NONLINEAR CONTROL: MODELING
OUTLINE

We shall describe

• ordinary differential equations (ODEs) modeling nonlinear continuous state and continuous time systems

• hybrid automata and piecewise affine systems modeling hybrid systems
NONLINEAR CONTINUOUS STATE SYSTEMS

An ordinary differential equation is a mathematical model of a continuous state continuous time system:
\[ \dot{x}(t) = f(x(t)) \]

\[ X = \mathbb{R}^n \quad \equiv \text{state space} \]
\[ f: \mathbb{R}^n \to \mathbb{R}^n \quad \equiv \text{vector field (assigns a “velocity” vector to each } x) \]

Given an initial set of states \( \text{Init} \subset X \), an execution (solution in the sense of Caratheodory) over the time interval \([0,T)\) is a function \( x: [0,T) \to \mathbb{R}^n \) such that:

• \( x(0) \in \text{Init} \)
• \( x \) is continuous and piecewise differentiable
• \( x(t) = x(0) + \int_0^t f(x(\tau))d\tau, \forall t \in [0,T) \)

If \( x \) is a solution, \( dx/dt(t) = f(x(t)) \) for all \( t \) for which \( x \) is differentiable
ODE SOLUTION: WELL-POSEDNESS?

\[ \dot{x}(t) = f(x(t)), \quad x(0) = x_0 \]

• Existence?
  local existence (solution exists over \([0, \delta))\)
  global existence (solution exists over \([0, \infty))\)

• Uniqueness?

Both critical for simulation...
ODE SOLUTION: EXISTENCE?

• \( \dot{x} = -\text{sgn}(x), \quad x(0) = 0 \)

No solution if \( x(0) = 0 \), because on any interval \([0, \delta)\), \( x \) cannot:
  
  remain zero, become positive, or become negative

Theorem [local existence]

If \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is \textit{continuous}, then \( \forall x_0 \) there exists at least a solution with \( x(0) = x_0 \) defined on some \([0, \delta)\).
ODE SOLUTION: UNIQUENESS?

- \( \dot{x} = x^{1/3}, \quad x(0) = 0 \)

\( x(t)=0 \) and \( x(t) = (2/3t)^{3/2} \) are two solutions

Def. \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is **Lipschitz continuous**, if in any bounded set \( A \) of \( \mathbb{R}^n \) there exists a constant \( L \) such that

\[
||f(x_1) - f(x_2)|| \leq L \ ||x_1 - x_2||, \ \forall \ x_1, x_2 \in A
\]

Theorem [local existence and uniqueness]

If \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is **Lipschitz continuous**, then \( \forall \ x_0 \) there exists a single solution with \( x(0) = x_0 \) defined on some \([0, \delta] \).
ODE SOLUTION: GLOBAL EXISTENCE AND UNIQUENESS?

- \( \dot{x} = -x^2, \quad x(0) = -1 \)

\( x(t) = \frac{1}{t-1} \) local solution on \([0,1)\) (tends to \(-\infty\) as \(t \uparrow 1\))

Def. \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is globally Lipschitz continuous, if there exists a constant \( L \) such that

\[ ||f(x_1) - f(x_2)|| \leq L \ ||x_1 - x_2||, \ \forall \ x_1, x_2 \in \mathbb{R}^n \]

Theorem [global existence and uniqueness]
If \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is globally Lipschitz continuous, then \( \forall \ x_0 \) there exists a single solution with \( x(0) = x_0 \) defined on \([0,\infty)\).
Hybrid automaton modeling a hybrid system
EXAMPLE: THERMOSTAT

Temperature in a room controlled by a thermostat switching a heater on and off

Dynamics of the temperature $x$ (in °C):

heater on: $\dot{x} = -0.2x + 6 \quad (x \to 30)$

heater off: $\dot{x} = -0.2x \quad (x \to 0)$

Goal: regulate the temperature around 20°C

Strategy: turn the heater from OFF to ON as soon as $x \leq 18$

turn the heater from ON to OFF as soon as $x \geq 22$

hysteretic behavior
EXAMPLE: THERMOSTAT

• Continuous dynamics
  \[ \dot{x} = -0.2x \]
  \[ \dot{x} = -0.2x + 6 \]
  linear ODEs describing the temperature evolution

• Discrete dynamics

\[ Q = \{ \text{ON, OFF} \} \]
\[ \text{ON} = \Phi(\text{OFF}, e_1) \quad e_1 = [x \leq 18] \]
\[ \text{OFF} = \Phi(\text{ON}, e_2) \quad e_2 = [x \geq 22] \]
EXAMPLE: THERMOSTAT

\[ x \geq 18 \]

\text{OFF}

\[ \dot{x} = -0.2x \]
\[ x \geq 18 \]

\[ x := x \]

\[ x \leq 18 \]

\[ x := x \]

\text{ON}

\[ \dot{x} = -0.2x + 6 \]
\[ x \leq 22 \]

\[ x \geq 22 \]
EXAMPLE: THERMOSTAT

Initialization of the system:
\[
\text{Init} = \{(\text{OFF}, x): x \geq 18 \} \cup \{(\text{ON}, x): x \leq 22 \}
\]
EXAMPLE: THERMOSTAT

Reset of the continuous state $x$ due to the transition

$$x := x \rightarrow x := x^-$$
EXAMPLE: THERMOSTAT

Reset of the continuous state $x$ due to the transition

$$x := x \rightarrow x := x^-$$

right continuous: $x(t) = x^+(t)$, $\forall t$

$$x : [0, \infty) \rightarrow \mathbb{R} \text{ piecewise continuous}$$

$$x^- : (0, \infty) \rightarrow \mathbb{R} \quad x^-(t) := \lim_{\varepsilon \to 0^+} x(t - \varepsilon)$$

$$x^+ : [0, \infty) \rightarrow \mathbb{R} \quad x^+(t) := \lim_{\varepsilon \to 0^+} x(t + \varepsilon)$$
EXAMPLE: THERMOSTAT

If $x$ exits the Domain, then a discrete transition must occur

Guard conditions enable the discrete transition

OFF
\[ \dot{x} = -0.2x \]
\[ x \geq 18 \]

ON
\[ \dot{x} = -0.2x + 6 \]
\[ x \leq 22 \]
EXAMPLE: THERMOSTAT

If $x$ exits the Domain, then a discrete transition must occur.

Guard conditions enable the discrete transition

$x = 18$

$\rightarrow$ transition from OFF to ON occurs when $x = 18$
EXAMPLE: THERMOSTAT

If $x$ exits the Domain, then a discrete transition must occur.

Guard conditions enable the discrete transition:

$x = 22$

$\Rightarrow$ transition from ON to OFF occurs when $x = 22$
EXAMPLE: THERMOSTAT

\[
x \leq 18
\]

\[\begin{align*}
\text{OFF} & : \dot{x} = -0.2x \\
& \quad x \geq 17
\end{align*}\]

\[\begin{align*}
\text{ON} & : \dot{x} = -0.2x + 6 \\
& \quad x \leq 23
\end{align*}\]

\[
x \geq 22
\]

\[
\text{Domain(OFF)} \cap \text{Guard(OFF,ON)} = [17, \infty) \cap (-\infty, 18] = [17,18]
\rightarrow \text{transition from OFF to ON for } x \in [17,18]
\]

\[
\text{Domain(ON)} \cap \text{Guard(ON,OFF)} = (-\infty, 23] \cap [22, \infty) = [22,23]
\rightarrow \text{transition from ON to OFF for } x \in [22,23]
\]
EXAMPLE: THERMOSTAT

- Evolution of the temperature $x$ starting from the initial condition $x(0) = 5$ with the heater ON.
- Multiple executions starting from an initial condition $x(0) = 5$ with the heater ON.
- Evolution of the heater status starting from the initial condition $x(0) = 5$ with the heater ON.

→ Non-deterministic hybrid system.
A finite automaton is a mathematical model of an event-driven discrete system with finite state space:

\[ M = (Q, \Sigma, R) \]

- \( Q = \{ q_1, q_2, \ldots, q_N \} \equiv \text{finite set of states} \)
- \( \Sigma = \{ a, b, c, \ldots \} \equiv \text{finite set (alphabet) of input symbols (events)} \)
- \( R: Q \times \Sigma \rightarrow 2^Q \equiv \text{transition (set-valued) function} \)

where \( 2^Q \) denotes the power set of \( Q = \text{set of all subsets of } Q \)

\( R(q, \sigma) \subseteq Q \) is the set of all states the system can transit to from state \( q \) under the input symbol \( \sigma \)
FINITE AUTOMATA

\[ M = (Q, \Sigma, R) \]

- \( Q = \{ q_1, q_2, \ldots, q_N \} \equiv \text{finite set of states} \)
- \( \Sigma = \{ e_1, e_2, \ldots, e_m \} \equiv \text{finite set (alphabet) of input symbols (events)} \)
- \( R: Q \times \Sigma \rightarrow 2^Q \equiv \text{transition (set-valued) function} \)

Example: \( Q = \{ 1, 2, 3 \} \quad \Sigma = \{ a, b \} \)

\[
\begin{array}{c|cc|c}
 s & \sigma & R(s,\sigma) \\
\hline
 1 & a & \{ 2 \} \\
 1 & b & \emptyset \\
 2 & a & \emptyset \\
 2 & b & \{ 1, 3 \} \\
 3 & a & \{ 1 \} \\
 3 & b & \emptyset \\
\end{array}
\]

Graphical representation: 

1. \( a \rightarrow 2 \)
2. \( b \rightarrow 1 \)
3. \( a \rightarrow 3 \)
4. \( b \rightarrow 3 \)
**FINITE AUTOMATA: EXECUTION**

\[ M = (Q, \Sigma, R) \]

- \( Q = \{ q_1, q_2, \ldots, q_N \} \equiv \text{finite set of states} \)
- \( \Sigma = \{ e_1, e_2, \ldots, e_m \} \equiv \text{finite set (alphabet) of input symbols (events)} \)
- \( R: Q \times \Sigma \to 2^Q \equiv \text{transition (set-valued) function} \)

Given an initial set of states \( \text{Init} \subset Q \),

**M accepts an input sequence** \( \{ \sigma_0, \sigma_1, \sigma_2, \ldots \} \), with \( \sigma_i \in \Sigma, \ \forall \ i \geq 0 \), if there exists a sequence of states \( \{ s_0, s_1, s_2, \ldots \} \), \( s_i \in Q, \ \forall \ i \geq 0 \), such that:

- \( s_0 \in \text{Init} \)
- \( s_{i+1} \in R(s_i, \sigma_i) \ \forall \ i \geq 0 \)

\( \{ s_0, s_1, s_2, \ldots \} \) in the definition above is called an execution of **M** under the input sequence \( \{ \sigma_0, \sigma_1, \sigma_2, \ldots \} \)
An example:

Init = \{1\}

1,2,1 is an execution under input a, b
1,2,3,1,2 under input a,b,a,a

a,b,b is not accepted as an input
A finite automaton $M = (Q, \Sigma, R)$ is deterministic if

$$|R(q, \sigma)| \leq 1 \text{ for all } q \in Q, \sigma \in \Sigma, |R(q, \sigma)| = \text{cardinality of } R(q, \sigma)$$

$\rightarrow$ at most one execution given an initial state and an input string

Otherwise, it is nondeterministic.
A finite automaton $M = (Q, \Sigma, R)$ has a blocking state $q \in Q$ if $R(q, \sigma) = \emptyset$, for all $\sigma \in \Sigma$.
HYBRID AUTOMATON

A hybrid automaton is a mathematical model for a dynamical system whose state \( s = (q, x) \) consists of two components:

- continuous state \( x \) taking values in \( X = \mathbb{R}^n \)
- discrete state (mode) \( q \) taking values in \( Q = \{q_1, q_2, \ldots, q_N\} \)

Hybrid state space:

\[ S = Q \times X \] (N copies of \( X \))

for each value of \( q \), the admissible values for \( x \) are only a subset of \( X \)

\( \rightarrow \text{Dom}(q) = \text{domain} \) for \( x \) within mode \( q \)
$X = \mathbb{R}^2$

$q = q_1$

$Dom(q_1)$

$q = q_2$

$Dom(q_2)$
HYBRID AUTOMATON

Dynamics of $q$ and $x$ are correlated:

- $x$ follows an ODE within $\text{Dom}(q)$. The ODE depends on the value of mode $q$. 
Dynamics of q and x are correlated:

- x follows an ODE within $\text{Dom}(q)$. The ODE depends on the value of mode q
- q follows a finite automaton with transitions triggered by events given by x entering subsets of X called Guards
  \[ \text{G}(q,q') = \text{values for } x \text{ enabling the transition from } q \text{ to } q' \]
- when a transition from q to q' takes place, x is reset within $\text{Dom}(q')$ by some Reset map
  \[ \text{R}((q,q'),x) = \text{reset values for } x^+ \text{ when a transition from } q \text{ to } q' \text{ occurs at } x \]
$q = q_1$

$G(q_1, q_2)$

$q = q_2$

$Dom(q_2)$

$R((q_1, q_2), x)$

$R((q_2, q_3), x)$

$G(q_2, q_3)$
HYBRID AUTOMATA: FORMAL DEFINITION

A hybrid automaton $H$ is a collection

$$H = (Q, X, f, Init, Dom, E, G, R)$$

- $Q = \{q_1, q_2, \ldots\}$ is a set of discrete states (modes)
- $X = \mathbb{R}^n$ is the continuous state space
- $f: Q \times X \rightarrow \mathbb{R}^n$ is a set of vector fields on $X$
  - for each $q \in Q$, $f(q, \cdot)$ is the vector field of the ODE governing the evolution of $x$
  - for each $q \in Q$, $f(q, \cdot)$ is assumed to be globally Lipschitz continuous
HYBRID AUTOMATA: FORMAL DEFINITION

A hybrid automaton $H$ is a collection

$$H = (Q, X, f, Init, Dom, E, G, R)$$

- $Q = \{q_1, q_2, \ldots\}$ is a set of discrete states (modes)
- $X = \mathbb{R}^n$ is the continuous state space
- $f: Q \times X \rightarrow \mathbb{R}^n$ is a set of vector fields on $X$
- $Init \subseteq Q \times X$ is a set of initial states
- $Dom: Q \rightarrow 2^X$ assigns to each $q \in Q$ a domain $Dom(q)$ of $X$
- $E \subseteq Q \times Q$ is a set of transitions (edges)
  - each $e \in E$ is of the form $e=(q, q')$ and represents a transition from $q$ to $q'$ (edge in the directed graph with vertex set $Q$)
HYBRID AUTOMATA: FORMAL DEFINITION

A hybrid automaton $H$ is a collection

\[ H = (Q, X, f, \text{Init}, \text{Dom}, E, G, R) \]

- $Q = \{q_1, q_2, \ldots\}$ is a set of discrete states (modes)
- $X = \mathbb{R}^n$ is the continuous state space
- $f: Q \times X \to \mathbb{R}^n$ is a set of vector fields on $X$
- $\text{Init} \subseteq Q \times X$ is a set of initial states
- $\text{Dom}: Q \to 2^X$ assigns to each $q \in Q$ a domain $\text{Dom}(q)$ of $X$
- $E \subseteq Q \times Q$ is a set of transitions (edges)
- $G: E \to 2^X$ is a set of guards (guard condition)
  - For each $e = (q, q')$, whenever $x$ reaches $G(e)$ from within $\text{Dom}(q)$, transition $e$ is enabled
HYBRID AUTOMATA: FORMAL DEFINITION

A hybrid automaton $H$ is a collection

$$H = (Q, X, f, Init, Dom, E, G, R)$$

- $Q = \{q_1, q_2, \ldots\}$ is a set of discrete states (modes)
- $X = \mathbb{R}^n$ is the continuous state space
- $f: Q \times X \to \mathbb{R}^n$ is a set of vector fields on $X$
- $Init \subseteq Q \times X$ is a set of initial states
- $Dom: Q \to 2^X$ assigns to each $q \in Q$ a domain $Dom(q)$ of $X$
- $E \subseteq Q \times Q$ is a set of transitions (edges)
- $G: E \to 2^X$ is a set of guards (guard condition)
- $R: E \times X \to 2^X$ is a set of reset maps
  - for each $e = (q, q') \in E$ and $x \in Dom(q)$, $R(e, x) \subseteq Dom(q')$ is the set of values that $x$ can be reset to after the transition $e$
HYBRID AUTOMATA: FORMAL DEFINITION

A hybrid automaton $H$ is a collection

$$H = (Q, X, f, \text{Init}, \text{Dom}, E, G, R)$$

- $Q = \{q_1, q_2, \ldots\}$ is a set of discrete states (modes)
- $X = \mathbb{R}^n$ is the continuous state space
- $f: Q \times X \rightarrow \mathbb{R}^n$ is a set of vector fields on $X$
- $\text{Init} \subseteq Q \times X$ is a set of initial states
- $\text{Dom}: Q \rightarrow 2^X$ assigns to each $q \in Q$ a domain $\text{Dom}(q)$ of $X$
- $E \subseteq Q \times Q$ is a set of transitions (edges)
- $G: E \rightarrow 2^X$ is a set of guards (guard condition)
- $R: E \times X \rightarrow 2^X$ is a set of reset maps
HYBRID AUTOMATA: GRAPHICAL REPRESENTATION

- **Mode**: $q$
  - $\dot{x} = f(q, x)$
  - $\text{Dom}(q)$

- **Guard**: $x \in G(e)$

- **Transition**: $e = (q, q')$

- **Reset**: $x := z \in R(e, x)$

- **Mode**: $q'$
  - $\dot{x} = f(q', x)$
  - $\text{Dom}(q')$
EXAMPLE: THERMOSTAT

\[ x \geq 18 \]

OFF

\[ \dot{x} = -0.2 x \]

\[ x \geq 18 \]

x := x

ON

\[ \dot{x} = -0.2 x + 6 \]

\[ x \leq 22 \]

x := x

x := x

x \geq 22

\[ x \leq 22 \]

H = (Q,X,f,Init,Dom,E,G,R)

Q = \{OFF, ON\}; \quad X = \mathbb{R};

f(OFF,x) = -0.2 x; f(ON, x) = -0.2 x + 6

Init = \{ (OFF,x) : x \geq 18 \} \cup \{ (ON,x) : x \leq 22 \}

Dom(OFF) = [18, \infty); Dom(ON) = (-\infty, 22]

E = \{ (OFF,ON),(ON,OFF) \}

G((OFF,ON)) = (-\infty, 18]; G((ON,OFF)) = [22, \infty)

R(e,x) = \{ x \} for any e \in E
HYBRID AUTOMATA: EXECUTION

To define an execution of a hybrid system, we need to introduce first the notions of

• hybrid time set
• hybrid trajectory (which includes hybrid time set among its components)

We shall see that an execution of a hybrid automaton H is a hybrid trajectory that is “consistent” with the definition of H
HYBRID TIME SET

A hybrid time set is a finite or infinite sequence of intervals $\tau = \{I_i, i=0,1,\ldots, M \}$ such that

- $I_i = [\tau_i, \tau'_i]$ for $i < M$
- $I_M = [\tau_M, \tau'_M]$ or $I_M = [\tau_M, \tau'_M)$ if $M < \infty$
- $\tau'_i = \tau_{i+1}$
- $\tau_i \leq \tau'_i$
A hybrid time set is a finite or infinite sequence of intervals
\[ \tau = \{I_i, \ i=0,1,\ldots, \ M \} \] such that

- \( I_i = [\tau_i, \tau_i'] \) for \( i < M \), where \( \tau_i \) represent times of discrete transitions
- \( I_M = [\tau_M, \tau_M'] \) or \( I_M = [\tau_M, \tau_M) \) if \( M<\infty \)
- \( \tau_i' = \tau_{i+1} \) for consecutive intervals, without gaps
- \( \tau_i \leq \tau_i' \) intervals can be degenerate to represent multiple transitions at the same time
HYBRID TIME SET

A hybrid time set is a finite or infinite sequence of intervals
\( \tau = \{ I_i, \ i=0,1,\ldots, \ M \} \) such that

- \( I_i = [\tau_i, \tau_i'] \) for \( i < M \)
- \( I_M = [\tau_M, \tau_M'] \) or \( I_M = [\tau_M, \tau_M) \) if \( M < \infty \)
- \( \tau_i' = \tau_{i+1} \)
- \( \tau_i \leq \tau_i' \)

\( t_1 \prec t_2 : \) “\( t_1 \) precedes \( t_2 \)”

\( t_1 \prec t_2 \prec t_3 \prec t_4 \)

the elements of \( \tau \) are linearly ordered
HYBRID TIME SET: LENGTH

Two notions of length for a hybrid time set $\tau = \{I_i, i=0,1,\ldots, M\}$:

- **Discrete extent:**
  \[<\tau> = M + 1\]  \(\text{number of discrete transitions}\)

- **Continuous extent:**
  \[||\tau|| = \Sigma_{i=0,1,\ldots,M} |\tau_i' - \tau_i|\]  \(\text{total duration of intervals in } \tau\)

\[<\tau> = 4\]
\[||\tau|| = \tau_3' - \tau_0\]
HYBRID TIME SET: CLASSIFICATION

A hybrid time set $\tau = \{ I_i, i=0,1,\ldots, M \}$ is

- Finite: if $<\tau>$ is finite and $I_M = [\tau_M, \tau_M']$
- Infinite: if $<\tau>$ is infinite or $||\tau||$ is infinite
- Zeno: if $<\tau>$ is infinite but $||\tau||$ is finite
A hybrid trajectory is a triple \((\tau, q, x)\) that consists of:

- A hybrid time set \(\tau = \{l_i, i=0,1,..., M\}\)
- Two sequences of functions \(q = \{q_i(\cdot), i=0,1,..., M\}\) and \(x = \{x_i(\cdot), i=0,1,..., M\}\) such that
  
  \[
  q_i: l_i \rightarrow Q \\
  x_i: l_i \rightarrow X
  \]

Remarks:

- mixture of the two notions of continuous and discrete evolution
- the hybrid signals \(x\) and \(q\) can take multiple values at the same time instant
HYBRID AUTOMATA: EXECUTION

A hybrid trajectory \((\tau, q, x)\) is an execution (solution) of the hybrid automaton \(H = (Q,X,f,Init,Dom,E,G,R)\) if it satisfies the following conditions:

• **Initial condition:** \((q_0(\tau_0), x_0(\tau_0)) \in Init\)

• **Continuous evolution:**
  for all \(i\) such that \(\tau_i < \tau_i'\)
  
  \(q_i: I_i \rightarrow Q\) is constant
  
  \(x_i: I_i \rightarrow X\) is the solution to the ODE associated with \(q_i(\tau_i)\)
  
  \(x_i(t) \in \text{Dom}(q_i(\tau_i)), t \in [\tau_i, \tau_i']\)

• **Discrete evolution:**

  \((q_i(\tau_i'),q_{i+1}(\tau_i+1)) \in E\) transition is feasible
  
  \(x_i(\tau_i') \in G(q_i(\tau_i'),q_{i+1}(\tau_i+1))\) guard condition satisfied
  
  \(x_{i+1}(\tau_i+1) \in R((q_i(\tau_i'),q_{i+1}(\tau_i+1)),x_i(\tau_i'))\) reset condition satisfied
EXAMPLE: THERMOSTAT

H = (Q,X,f,Init,Dom,E,G,R)

Q = \{OFF, ON\}; \quad X = \mathbb{R};

f(OFF,x) = -0.2 \cdot x; \quad f(ON, x) = -0.2 \cdot x + 6

Init = \{(OFF,x) : x \geq 18\} \cup \{(ON,x) : x \leq 22\}

Dom(OFF) = [18, \infty); \quad Dom(ON) = (-\infty, 22]

E = \{(OFF,ON),(ON,OFF)\}

G((OFF,ON)) = (-\infty,18]; \quad G((ON,OFF)) = [22,\infty)

R(e,x) = \{x\} \text{ for any } e \in E
EXAMPLE: THERMOSTAT

infinite hybrid time set

evolution of the heater status starting from the initial condition \( x(0) = 5 \) with the heater ON

evolution of the temperature \( x \) starting from the initial condition \( x(0) = 5 \) with the heater ON
EXAMPLE: THERMOSTAT

Infinite hybrid time set

Execution accepted by the thermostat hybrid automaton

Evolution of the heater status starting from the initial condition $x(0) = 5$ with the heater ON

Evolution of the temperature $x$ starting from the initial condition $x(0) = 5$ with the heater ON
HYBRID AUTOMATA EXECUTIONS

What can go wrong?

Problems of the ODE solution (existence, uniqueness, finite escape) avoided by the globally Lipschitz assumption

Problems due the hybrid nature!!!

• Zeno
• chattering
• blocking
• nondeterministic
HYBRID EXECUTION: ZENO

Let \((\tau, q, x)\) be an execution of \(H\).

\((\tau, q, x)\) is called Zeno execution if

\[ \tau = \{l_i, i=0,1,\ldots, M \} \]

is Zeno (infinite number of discrete transitions in finite time)
EXAMPLE: BOUNCING BALL

State of the system given by position $x_1 := y$ and velocity $x_2 := \frac{dy}{dt}$

- $[x_1 > 0]$ or $[x_1 = 0$ and $x_2 \geq 0]$

2 situations:
- a) ball flying in the air
- b) ball hitting the ground

$\begin{align*}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= -g
\end{align*}$

$x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \dot{x} = f(x)$
EXAMPLE: BOUNCING BALL

2 situations:

a) ball flying in the air
b) ball hitting the ground

State of the system given by position $x_1 := y$ and velocity $x_2 := \frac{dy}{dt}$

$x_1 = 0$ and $x_2 < 0$

$x_1^+(t) = x_1^-(t) = 0$

$x_2^+(t) = -cx_2^-(t), c \in [0,1]$
EXAMPLE: BOUNCING BALL

\[ x_1 = 0 \land x_2 < 0 \]

\[ \dot{x}_1 = x_2 \]

\[ \dot{x}_2 = -g \]

\[ x_1 > 0 \lor (x_1 = 0 \land x_2 \geq 0) \]

\[ x_2 := -cx_2 \]

\[ x_1 \geq 0 \]

\( H = (Q,X,f,Init,Dom,E,G,R) \)

\( Q = \{ \text{fly} \}; X = \mathbb{R}^2; \)

\( f(\text{fly},x) = [x_2, -g]^T \)

\( Init = \{(\text{fly},(x_1,x_2)) : x_1 \geq 0\} \)

\( \text{Dom}(\text{fly}) = \{(x_1,x_2) : x_1 > 0\} \cup \{(x_1,x_2) : x_1 = 0, x_2 \geq 0\} \)

\( E = \{(\text{fly},\text{fly})\} \)

\( G(e) = G((\text{fly},\text{fly})) = \{(x_1,x_2) : x_1 = 0, x_2 < 0\} \)

\( R(e,(x_1,x_2)) = \{(x_1, -c x_2)\} \)
EXAMPLE: BOUNCING BALL

\[ x_1 = 0 \land x_2 < 0 \]

\[ x_2 := -cx_2 \quad x_1 \geq 0 \]

\[ \dot{x}_1 = x_2 \]

\[ \dot{x}_2 = -g \]

\[ x_1 > 0 \lor (x_1 = 0 \land x_2 \geq 0) \]

\[ x_1(0) = 0, \quad x_2(0) > 0 \]

\[ c \in (0,1) \]
EXAMPLE: BOUNCING BALL

\[ x_2 := -cx_2 \]

\[ x_1 \geq 0 \]

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -g \]

\[ x_1 > 0 \lor (x_1 = 0 \land x_2 \geq 0) \]

\[ x_1(0) = 0, \quad x_2(0) > 0 \]

Infinite number of transitions in finite time
\[ \rightarrow \text{Zeno hybrid system} \]

(Paradox of Achilles and the turtle by Zeno of Elea born around 490 BC)

\[ c \in (0,1) \]
EXAMPLE: BOUNCING BALL

\[ x_1 = 0 \land x_2 < 0 \]

\[ x_2 := -cx_2 \]

\[ x_1 \geq 0 \]

\[ c \in (0,1) \]

\[ x_1(0) = 0, \; x_2(0) > 0 \]

\[ \tau_0' = 2 \frac{x_2(0)}{g} \]

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -g \\
x_1 > 0 \lor (x_1 = 0 \land x_2 \geq 0)
\end{align*}
\]
EXAMPLE: BOUNCING BALL

\[ x_1 = 0 \land x_2 < 0 \]

\[ x_2 := -cx_2 \]

\[ x_1 \geq 0 \]

\[ \dot{x}_1 = x_2 \quad \dot{x}_2 = -g \]

\[ x_1 > 0 \lor (x_1 = 0 \land x_2 \geq 0) \]

\[ c \in (0,1) \]

\[ x_1(0) = 0, \ x_2(0) > 0 \]

\[ \tau_0' = \frac{2x_2(0)}{g} \]

\[ \tau_\infty = \tau_0' + c \tau_0' + c^2\tau_0' + \ldots = \frac{1}{1-c} \tau_0' = \frac{1}{1-c} \left( \frac{2x_2(0)}{g} \right) \]
HYBRID EXECUTION: CHATTERING

Let \((\tau, q, x)\) be an execution of H. 
\((\tau, q, x)\) is called **chattering execution** if

- \(\tau = \{l_i, i=0,1,\ldots, M\}\) is Zeno (infinite number of discrete transitions in finite time)
- After some \(k \geq 0\), all intervals \(l_i, i \geq k\), are singletons

**Execution:**
\[
\begin{align*}
\tau &= \{[0,1],[1,1],[1,1],\ldots\} \\
q &= \{q_1,q_2,q_1,q_2, \ldots\} \\
x &= \{t-1, 0, 0, 0, \ldots\}
\end{align*}
\]
A hybrid automaton is Zeno if it accepts some Zeno execution

- difficult to simulate and analyse
- regularization methods to eliminate Zeno behavior
TEMPORAL REGULARIZATION: BOUNCING BALL

• each bounce takes $\varepsilon > 0$ time units
• an infinite number of discrete transitions cannot occur in finite time
TEMPORAL REGULARIZATION: BOUNCING BALL

**bounc**

\[
\begin{align*}
\dot{x}_1 &= 0 \\
\dot{x}_2 &= 0 \\
\dot{x}_3 &= 1 \\
x_3 &\leq \epsilon \\
x_3 &\geq \epsilon
\end{align*}
\]

\[x_3 := 0\]

**fly**

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -g \\
\dot{x}_3 &= 0 \\
x_1 &> 0 \lor (x_1 = 0 \land x_2 \geq 0) \\
x_1 &\geq 0
\end{align*}
\]

\[x_2 := -cx_2\]
TEMPORAL REGULARIZATION: BOUNCING BALL

\[ \varepsilon = 0.1 \]
Executions get extended beyond the Zeno time $\tau_{\infty}$

In the limit, as $\varepsilon \to 0$, the execution of the regularized hybrid automaton converges to

- the Zeno execution for $t < \tau_{\infty}$
- $x_1(t) = x_2(t) = 0$ for $t \geq \tau_{\infty}$
HYBRID AUTOMATA: BLOCKING vs. NON-BLOCKING

A hybrid automaton $H = (Q, X, f, \text{Init}, \text{Dom}, E, G, R)$ is non-blocking if for all initial states $(q, x) \in \text{Init}$ there is an infinite execution starting at $(q, x)$.

The execution starting from $(q_1, -1)$ gets stuck at $x = 0$ after 1 time unit.

→ blocking hybrid automaton
A hybrid automaton $H = (Q,X,f,Init,Dom,E,G,R)$ is non-blocking if for all initial states $(q,x) \in Init$ there is an infinite execution starting at $(q,x)$.

Multiple infinite executions starting from $(q_1,-3) \rightarrow$ non blocking
NON-BLOCKING HYBRID AUTOMATA

Given a hybrid automaton $H = (Q, X, f, \text{Init}, \text{Dom}, E, G, R)$, transition states are those states from which continuous evolution is impossible:

$$\text{Trans} := \{(q', x') \in Q \times X : \forall \delta > 0, \exists t \in [0, \delta) \text{ such that } x(t) \notin \text{Dom}(q')\}$$

where $x(t)$ is the solution of $\frac{dx}{dt} = f(q', x)$ with $x(0) = x'$

A hybrid automaton is non-blocking if:

1. $f(q, \cdot)$ is Lipschitz for each $q \in Q$
2. for each $(q, x) \in \text{Trans}$, there exists $q'$ such that $(q, q') \in E$ and $x \in G(q, q')$

(a discrete transition should be possible from the Trans states)

Remark: not a necessary condition, since transition states are not necessarily reached from the initial states
NON-BLOCKING HYBRID AUTOMATA

\{(q_2,x): x \leq 0\} \subseteq \text{Trans}

trivial for x < 0 (already outside \text{Dom}(q_2)),
for x = 0 it would exit the domain

No discrete transition possible from q_2

Still… non blocking!
HYBRID AUTOMATA: DETERMINISTIC vs. NONDETERMINISTIC

An execution is maximal if “it cannot be extended any further”

A hybrid automaton is deterministic if for each initial state \((q,x) \in \text{Init}\) there exists at most one maximal execution starting at \((q,x)\).

Remarks:

1. infinite executions are maximal.
2. to make the notion of maximal execution precise, we should introduce the notion of prefix and a partial order on the hybrid time sets and on the set of executions of a hybrid automaton (see the Notes by John Lygeros)
What causes non-determinism?

- Choice between continuous evolution and discrete jump
- Jump to multiple modes
- Multiple reset positions
A hybrid automaton is deterministic if

1. If $x \in G(q,q')$ for some $(q,q') \in E$, then $(q,x) \in \text{Trans}$
   
   (jump only when continuous evolution not possible)

2. If $(q,q'), (q,q'') \in E$, then $G(q,q') \cap G(q,q'') = \emptyset$
   
   (no multiple jumps possible)

3. If $(q,q') \in E$ and $x \in G(q,q')$, then $R((q,q'),x)$ is a singleton
   
   (only one reset position when jumping)
EXISTENCE AND UNIQUENESS OF EXECUTIONS

A hybrid automaton has a unique infinite execution for each initial state if it is non-blocking and deterministic.

Sufficient condition given by the putting together the sufficient conditions for the hybrid automaton to be non-blocking and deterministic.
EXAMPLE: SWITCHED LINEAR SYSTEM

\[ H = (Q, X, f, Init, \text{Dom}, E, G, R) \]

- \( Q = \{q_1, q_2\} \quad X = \mathbb{R}^2 \)
- \( f(q_1, x) = A_1 x \) and \( f(q_2, x) = A_2 x \) with:
  \[ A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix} \]
- \( \text{Init} = Q \times \{x \in X : ||x|| > 0\} \)
- \( \text{Dom}(q_1) = \{x \in X : x_1 x_2 \leq 0\} \quad \text{Dom}(q_2) = \{x \in X : x_1 x_2 \geq 0\} \)
- \( E = \{(q_1, q_2), (q_2, q_1)\} \)
- \( G((q_1, q_2)) = \{x \in X : x_1 x_2 > 0\} \quad G((q_2, q_1)) = \{x \in X : x_1 x_2 < 0\} \)
- \( R((q_1, q_2), x) = R((q_2, q_1), x) = \{x\} \)
EXAMPLE: SWITCHED LINEAR SYSTEM

\[ \dot{x} = A_1 x \]
\[ \dot{x} = A_2 x \]

\[ \dot{x} = A_2 x \]
\[ \dot{x} = A_1 x \]
EXAMPLE: SWITCHED LINEAR SYSTEM

\[ \dot{x} = A_1 x \]

\[ \dot{x} = A_2 x \]
EXAMPLE: SWITCHED LINEAR SYSTEM

$q_1$: quadrants 2 and 4
$q_2$: quadrants 1 and 3
EXAMPLE: SWITCHED LINEAR SYSTEM

\[ H = (Q,X,f,\text{\textit{Init}},\text{Dom},E,G,R) \]

- \( Q = \{q_1, q_2\} \)  \( X = \mathbb{R}^2 \)
- \( f(q_1,x) = A_1x \) and \( f(q_2,x) = A_2x \) with:
  \[ A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix} \]
- \( \text{\textit{Init}} = Q \times \{x \in X: ||x|| > 0\} \)
- \( \text{Dom}(q_1) = \{x \in X: x_1x_2 \leq 0\} \) \( \text{Dom}(q_2) = \{x \in X: x_1x_2 \geq 0\} \)
- \( E = \{(q_1,q_2),(q_2,q_1)\} \)
- \( G((q_1,q_2)) = \{x \in X: x_1x_2 > 0\} \) \( G((q_2,q_1)) = \{x \in X: x_1x_2 < 0\} \)
- \( R((q_1,q_2),x) = R((q_2,q_1),x) = \{x\} \)

Is it deterministic?
A hybrid automaton is deterministic if

1. If \( x \in G(q,q') \) for some \( (q,q') \in E \), then \( (q,x) \in \text{Trans} \)
   (jump only when continuous evolution not possible)

2. If \( (q,q'), (q,q'') \in E \), then \( G(q,q') \cap G(q,q'') = \emptyset \)
   (no multiple jumps possible)

3. If \( (q,q') \in E \) and \( x \in G(q,q') \), then \( R((q,q'),x) \) is a singleton
   (only one reset position when jumping)
HYBRID AUTOMATA: FORMAL DEFINITION

A hybrid automaton \( H \) is a collection

\[
H = (Q, X, f, Init, \text{Dom}, E, G, R)
\]

- \( Q = \{q_1, q_2, \ldots\} \) is a set of discrete states (modes)
- \( X = \mathbb{R}^n \) is the continuous state space
- \( f: Q \times X \rightarrow \mathbb{R}^n \) is a set of vector fields on \( X \)
- \( Init \subseteq Q \times X \) is a set of initial states
- \( \text{Dom}: Q \rightarrow 2^X \) assigns to each \( q \in Q \) a domain \( \text{Dom}(q) \) of \( X \)
- \( E \subseteq Q \times Q \) is a set of transitions (edges)
- \( G: E \rightarrow 2^X \) is a set of guards (guard condition)
- \( R: E \times X \rightarrow 2^X \) is a set of reset maps

No input and output variables…. 
OPEN HYBRID AUTOMATA

An open hybrid automaton $H$ is a collection

$$H = (Q,X,V,W,f,h,Init,\text{Dom},E,G,R)$$

- $Q = \{q_1,q_2, \ldots\}$ is a set of discrete states (modes)
- $X = \mathbb{R}^n$ is the continuous state space
- $V = \Sigma \times U$ is the input space (discrete & continuous)
- $W = \Psi \times Y$ is the output space (discrete & continuous)
- $f: Q \times X \times V \rightarrow \mathbb{R}^n$ is a set of vector fields on $X$
- $h: Q \times X \rightarrow W$ is an output map
- $Init \subseteq Q \times X$ is a set of initial states
- $\text{Dom}: Q \rightarrow 2^{X \times V}$ assigns to each $q \in Q$ a domain $\text{Dom}(q)$ of $X \times V$
- $E \subseteq Q \times Q$ is a set of transitions (edges)
- $G: E \rightarrow 2^{X \times V}$ is a set of guards (guard condition)
- $R: E \times X \times V \rightarrow 2^X$ is a set of reset maps