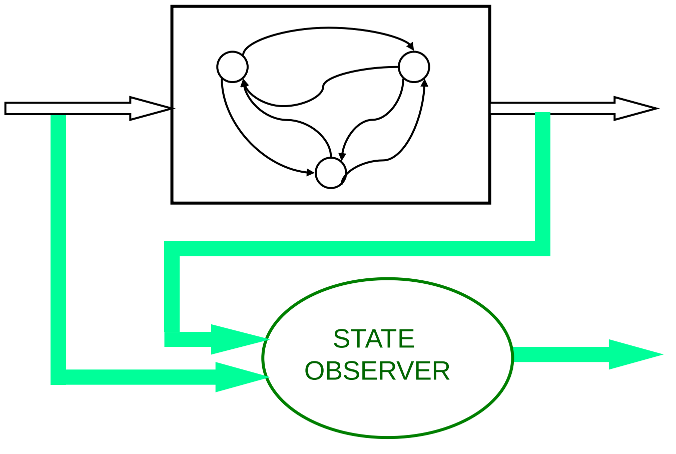
Hybrid Systems Course
Observer design for hybrid systems

Maria Prandini

DEIB - Politecnico di Milano
E-mail: maria.prandini@polimi.it

OBSERVER DESIGN PROBLEM

Goal: recover the state of a system from its input and output



OUTLINE

- observer design for continuous time linear systems
- observer design for switched linear systems with known switchings
- observer design for hybrid systems

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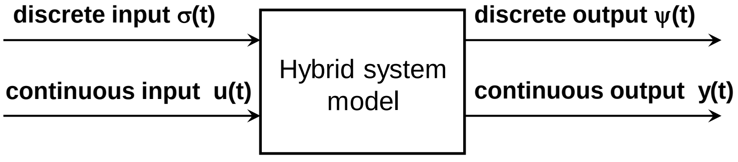
HYBRID SYSTEM



switched linear system with minimum and maximum dwell times between consecutive switchings

SWITCHED SYSTEMS vs. HYBRID AUTOMATA

- switched systems can be seen as a higher-level abstraction of hybrid automata (details of the discrete behavior neglected)
- simpler to describe but with more solutions than the original hybrid automata (conservative analysis results)

CONTINUOUS DYNAMICS

$$\dot{x}(t) = A_q x(t) + B_q u(t)$$
$$y(t) = C_q x(t)$$

$u(t) \in \Re^m \equiv$ continuous input

$y(t) \in \Re^p \equiv$ continuous output

$x(t) \in \Re^n \equiv$ continuous state

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Assumption:

(A_q, C_q) observable for all $q \in Q$

DISCRETE DYNAMICS

$$q(k+1) \in \varphi(q(k), \sigma(k+1))$$
$$\sigma(k+1) \in \Phi(q(k), x(t_{k+1}), u(t_{k+1}))$$
$$\psi(k+1) = \eta(q(k), \sigma(k+1), q(k+1))$$

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where

$q(k) \in Q \equiv$ finite set of states
$\sigma(k) \in \Sigma \equiv$ finite set of input symbols (events)
$\psi(k) \in \Psi \equiv$ finite set of output symbols

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$\varphi: Q \times \Sigma \rightarrow 2^Q \equiv$ transition (set-valued) function

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$\Phi: Q \times X \times U \rightarrow 2^\Sigma \equiv$ function specifying admissible events

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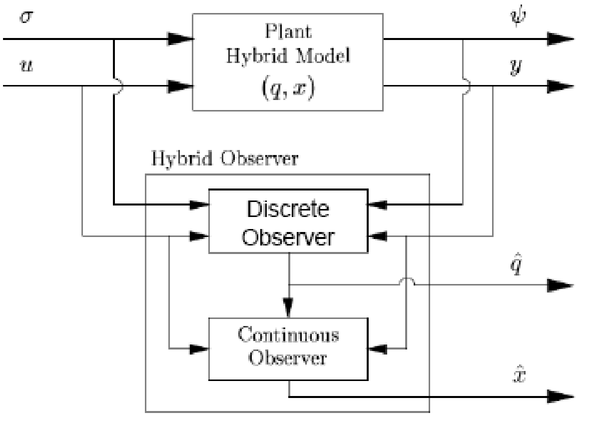
Note:
the continuous part affects the discrete part via function Φ

HYBRID OBSERVER



HYBRID OBSERVER DESIGN

- the discrete observer provides an estimate of the discrete state
- the continous observer uses this information to estimate the continuous state

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- the continous observer uses this information to estimate the continuous state

Idea:

If the discrete observer correctly identifies the discrete state in finite time, then, after that time, the switching signal is available to the continuous observer

FINITE AUTOMATA

Finite automaton (with input and output):

$$M = (Q, \Sigma, \Psi, \varphi, \phi, \eta)$$

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$\Sigma \equiv$ finite set of input symbols (events)
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Ψ $\equiv$ finite set of output symbols
φ: $Q \times \Sigma \rightarrow 2^Q$ $\equiv$ transition (set-valued) function
ϕ: $Q \rightarrow 2^\Sigma$ $\equiv$ function specifying the admissible events
η: $Q \times \Sigma \times Q \rightarrow \Psi$ $\equiv$ output function

$$q(k+1) \in \varphi(q(k), \sigma(k+1))$$
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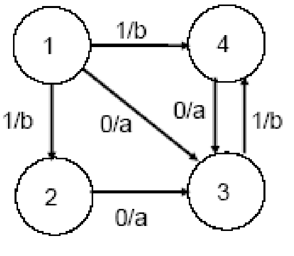

$$Q = \{1, 2, 3, 4\}$$
$$\Sigma = \{0, 1\}$$
$$\Psi = \{a, b\}$$

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Def. [alive automaton]:
M is *alive* if it has no blocking state [for any $q \in Q\ \exists\ \sigma \in \phi(q)$ and $q' \in Q$ such that $q' = \varphi(q,\sigma)$]

CURRENT-STATE OBSERVABILITY NOTION

Definition [current-state observability]:

$M = (Q, \Sigma, \Psi, \varphi, \phi, \eta)$ is *current-state observable* if there exists an integer *K* such that, for any unknown initial state $q(0) = q_0 \in Q$ and for every input sequence $\sigma(1), \sigma(2), \ldots, \sigma(i)$, the location $q(i)$ can be determined from the output sequence $\psi(1), \psi(2), \ldots, \psi(i)$, for every $i \geq K$.

CURRENT-STATE OBSERVER

The current-state observer for M = (Q, Σ, Ψ, φ, ϕ, η) is a finite automaton M_O = (Q_o, Σ_o, Ψ_o, φ_o, η_o) with

$Q_o \subseteq 2^Q$
$\Sigma_o = \Psi$
$\Psi_o = Q_o$

$$\hat{q}(k+1) = \varphi_O(\hat{q}(k), \psi(k+1))$$

$$\psi_O(k+1) = \hat{q}(k+1)$$

initialized with $\hat{q}(0) = Q$ (the initial state q(0) is not known)

CURRENT-STATE OBSERVER

Observer transition function φ_o: $Q_o \times \Sigma_o \to 2^{Q_o}$

- one could set $Q_o = 2^Q$ and consider all subsets $S \subseteq Q$ and all symbols $\psi \in \Sigma_o = \Psi$ and compute $\varphi_O(S, \psi)$

  $$\varphi_O(S, \psi) = \{q \in Q : \exists s \in S, \sigma \in \phi(s) \text{ s.t. } q \in \varphi(s, \sigma), \psi = \eta(s, \sigma, q)\}$$

  (set of successors of the states in S, compatibly with the observation ψ)

  2^Q and Ψ finite → the procedure is guaranteed to terminate

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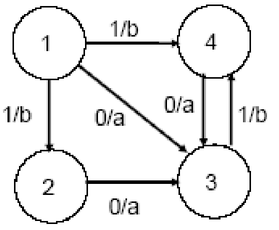
- a more efficient method is to build the current-state observation tree
  - start with $S_0 = Q$ (the initial state estimate),
  - compute $S_{1,i} = \varphi_o(S_0, \psi_i), \forall \psi_i \in \Psi$
  - for each $S_{1,i}$ determine $S_{2,ij} = \varphi_o(S_{1,i}, \psi_j), \forall \psi_j \in \Psi$
  - and so on…

CURRENT-STATE OBSERVER: AN EXAMPLE



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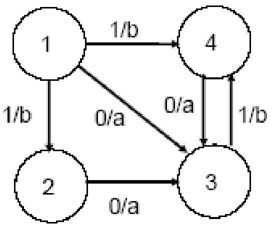

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Current-state observation tree

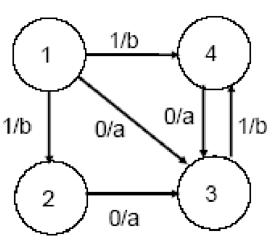
$\{1,2,3,4\}$

$\psi = a$: $\{3\}$ — $\psi = b$: $\{4\}$ — $\psi = a$: $\{3\}$

$\psi = b$: $\{2,4\}$ — $\psi = a$: $\{3\}$

$Q_o = \{\{1,2,3,4\}, \{2,4\}, 3, 4\}$

CURRENT-STATE OBSERVER: AN EXAMPLE



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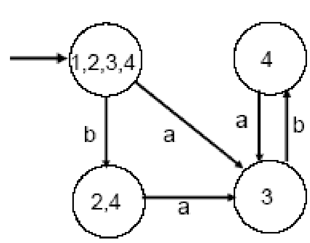
Current-state observer



CURRENT-STATE OBSERVABILITY

Theorem: Given M = (Q, Σ, Ψ, φ, ϕ, η), let M_O = (Q_o, Σ_o, Ψ_o, φ_o, η_o) be the corresponding observer. An alive M is current-state observable if and only if:

- the set $Q \cap Q_o$ is nonempty

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- the set $Q \cap Q_o$ is nonempty
- every primary cycle $Q_c^i \subset Q_o$ includes at least one state in Q, that is $Q_c^i \cap Q$ is nonempty

Definition [primary cycle]: A primary cycle is a path from some node to itself which do not cross the same node twice

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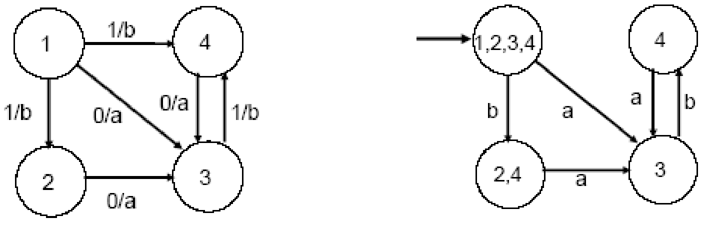
- the set $Q \cap Q_o$ is nonempty
- every primary cycle $Q_c^i \subset Q_o$ includes at least one state in Q, that is $Q_c^i \cap Q$ is nonempty
- the subset $Q \cap Q_o$ is φ_o-invariant
  (this jointly with the previous condition means that cycles are all made of singletones)

Definition [primary cycle]: A primary cycle is a path from some node to itself which do not cross the same node twice

CURRENT-STATE OBSERVER: AN EXAMPLE

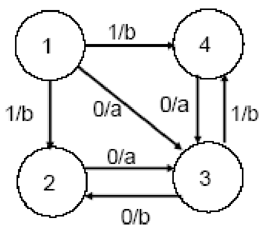


$Q = \{1, 2, 3, 4\}$ $\quad Q_o = \{\{1, 2, 3, 4\}, \{2, 4\}, 3, 4\}$

$Q \cap Q_o = \{3, 4\} \neq \emptyset$
$Q_c^1 = \{3, 4\} \subset Q_o \quad Q_c^1 \cap Q = \{3, 4\} \neq \emptyset$
$Q \cap Q_o = \{3, 4\} \rightarrow \varphi_o - invariant$

} current-state observable

CURRENT-STATE OBSERVER: AN EXAMPLE



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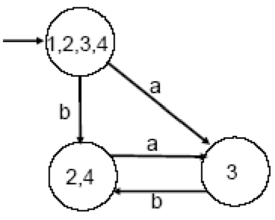

$Q_O = \{\{1, 2, 3, 4\}, \{2, 4\}, 3\}$

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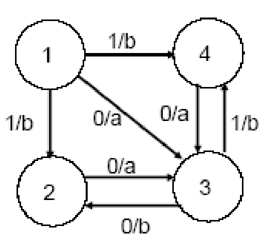

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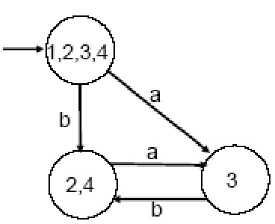

$Q_O = \{\{1, 2, 3, 4\}, \{2, 4\}, 3\}$

$Q \cap Q_O = \{3\}$

$Q_c^1 = \{\{2, 4\}, 3\}$ $\quad Q_c^i \cap Q = \{3\} \neq \emptyset$

$Q \cap Q_0 = \{3\}$ is not φ_o-invariant

} not current-state observable

CURRENT-STATE OBSERVABILITY

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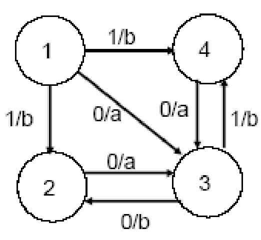
M = (Q, Σ, Ψ, φ, ϕ, η) is *current-state observable* if there exists an integer *K* such that, for any unknown initial state $q(0) = q_0 \in Q$ and for every input sequence $\sigma(1), \sigma(2), \ldots, \sigma(i)$, the location q(*i*) can be determined from the output sequence $\psi(1), \psi(2), \ldots, \psi(i)$, for every $i \geq K$.

Remark: If also the input sequence $\sigma(1), \sigma(2), \ldots, \sigma(i)$ is available, then one can replace $\psi(1), \psi(2), \ldots, \psi(i)$ with $(\psi(1), \sigma(1)), (\psi(2), \sigma(2)), \ldots, (\psi(i), \sigma(i))$

CURRENT-STATE OBSERVER: AN EXAMPLE



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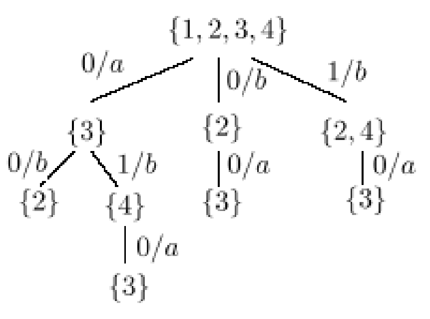

$Q_o = \{\{1, 2, 3, 4\}, \{2, 4\}, 2, 3, 4\}$

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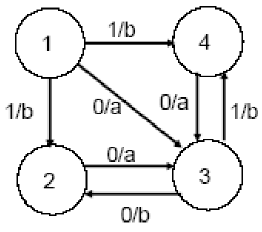

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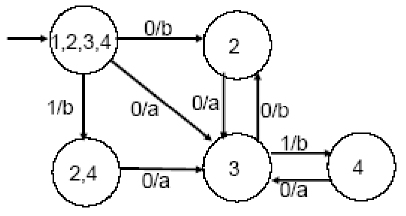

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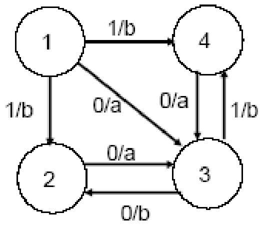

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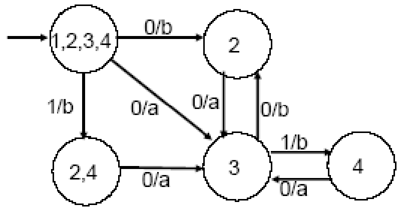


$Q_o = \{2, 3, 4, \{2, 4\}, \{1, 2, 3, 4\}\}$

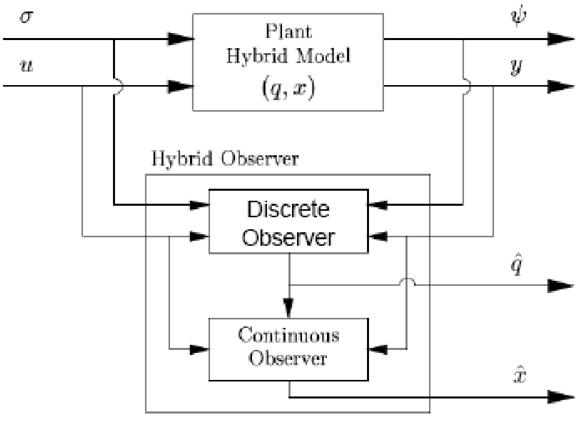
$Q \cap Q_o = \{2, 3, 4\} \neq \emptyset$

$Q_c^1 = \{2, 3\} \subset Q, \quad Q_c^2 = \{3, 4\} \subset Q$

$Q \cap Q_o = \{2, 3, 4\} \rightarrow \varphi_o - invariant$

current-state observable

HYBRID OBSERVER



HYBRID OBSERVER DESIGN

Consider the finite automaton M = (Q, Σ, Ψ, φ, ϕ, η) associated with the hybrid system

$$q(k+1) \in \varphi(q(k), \sigma(k+1))$$

$$\sigma(k+1) \in \phi(q(k))$$

$$\psi(k+1) = \eta(q(k), \sigma(k+1), q(k+1))$$

where the function ϕ: Q $\rightarrow 2^{\Sigma \cup \{\varepsilon\}}$ specifying the admissible events is defined as

$$\phi(q) := \bigcup_{x \in X, u \in U} \Phi(q, x, u)$$

HYBRID OBSERVER DECOUPLED DESIGN

If the finite automaton M = (Q, Σ, Ψ, φ, ϕ, η) associated with the hybrid system is current-state observable, then, after a finite time T = K $\tau_{D,max}$ the discrete state is available:

$$\hat{q}(t) = q(t), t \geq T$$

HYBRID OBSERVER DECOUPLED DESIGN

If the finite automaton M = (Q, Σ, Ψ, φ, ϕ, η) associated with the hybrid system is current-state observable, then, after a finite time T = K $\tau_{D,max}$ the discrete state is available:

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A switching observer can then be designed for the switched linear system with minimum dwell time $\tau_{D,min}$

$$\dot{x}(t) = A_{q(t)}x(t) + B_{q(t)}u(t)$$
$$y(t) = C_{q(t)}x(t)$$

that consistently estimates the continuous state x(t), with an exponential convergent rate

HYBRID OBSERVER COUPLED DESIGN

What if the finite automaton M = (Q, Σ, Ψ, φ, ϕ, η) associated with the hybrid system is not current-state observable?

HYBRID OBSERVER COUPLED DESIGN

What if the finite automaton M = (Q, Σ, Ψ, φ, ϕ, η) associated with the hybrid system is not current-state observable?

Idea:

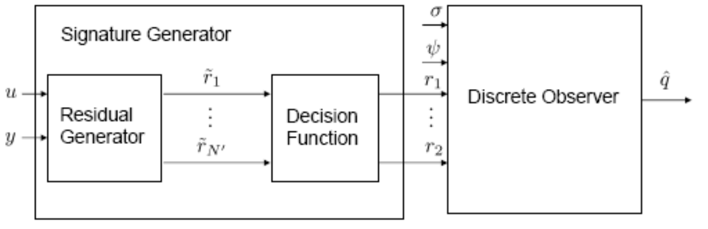
use the continuous input and output to produce additional discrete output symbols (the “signatures”) to make the automaton current-state observable

SIGNATURE GENERATOR

Use a bank of observers matched to the dynamics in the subset of discrete states of interest

$$\dot{\hat{x}}(t) = (A_q - L_q C_q)\hat{x}(t) + B_q u(t) + L_q\, y(t)$$

$$\tilde{r}_q(t) = C_q \hat{x}(t) - y(t)$$ ← residuals as in Fault Detection



$$r_j(t) = \begin{cases} true & \text{if } \|\tilde{r}_j(t)\| \leq \varepsilon \\ false & \text{if } \|\tilde{r}_j(t)\| > \varepsilon \end{cases}$$

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What if the finite automaton M = (Q, Σ, Ψ, φ, ϕ, η) associated with the hybrid system is not current-state observable?

Idea:

use the continuous input and output to produce additional discrete output symbols (the "signatures") to make the automaton current-state observable

Note that this requires some time. During such a time the switching observer for the continuous state generally causes the estimation error to growth, which needs to be accounted for in the design of the observer gains

HYBRID OBSERVER COUPLED DESIGN

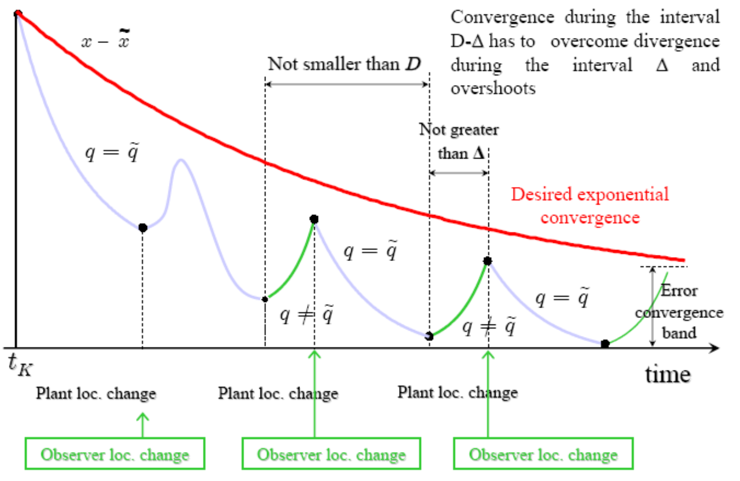

Figure taken from Marika Di Benedetto lecture in 2006

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