Hybrid Systems Course
Observer design for hybrid systems

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OBSERVER DESIGN PROBLEM
Goal: recover the state of a system from its input and output
MOTIVATION FOR OBSERVER DESIGN

• Need to monitor the evolution of the system

• Control algorithms require full state feedback

... but measuring the complete state of the system may be not economically feasible or even possible

OUTLINE

• observer design for continuous time linear systems

• observer design for switched linear systems with known switchings

• observer design for hybrid systems
OUTLINE

• observer design for continuous time linear systems

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• observer design for hybrid systems

CONTINUOUS TIME LINEAR SYSTEMS

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \]

\( u(t) \in \mathbb{R}^m \equiv \text{input} \)
\( y(t) \in \mathbb{R}^p \equiv \text{output} \)
\( x(t) \in \mathbb{R}^n \equiv \text{state} \)
CONTINUOUS TIME LINEAR SYSTEMS

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \]

\[ u(t) \in \mathbb{R}^m \equiv \text{input} \]
\[ y(t) \in \mathbb{R}^p \equiv \text{output} \]
\[ x(t) \in \mathbb{R}^n \equiv \text{state} \]

OBSERVABILITY NOTION

“possibility of reconstructing the state from past input and output measurements”

Definition [indistinguishable states]

\( x_1 \) and \( x_2 \) are indistinguishable if the output associated with \( x(0) = x_1 \) and \( x(0) = x_2 \) is identical for any input \( u(t), t \geq 0 \)

\[ y_1(t) = Ce^{At}x_1 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau \]
\[ y_2(t) = Ce^{At}x_2 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau \]
\[ y_1(t) - y_2(t) = Ce^{At}(x_1 - x_2) \]
OBSERVABILITY NOTION

“possibility of reconstructing the state from past input and output measurements”

Definition [unobservable state]
x is unobservable if it is indistinguishable from the origin

\[ Ce^{At}x = 0, \quad t \geq 0 \]
OBSERVABILITY NOTION AND CHARACTERIZATION

“possibility of reconstructing the state from past input and output measurements”

Definition [unobservable state]

- $x$ is unobservable if it is indistinguishable from the origin

\[ x \text{ belongs to the null space of the observability matrix } O_n \]

Further definitions:

- $x$ is observable if it is not unobservable

The set of all the unobservable states is the unobservable subspace, the orthogonal subspace is called observable subspace
OBSERVABILITY NOTION AND CHARACTERIZATION

“possibility of reconstructing the state from past input and output measurements”

Definition [observable system]
A system is observable if all states $x \neq 0$ are observable

\[
\begin{bmatrix}
A & C
\end{bmatrix}
\]

the observability matrix $O_n$ has maximum rank $(n)$

Remark:
if the observability matrix $O_n$ has maximum rank $(n)$, then, the pair $(A, C)$ is called observable

OBSERVABILITY NOTION AND CHARACTERIZATION

Key property:

$(A, C)$ is observable

\[
\begin{bmatrix}
A & C
\end{bmatrix}
\]

one can select matrix $L$ such that $A-LC$ has arbitrarily chosen eigenvalues
KALMAN DECOMPOSITION

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \]

similarity transformation for the decomposition into observable and unobservable part

\[ w := T_0 x \]
**KALMAN DECOMPOSITION**

\[
\begin{bmatrix}
\dot{w}_o(t) \\
\dot{w}_{no}(t)
\end{bmatrix} =
\begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
w_o(t) \\
w_{no}(t)
\end{bmatrix}
+ 
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u(t)
\]

\[
y(t) =
\begin{bmatrix}
C_1 \\
0
\end{bmatrix}
\begin{bmatrix}
w_o(t) \\
w_{no}(t)
\end{bmatrix}
\]

**OBSERVABILITY NOTION**

“possibility of reconstructing the state from past input and output measurements”

**Definition [detectable system]**

A system is detectable if the eigenvalues of the unobservable part have all strictly negative real part

**Remark:**

in this case, the pair \((A,C)\) is detectable
ASYMPTOTIC OBSERVER

“possibility of reconstructing the state from past input and output measurements”

Definition [asymptotic observer]:
An asymptotic observer is a system that consistently estimates the state $x(t)$ based on the input and output measurements $u(\tau)$ and $y(\tau)$, $0 \leq \tau \leq t$, for any (unknown) initial condition $x_0$ and for any input $u(\cdot)$:

$$\lim_{t \to \infty} \|\hat{x}(t) - x(t)\| = 0, \quad \forall x(0) = x_0 \in \mathbb{R}^n, \forall u(\cdot)$$

Remark: Also called Luenberger observer
\[
\dot{x}(t) = A\dot{x}(t) + Bu(t) + L(y(t) - \hat{y}(t))
\]
\[
\hat{y}(t) = C\hat{x}(t)
\]
observer gain
DYNAMICS OF THE STATE ESTIMATION ERROR

\[ e(t) := x(t) - \hat{x}(t) \]

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) =Cx(t) \]

\[ \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t)) \]
\[ \hat{y}(t) = C\hat{x}(t) \]
DYNAMICS OF THE STATE ESTIMATION ERROR

\[ e(t) := x(t) - \hat{x}(t) \]

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \]

\[ \dot{x}(t) = A\hat{x}(t) + Bu(t) + L(Cx(t) - C\hat{x}(t)) \]
\[ \hat{y}(t) = C\hat{x}(t) \]

\[ \dot{e}(t) = (A - LC) e(t) \]

ASYMPTOTIC OBSERVER

\[ \dot{e}(t) = (A - LC) e(t) \]

If \( A \) is Hurwitz, one can set \( L = 0 \) and obtain that the estimation error converges exponentially to zero.

This means that one can just duplicate the system dynamics, without using the output measurements. The rate of convergence will be determined by the real part of the eigenvalues of \( A \) (see the lecture on Lyapunov stability).
ASYMPTOTIC OBSERVER

Theorem:

If \((A,C)\) is detectable, then, \(L\) can be designed so that \(A-LC\) is Hurwitz and, hence, the estimation error converge exponentially to zero:

\[ ||e(t)|| \leq \mu e^{-\lambda_0 t} ||e(0)||, \quad t \geq 0, \quad \forall e(0) = e_0 \in \mathbb{R}^n \]

Sketch of the proof:

- the system can be decomposed in observable/unobservable part
ASYMPTOTIC OBSERVER

Theorem:
If \((A,C)\) is detectable, then, \(L\) can be designed so that \(A-LC\) is Hurwitz and, hence, the estimation error converge exponentially to zero:

\[
\|e(t)\| \leq \mu e^{-\lambda_0 t} \|e(0)\|, \quad t \geq 0, \quad \forall e(0) = e_0 \in \mathbb{R}^n
\]

Sketch of the proof:
- the system can be decomposed in observable/unobservable part
- the observable part can be reconstructed from the output
- the unobservable part is asymptotically stable, hence it can be reconstructed just by duplicating the corresponding system dynamics
KALMAN DECOMPOSITION

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \]

similarity transformation for the decomposition into observable and unobservable part

\[ w := T_0 x \]

KALMAN DECOMPOSITION

\[
\begin{bmatrix}
\dot{\omega}_o(t) \\
\dot{\omega}_{no}(t)
\end{bmatrix} =
\begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
\omega_o(t) \\
\omega_{no}(t)
\end{bmatrix} +
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u(t)
\]

\[ y(t) = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} \omega_o(t) \\
\omega_{no}(t) \end{bmatrix} \]
This sub-system is observable, not necessarily stable

\[ \begin{aligned}
\dot{w}_o(t) &= A_{11}w_o(t) + B_1u(t) \\
y(t) &= C_1w_o(t)
\end{aligned} \]

The output contains info to reconstruct the state \( w_o(t) \):
an exponentially stable dynamics can be imposed to the state estimation error
KALMAN DECOMPOSITION: UNOBSERVABLE PART

\[ \dot{w}_{no}(t) = A_{22} w_{no}(t) + A_{21} \dot{w}_{o}(t) + B_2 u(t) \]

the unobservable state \( w_{no}(t) \) does not affect the output

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the contribution of the unknown and unobservable initial condition \( w_{no}(0) \) exponentially converges to zero
DYNAMICS OF THE ESTIMATION ERROR

\[ e_w := T_o e \]

\[ e_w(t) = T_o (A - LC) T_o^{-1} e_w(t) \]

\[ = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} C_1 & 0 \end{bmatrix} e_w(t) \]

\[ = \begin{bmatrix} A_{11} - L_1 C_1 \\ A_{21} - L_2 C_1 \end{bmatrix} \begin{bmatrix} 0 \\ A_{22} \end{bmatrix} e_w(t) \]

- Eigenvalues of \( A_{11} - L_1 C_1 \)
can be arbitrarily selected
- Since \((A_{11}, C_1)\) is observable
- Eigenvalues of the unobservable part keep fixed

ASYMPTOTIC OBSERVER

Theorem:

If \((A, C)\) is detectable, then, L can be designed so that \(A - LC\) is Hurwitz and, hence, the estimation error converges exponentially to zero:

\[ ||e(t)|| \leq \mu e^{-\lambda t} ||e(0)||, \quad t \geq 0, \quad \forall e(0) = e_0 \in \mathbb{R}^n \]

Remark: The convergence rate can be arbitrarily chosen if and only if \((A, C)\) is observable.
OUTLINE

• observer design for continuous time linear systems

• observer design for switched linear systems with known switchings

• observer design for hybrid systems

SWITCHED LINEAR SYSTEMS (WITH INPUT/OUTPUT)

\[
\begin{align*}
\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\
y(t) &= C_{\sigma(t)}x(t)
\end{align*}
\]

Switching occurs within the family of systems:

\[
\begin{align*}
\dot{x}(t) &= A_qx(t) + B_qu(t) \\
y(t) &= C_qx(t)
\end{align*}
\]

\[q \in Q = \{1, 2, \ldots, m\}\]
SWITCHED LINEAR SYSTEMS (WITH INPUT/OUTPUT)

\[
\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) \\
y(t) = C_{\sigma(t)} x(t)
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\[
\dot{x}(t) = A_q x(t) + B_q u(t) \\
y(t) = C_q x(t)
\]
\[q \in Q = \{1, 2, \ldots, m\}\]

Assumptions:
(i) the switching signal \(\sigma : [0, \infty) \rightarrow Q\) is available as (discrete) output signal
(ii) \((A_q, C_q)\) detectable for all \(q \in Q\)
SWITCHING OBSERVER

\[
\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + L_{\sigma(t)}(y(t) - \hat{y}(t)) \\
\hat{y}(t) = C_{\sigma(t)}\hat{x}(t)
\]

observer time-varying gain

DYNAMICS OF THE STATE ESTIMATION ERROR

\[
e(t) := x(t) - \hat{x}(t)
\]

system
\[
\begin{align*}
\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\
y(t) &= C_{\sigma(t)}x(t)
\end{align*}
\]

observer
\[
\begin{align*}
\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + L_{\sigma(t)}(y(t) - \hat{y}(t)) \\
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DYNAMICS OF THE STATE ESTIMATION ERROR

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\[
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\dot{x}(t) &= A_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}u(t) + L_{\sigma(t)}(y(t) - \hat{y}(t)) \\
\hat{y}(t) &= C_{\sigma(t)}\hat{x}(t)
\end{align*}
\]

\[ \dot{e}(t) = (A_{\sigma(t)} - L_{\sigma(t)}C_{\sigma(t)})e(t) \]

\[ A_q - L_qC_q \text{ Hurwitz for all } q \in Q \text{ does not guarantee that } \]
\[ e(t) \to 0, \forall e(0), \forall \sigma: [0, \infty) \to Q \text{ (GUAS of equilibrium } e = 0) \]
Theorem: If there exists \( P = P^T > 0 \) such that
\[
P(A_q - L_q C_q) + (A_q - L_q C_q)^T P < 0
\]
\( \forall q \in Q = \{1, 2, \ldots, m\} \)
then, the switching observer consistently estimates the continuous state of the switched system, for any \( e(0) \) and for any \( \sigma: [0, \infty) \rightarrow Q \).

Proof. \( V(e) = e^T P e \) is a radially unbounded common Lyapunov function at the equilibrium \( e = 0 \). Then, \( e = 0 \) is GUAS.
Theorem: If there exists $P = P^T > 0$ such that
$$P(A_q - L_qC_q) + (A_q - L_qC_q)^T P < 0$$
for all $q \in Q = \{1, 2, \ldots, m\}$
then, the switching observer consistently estimates the continuous state of the switched system, for any $e(0)$ and for any $\sigma: [0, \infty) \rightarrow Q$.

Proof. $V(e) = e^T P e$ is a radially unbounded common Lyapunov function at the equilibrium $e = 0$. Then, $e = 0$ is GUAS.

Note: exponential convergence can also be proven [see the lecture on Lyapunov stability]

SWITCHING OBSERVER DESIGN

We are designing the observer, and its gains…
We are designing the observer, and its gains...

We should choose the observer gains $L_1, L_2, \ldots, L_m$ such that there exists $P = P^T > 0$ satisfying

$$P(A_q - L_q C_q) + (A_q - L_q C_q)^T P < 0$$

$$\forall q \in Q = \{1, 2, \ldots, m\}$$

Apparently not an easy problem because of the terms $P L_i$. 

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SWITCHING OBSERVER DESIGN

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$$P(A_q - L_q C_q) + (A_q - L_q C_q)^T P < 0$$

$\forall q \in Q = \{1, 2, \ldots, m\}$

Linear Matrix Inequalities (LMI) reformulation:

By setting $L_q = P^{-1} Y_q$, we have $PL_q = Y_q$.

The problem can then be rephrased as that of determining $P = P^T > 0$ and $Y_1, Y_2, \ldots, Y_m$ such that

$$PA_q - Y_q C_q + A_q^T P - C_q^T Y_q^T < 0$$

$\forall q \in Q = \{1, 2, \ldots, m\}$
SWITCHING OBSERVER DESIGN

We are designing the observer, and its gains...

We should choose the observer gains $L_1$, $L_2$, ..., $L_m$ such that there exists $P = P^T > 0$ satisfying

$$P(A_q - L_q C_q) + (A_q - L_q C_q)^T P < 0$$

$\forall q \in Q = \{1, 2, \ldots, m\}$

Linear Matrix Inequalities (LMI) reformulation:

By setting $L_q = P^{-1} Y_q$, we have $PL_q = Y_q$

The problem can then be rephrased as that of determining $P = P^T > 0$ and $Y_1$, $Y_2$, ..., $Y_m$ such that

$$PA_q - Y_q C_q + A_q^T P - C_q^T Y_q^T < 0$$

$\forall q \in Q = \{1, 2, \ldots, m\}$  \quad \{LMIs in $P$ and $Y_q$\}

Gains are then recovered by $L_q = P^{-1} Y_q$
SWITCHED LINEAR SYSTEMS (WITH INPUT/OUTPUT)

\[
\begin{align*}
\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\
y(t) &= C_{\sigma(t)}x(t)
\end{align*}
\]

Switching occurs within the family of systems:

\[
\begin{align*}
\dot{x}(t) &= A_qx(t) + B_qu(t) \\
y(t) &= C_qx(t)
\end{align*}
\]

\[q \in Q = \{1, 2, \ldots, m\}\]

Assumptions:
(i) the switching signal \(\sigma: [0, \infty) \rightarrow Q\) is available as (discrete) output signal
(ii) \((A_q, C_q)\) observable for all \(q \in Q\)
(iii) known minimum dwell time \(\tau_D > 0\) between consecutive switchings

SWITCHING OBSERVER

\[
\begin{align*}
\dot{x}(t) &= A_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}u(t) + L_{\sigma(t)}(y(t) - \hat{y}(t)) \\
\hat{y}(t) &= C_{\sigma(t)}\hat{x}(t)
\end{align*}
\]

observer time-varying gain
SWITCHING OBSERVER DESIGN

Idea:

design the switching observer gains $L_1, L_2, \ldots, L_m$ such that the dynamics of the estimation error

$$
\dot{e}(t) = (A_{\sigma(t)} - L_{\sigma(t)}C_{\sigma(t)})e(t)
$$

is contractive over each switching time interval, and, hence, $e(t) \to 0$, for any $e(0)$ and for any $\sigma: [0, \infty) \to Q$ with minimum dwell time $\tau_D$.

Note:

stability under slow switching condition is forced by making the error dynamics fast compared with the given $\tau_D$.

SQUASHING LEMMA

Suppose $(A,C)$ observable. Let $\tau_D > 0$.

Then, for any $\rho > 0$ there exists $\alpha > 0$ and $L$ such that

$$
\|e^{(A-LC)t}\| \leq \rho e^{-\alpha(t-\tau_D)} , t \geq 0
$$
SQUASHING LEMMA

Suppose \((A, C)\) observable. Let \(\tau_D > 0\).
Then, for any \(\rho > 0\) there exists \(\alpha > 0\) and \(L\) such that

\[
\|e^{(A-LC)t}\| \leq \rho e^{-\alpha(t-\tau_D)}, \ t \geq 0
\]

Proof.
The eigenvalues of \(A-LC\) can be arbitrarily selected. Choose \(L\) so that they are distinct, real, strictly negative with \(\lambda_i < -\alpha\).

Then, \(M := A-LC\) is diagonalizable
\(M = T M_d T^{-1}\) with \(M_d\) diagonal \(\Rightarrow e^{Mt} = T e^{M_d t} T^{-1}\)
SQUASHING LEMMA

Suppose \((A,C)\) observable. Let \(\tau_D > 0\).
Then, for any \(\rho > 0\) there exists \(\alpha > 0\) and \(L\) such that

\[
\left\| e^{(A-LC)t} \right\| \leq \rho e^{-\alpha(t-\tau_D)}, \ t \geq 0
\]

Proof.

The eigenvalues of \(A-LC\) can be arbitrarily selected. Choose \(L\) so that they are distinct, real, strictly negative with \(\lambda_i < -\alpha\).
Then, \(M := A-LC\) is diagonalizable
\(M = T M_d T^{-1}\) with \(M_d\) diagonal \(\Rightarrow e^{Mt} = T e^{M_d t} T^{-1}\)

\[
\left\| e^{Mt} \right\| \leq \|T\| \|T^{-1}\| e^{-\alpha t}
\]

SQUASHING LEMMA

Suppose \((A,C)\) observable. Let \(\tau_D > 0\).
Then, for any \(\rho > 0\) there exists \(\alpha > 0\) and \(L\) such that

\[
\left\| e^{(A-LC)t} \right\| \leq \rho e^{-\alpha(t-\tau_D)}, \ t \geq 0
\]

Proof. [cont’d]

\[
\left\| e^{(A-LC)t} \right\| \leq \|T(\alpha)\| \|T^{-1}(\alpha)\| e^{-\alpha t}
\]
Choose \(\alpha > 0\) such that

\[
\|T(\alpha)\| \|T^{-1}(\alpha)\| \leq \rho e^{\alpha \tau_D}
\]
[such \(\alpha\) exists since \(T(\alpha)\) and \(T^{-1}(\alpha)\) are rational]
This concludes the proof.
SQUASHING LEMMA

Suppose (A,C) observable. Let $\tau_D > 0$.
Then, for any $\rho > 0$ there exists $\alpha > 0$ and $L$ such that

$$\|e^{(A-LC)t}\| \leq \rho e^{-\alpha(t-\tau_D)}, \; t \geq 0$$

Statement:
If $0 < \rho < 1$, then, during each switching time interval $[t_i, t_{i+1})$ with $t_{i+1} - t_i \geq \tau_D$, the dynamics contracts of a factor at least equal to $\rho$. 

$$\|e^{(A-LC)(t_{i+1}-t_i)}\| \leq \rho e^{-\alpha(t_{i+1}-t_i-\tau_D)} \leq \rho$$
SWITCHING OBSERVER DESIGN

Fix $0 < \rho < 1$.

For any $q \in Q$, determine $L_q$ according to the Squashing Lemma with $\tau_D$ equal to the (known) minimum dwell time.
SWITCHING OBSERVER DESIGN

Fix $0 < \rho < 1$.

For any $q \in Q$, determine $L_q$ according to the Squashing Lemma with $\tau_D$ equal to the (known) minimum dwell time.

Then, the switching observer with gains $L_1, L_2, \ldots, L_m$ consistently estimates the continuous state of the switched system, for any $e(0)$ and for any $\sigma: [0, \infty) \to Q$ with minimum dwell time $\tau_D$.

Remarks:
- convergence to zero is actually exponential
- explicit bounds improving the Squashing Lemma result have been recently introduced
Bibliography