrarily chosen eigenvalues

KALMAN DECOMPOSITION

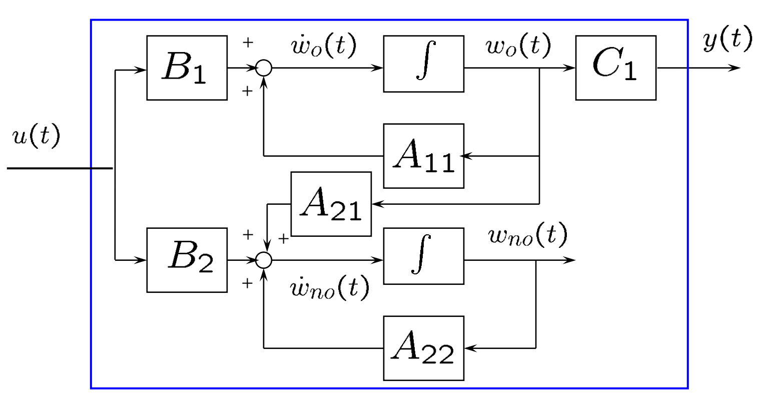


$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

similarity transformation for the decomposition into observable and unobservable part

$$w := \boxed{T_o} x$$

KALMAN DECOMPOSITION



$$\begin{bmatrix} \dot{w}_o(t) \\ \dot{w}_{no}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} w_o(t) \\ w_{no}(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} w_o(t) \\ w_{no}(t) \end{bmatrix}$$

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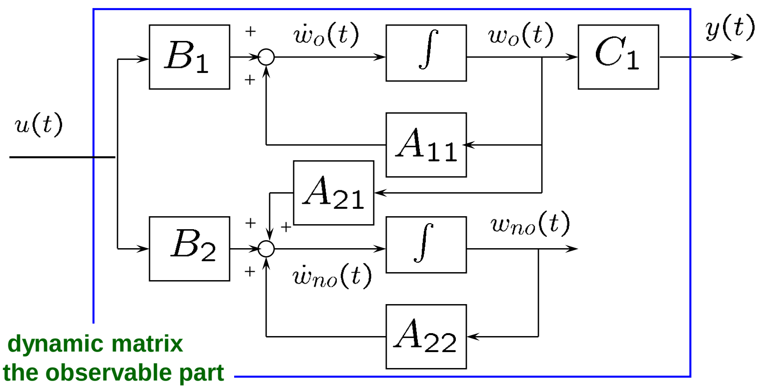

dynamic matrix
of the observable part

$$\begin{bmatrix} \dot{w}_o(t) \\ \dot{w}_{no}(t) \end{bmatrix} = \begin{bmatrix} \boxed{A_{11}} & 0 \\ A_{21} & \boxed{A_{22}} \end{bmatrix} \begin{bmatrix} w_o(t) \\ w_{no}(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

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dynamic matrix
of the unobservable part

OBSERVABILITY NOTION

“possibility of reconstructing the state from past input and output measurements”

Definition [detectable system]

A system is detectable if the eigenvalues of the unobservable part have all strictly negative real part

Remark:

in this case, the pair (A,C) is detectable

ASYMPTOTIC OBSERVER

"possibility of reconstructing the state from past input and output measurements"

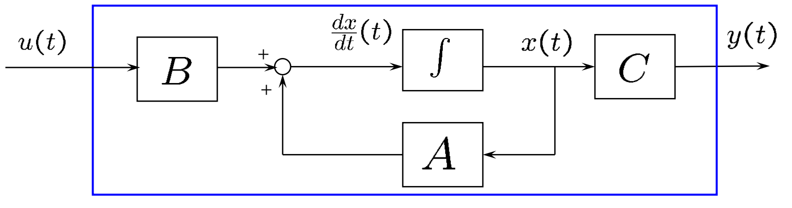
Definition [asymptotic observer]:

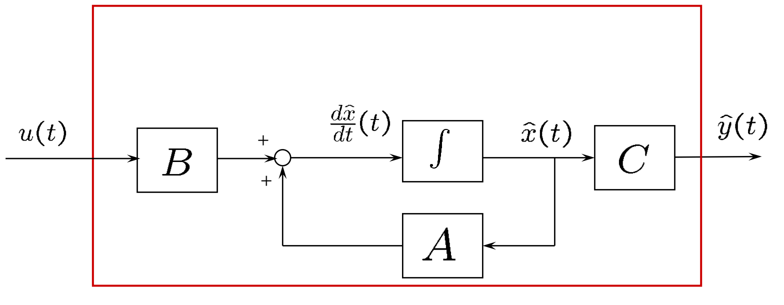
An asymptotic observer is a system that consistently estimates the state x(t) based on the input and output measurements u(τ) and y(τ), 0≤ τ ≤ t , for any (unknown) initial condition x_0 and for any input u(·):

$$\lim_{t \to \infty} \|\hat{x}(t) - x(t)\| = 0, \quad \forall x(0) = x_0 \in \Re^n, \forall u(\cdot)$$

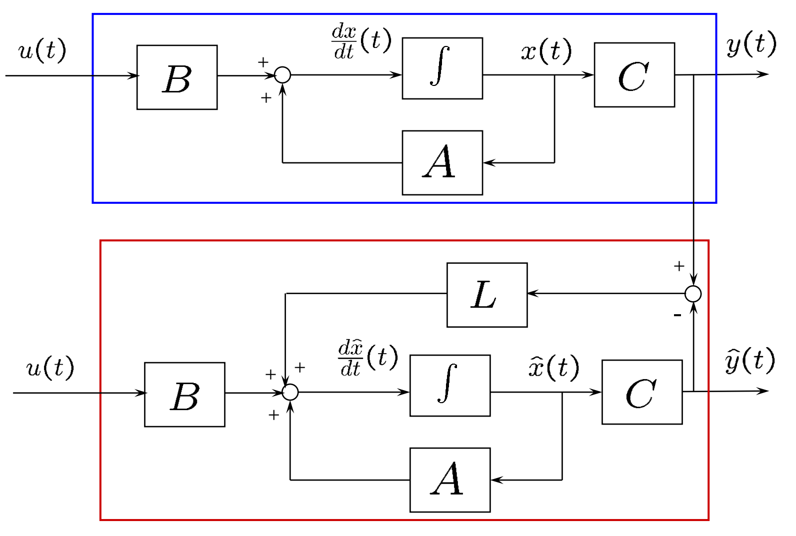
Remark: Also called Luenberger observer

ASYMPTOTIC OBSERVER





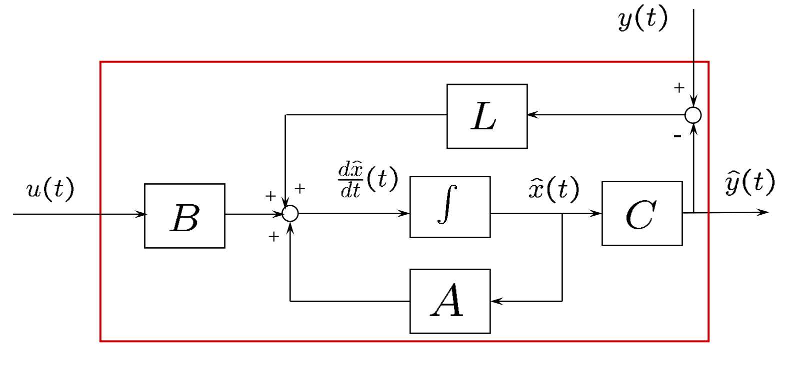
ASYMPTOTIC OBSERVER



ASYMPTOTIC OBSERVER

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + \boxed{L}(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = C\hat{x}(t)$$

observer gain



DYNAMICS OF THE STATE ESTIMATION ERROR

$$e(t) := x(t) - \widehat{x}(t)$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
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$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\,(y(t) - \hat{y}(t))$$
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$$\boxed{\dot{e}(t) = (A - LC)\, e(t)}$$

ASYMPTOTIC OBSERVER

$$\boxed{\dot{e}(t) = (A - LC)\, e(t)}$$

If A is Hurwitz, one can set L = 0 and obtain that the estimation error converges exponentially to zero.

This means that one can just duplicate the system dynamics, without using the output measurements. The rate of convergence will be determined by the real part of the eigenvalues of A (see the lecture on Lyapunov stability)

ASYMPTOTIC OBSERVER

Theorem:

If (A,C) is detectable, then, L can be designed so that A-LC is Hurwitz and, hence, the estimation error converge exponentially to zero:

$$\|e(t)\| \leq \mu e^{-\lambda_0 t}\|e(0)\|,\ t \geq 0, \quad \forall e(0) = e_0 \in \Re^n$$

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Sketch of the proof:

- the system can be decomposed in observable/unobservable part

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- the observable part can be reconstructed from the output

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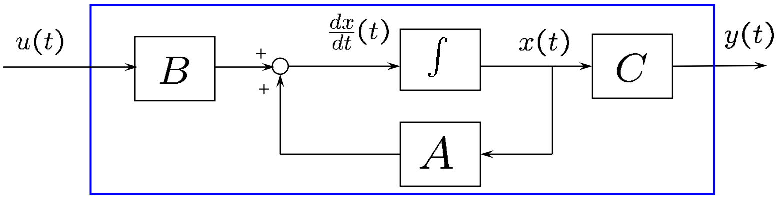
Sketch of the proof:

- the system can be decomposed in observable/unobservable part
- the observable part can be reconstructed from the output
- the unobservable part is asymptotically stable, hence it can be reconstructed just by duplicating the corresponding system dynamics

KALMAN DECOMPOSITION



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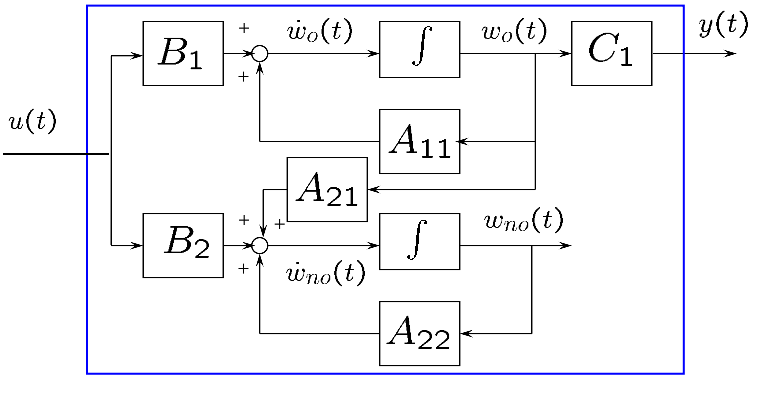

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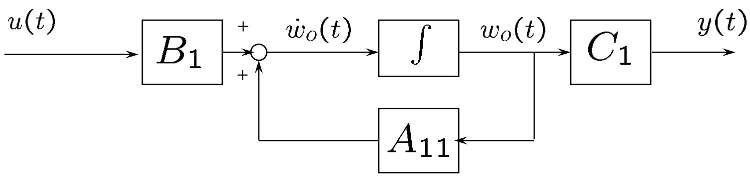
KALMAN DECOMPOSITION: OBSERVABLE PART



$$\begin{cases} \dot{w}_o(t) = A_{11} w_o(t) + B_1 u(t) \\ y(t) = C_1 w_o(t) \end{cases}$$

This sub-system is observable, not necessarily stable

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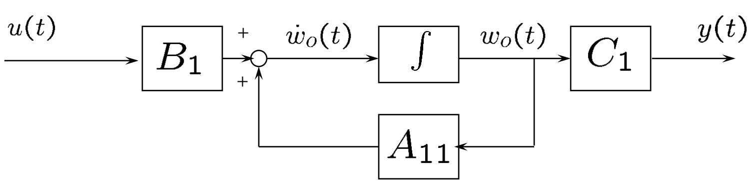
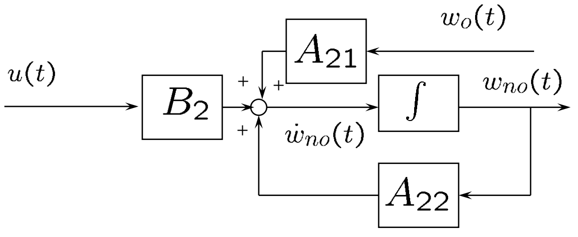


$$\begin{cases} \dot{w}_o(t) = A_{11} w_o(t) + B_1 u(t) \\ y(t) = C_1 w_o(t) \end{cases}$$

This sub-system is observable, not necessarily stable

the output contains info to reconstruct the state $w_o(t)$:
an exponentially stable dynamics can be imposed to the state estimation error

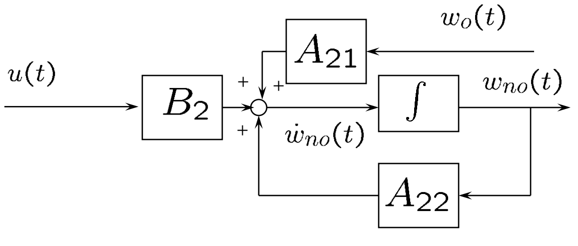
KALMAN DECOMPOSITION: UNOBSERVABLE PART



$$\dot{w}_{no}(t) = A_{22}\, w_{no}(t) + A_{21} w_o(t) + B_2 u(t)$$

the unobservable state $w_{no}(t)$ does not affect the output

KALMAN DECOMPOSITION: UNOBSERVABLE PART



$$\dot{w}_{no}(t) = \boxed{A_{22}}\, w_{no}(t) + \boxed{A_{21} w_o(t) + B_2 u(t)}$$

Hurwitz **known/reconstructed signals**

the contribution of the unknown and unobservable initial condition $w_{no}(0)$ exponentially converges to zero

DYNAMICS OF THE ESTIMATION ERROR

$$e_w := T_o\, e$$

$$\begin{aligned} \dot{e_w}(t) &= T_o\,(A - LC)\,T_o^{-1} e_w(t) \\ &= \left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} C_1 & 0 \end{bmatrix} \right) e_w(t) \\ &= \begin{bmatrix} \boxed{A_{11} - L_1 C_1} & 0 \\ A_{21} - L_2 C_1 & \boxed{A_{22}} \end{bmatrix} e_w(t) \end{aligned}$$

eigenvalues of A_{11}-L_1C_1 can be arbitrarily selected since (A_{11},C_1) is observable

eigenvalues of the unobservable part keep fixed

ASYMPTOTIC OBSERVER

Theorem:

If (A,C) is detectable, then, L can be designed so that A-LC is Hurwitz and, hence, the estimation error converges exponentially to zero:

$$\|e(t)\| \le \mu e^{-\lambda_0 t} \|e(0)\|, \; t \ge 0, \quad \forall e(0) = e_0 \in \Re^n$$

Remark: The convergence rate can be arbitrarily chosen if and only if (A,C) is observable.

OUTLINE

- observer design for continuous time linear systems
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- observer design for hybrid systems

SWITCHED LINEAR SYSTEMS (WITH INPUT/OUTPUT)

$$\boxed{\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) \end{aligned}}$$

Switching occurs within the family of systems:

$$\begin{aligned} \dot{x}(t) &= A_q x(t) + B_q u(t) \\ y(t) &= C_q x(t) \end{aligned} \qquad q \in Q = \{1, 2, \ldots, m\}$$

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Assumptions:

(i) the switching signal $\sigma : [0, \infty) \to Q$ is available as (discrete) output signal

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Assumptions:

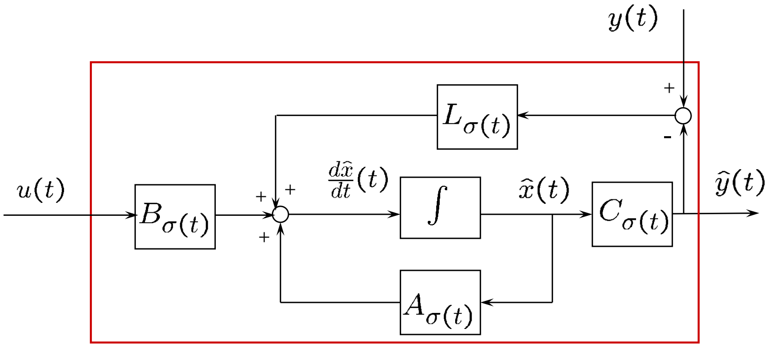
(i) the switching signal $\sigma : [0, \infty) \to Q$ is available as (discrete) output signal

(ii) (A_q, C_q) detectable for all $q \in Q$

SWITCHING OBSERVER

$$\begin{aligned}\dot{\hat{x}}(t) &= A_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}u(t) + \boxed{L_{\sigma(t)}}\,(y(t) - \hat{y}(t))\\ \hat{y}(t) &= C_{\sigma(t)}\hat{x}(t)\end{aligned}$$

observer time-varying gain



DYNAMICS OF THE STATE ESTIMATION ERROR

$$e(t) := x(t) - \hat{x}(t)$$

system $\quad \begin{cases}\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)\\ y(t) = C_{\sigma(t)}x(t)\end{cases}$

observer $\quad \begin{cases}\dot{\hat{x}}(t) = A_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}u(t) + L_{\sigma(t)}\,(y(t) - \hat{y}(t))\\ \hat{y}(t) = C_{\sigma(t)}\hat{x}(t)\end{cases}$

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$$\boxed{\dot{e}(t) = (A_{\sigma(t)} - L_{\sigma(t)} C_{\sigma(t)})\, e(t)}$$

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$$\boxed{\dot{e}(t) = (A_{\sigma(t)} - L_{\sigma(t)} C_{\sigma(t)})\, e(t)}$$

$A_q - L_q C_q$ Hurwitz for all $q \in Q$ does not guarantee that
$e(t) \to 0$, $\forall\, e(0)$, $\forall\, \sigma:[0,\infty) \to Q$ (GUAS of equilibrium e = 0)

SWITCHING OBSERVER

$$\dot{e}(t) = (A_{\sigma(t)} - L_{\sigma(t)} C_{\sigma(t)})\, e(t)$$

Theorem: If there exists $P = P^T > 0$ such that

$$P(A_q - L_q C_q) + (A_q - L_q C_q)^T P < 0$$
$$\forall q \in Q = \{1, 2, \ldots, m\}$$

then, the switching observer consistently estimates the continuous state of the switched system, for any e(0) and for any σ: [0,∞)→ Q.

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Proof. $V(e) = e^T P e$ is a radially unbounded common Lyapunov function at the equilibrium e =0. Then, e = 0 is GUAS.

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Proof. $V(e) = e^T P e$ is a radially unbounded common Lyapunov function at the equilibrium e =0. Then, e = 0 is GUAS.

Note: exponential convergence can also be proven [see the lecture on Lyapunov stability]

SWITCHING OBSERVER DESIGN

We are designing the observer, and its gains…

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We are designing the observer, and its gains…

We should choose the observer gains L_1, L_2, …, L_m such that there exists $P = P^T > 0$ satisfying

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Apparently not an easy problem beacuse of the terms $P L_i$

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Linear Matrix Inequalities (LMI) reformulation:

By setting $L_q = P^{-1} Y_q$, we have $PL_q = Y_q$

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Linear Matrix Inequalities (LMI) reformulation:

By setting $L_q = P^{-1} Y_q$, we have $PL_q = Y_q$

The problem can then be rephrased as that of determining $P = P^T > 0$ and Y_1, Y_2, …, Y_m such that

$$PA_q - Y_q C_q + A_q^T P - C_q^T Y_q^T < 0$$
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SWITCHING OBSERVER DESIGN

We are designing the observer, and its gains…

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Gains are then recovered by $L_q = P^{-1} Y_q$

SWITCHED LINEAR SYSTEMS (WITH INPUT/OUTPUT)

$$\boxed{\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) \end{aligned}}$$

Switching occurs within the family of systems:

$$\begin{aligned} \dot{x}(t) &= A_q x(t) + B_q u(t) \\ y(t) &= C_q x(t) \end{aligned} \qquad q \in Q = \{1, 2, \ldots, m\}$$

Assumptions:

(i) the switching signal $\sigma : [0, \infty) \rightarrow Q$ is available as (discrete) output signal

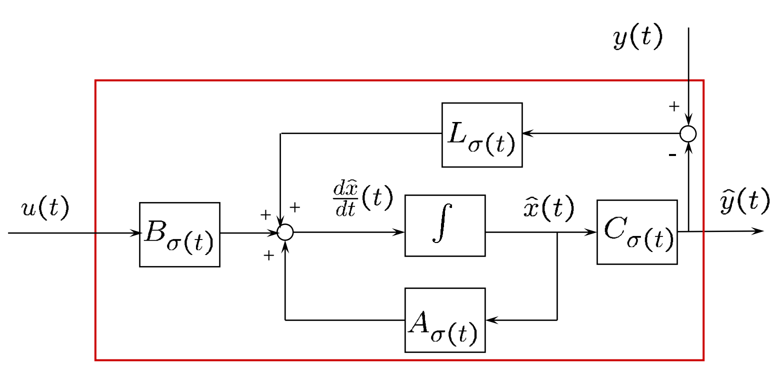
(ii) (A_q, C_q) <u>observable</u> for all q $\in$ Q

(iii) known minimum dwell time τ_D>0 between consecutive switchings

SWITCHING OBSERVER

$$\begin{aligned} \dot{\hat{x}}(t) &= A_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}u(t) + \boxed{L_{\sigma(t)}}(y(t) - \hat{y}(t)) \\ \hat{y}(t) &= C_{\sigma(t)}\hat{x}(t) \end{aligned}$$

observer time-varying gain



SWITCHING OBSERVER DESIGN

Idea:

design the switching observer gains L_1, L_2, …, L_m such that the dynamics of the estimation error

$$\dot{e}(t) = (A_{\sigma(t)} - L_{\sigma(t)} C_{\sigma(t)})\, e(t)$$

is contractive over each switching time interval, and, hence, $e(t) \to 0$, for any $e(0)$ and for any σ: $[0,\infty) \to Q$ with minimum dwell time τ_D.

Note:

stability under slow switching condition is forced by making the error dynamics fast compared with the given τ_D.

SQUASHING LEMMA

Suppose (A,C) observable. Let $\tau_D>0$.

Then, for any $\rho>0$ there exists $\alpha>0$ and L such that

$$\|e^{(A-LC)t}\| \leq \rho e^{-\alpha(t-\tau_D)}, t \geq 0$$

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$$\|e^{Mt}\| \leq \|T\| \|T^{-1}\| e^{-\alpha t}$$

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Proof. [cont'd]

$$\|e^{(A-LC)t}\| \leq \quad \|T(\alpha)\| \|T^{-1}(\alpha)\| \, e^{-\alpha t}$$

Choose $\alpha>0$ such that

$$\|T(\alpha)\| \|T^{-1}(\alpha)\| \leq \rho e^{\alpha \tau_D}$$

[such α exists since $T(\alpha)$ and $T^{-1}(\alpha)$ are rational]

This concludes the proof.

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If 0<ρ<1, then, during each switching time interval [t_i, t_{i+1}) with t_{i+1}-$t_i \geq \tau_D$, the dynamics contracts of a factor at least equal to ρ

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$$\|e^{(A-LC)(t_{i+1}-t_i)}\| \leq \rho e^{-\alpha(t_{i+1}-t_i-\tau_D)} \leq \rho$$

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Remarks:

- convergence to zero is actually exponential
- explicit bounds improving the Squashing Lemma result have been recently introduced

Bibliography

- A. Alessandri and P. Coletta.
  Switching observers for continuous-time and discrete-time linear systems
  In Proceedings of the American Control Conference Arlington, VA
  June, 2001