Switched systems and stability

Nonlinear Control

2019/20
SWITCHED SYSTEMS

• a family of systems

\[ \dot{x} = f_q(x), \; q \in Q = \{1, 2, \ldots, m\} \]
SWITCHED SYSTEMS

• a family of systems

\[ \dot{x} = f_q(x), \quad q \in Q = \{1, 2, \ldots, m\} \]

• a signal that orchestrates the switching between them
SWITCHED SYSTEMS AS HYBRID SYSTEMS

\[ \dot{x} = f_\sigma(x), \quad \sigma \in Q = \{1, 2\} \]

- discrete transition mechanism (domains and guards) depends on the switching signal
  - time-dependent switching
  - state-dependent switching
TIME-DEPENDENT SWITCHING

\[ \sigma : [0, \infty) \rightarrow Q \] (exogenous) switching signal

- piecewise constant function of time
- \( \sigma(t) \) specifies the system that is active at time \( t \)
TIME-DEPENDENT SWITCHING

\[ \dot{x} = f_\sigma(x), \quad \sigma \in Q = \{1, 2\} \quad \sigma : [0, \infty) \to Q \]

If the switching signal is arbitrary, then,
TIME-DEPENDENT SWITCHING

\[ \dot{x} = f_\sigma(x), \quad \sigma \in Q = \{1, 2\} \quad \sigma : [0, \infty) \to Q \]

If the switching signal is arbitrary, then,

\[ H = (Q, X, f, Init, Dom, E, G, R) \]

\[ Q = \{q_1, q_2\} \quad X = \mathbb{R}^n \]
\[ Init = Q \times X \]
\[ E = \{(q_1, q_2), (q_2, q_1)\} \]
\[ f(q_1, x) = f_1(x) \text{ and } f(q_2, x) = f_2(x) \]
\[ Dom(q_1) = Dom(q_2) = X \]
\[ G((q_1, q_2)) = G((q_2, q_1)) = X \]
\[ R((q_1, q_2), x) = R((q_2, q_1), x) = \{x\} \]
SWITCHED SYSTEMS vs. HYBRID AUTOMATA

• switched systems with time-dependent arbitrary switching can be seen as a higher-level abstraction of hybrid automata where the discrete transition mechanism is not specified

• simpler to describe but with more solutions than the original hybrid automata → conservative analysis results
the state space $X$ is partitioned into operating regions, each one associated to a system

- $\sigma(x)$ specifies the system that is active when the state is $x$
STATE-DEPENDENT SWITCHING

$\dot{x} = f_2(x)$

$\dot{x} = f_1(x)$

$\dot{x} = f_3(x)$

$\dot{x} = f_4(x)$

$X = \mathbb{R}^2$
EXAMPLE: SWITCHED LINEAR SYSTEM

\[ \dot{x} = A_1 x \]

\[ \dot{x} = A_2 x \]

\[ Cx = 0 \]
EXAMPLE: SWITCHED LINEAR SYSTEM

\[ Cx \geq 0 \]

\[ \dot{x} = A_1 x \]

\[ Cx \leq 0 \]

\[ \dot{x} = A_2 x \]

\[ Cx \geq 0 \]
EXAMPLE: SWITCHED LINEAR SYSTEM

\[ H = (Q, X, f, \text{Init}, \text{Dom}, E, G, R) \]

- \( Q = \{q_1, q_2\} \quad X = \mathbb{R}^2 \)
- \( f(q_1, x) = A_1 x \) and \( f(q_2, x) = A_2 x \)
- \( \text{Init} = Q \times X \)
- \( \text{Dom}(q_1) = \{x \in X : Cx \geq 0\} \quad \text{Dom}(q_2) = \{x \in X : Cx \leq 0\} \)
- \( E = \{(q_1, q_2), (q_2, q_1)\} \)
- \( G((q_1, q_2)) = \{x \in X : Cx \leq 0\} \quad G((q_2, q_1)) = \{x \in X : Cx \geq 0\}, \quad C^T \in \mathbb{R}^2 \)
- \( R((q_1, q_2), x) = R((q_2, q_1), x) = \{x\} \)
AUTONOMOUS vs CONTROLLED SWITCHING

• Autonomous switching
  – switching events are triggered by an external mechanism over which we do not have control

Examples:
unpredictable environmental factors
component failures
AUTONOMOUS vs CONTROLLED SWITCHING

• Autonomous switching
  – switching events are triggered by an external mechanism over which we do not have control

• Controlled switching
  – switching are imposed so as to achieve a desired behavior of the resulting system → switching control
SWITCHING CONTROL

The inner closed-loop system is a switched system.
SWITCHING CONTROL

Reasons for switching:

- nature of the control problem (system with different operation phases)

  Example: flight control system
SWITCHING CONTROL

Reasons for switching:

• large modeling uncertainty

Example: adaptive switching control

\[ \mathcal{P} = \text{admissible model set} \]

controller cover
SWITCHING CONTROL

Reasons for switching:

- sensor/actuator limitations

Example: quantized control
SWITCHED SYSTEMS: EQUILIBRIUM

\[
\dot{x} = f_\sigma(x)
\]

family of systems

\[
\dot{x} = f_q(x), \; q \in Q = \{1, 2, \ldots, m\}
\]

with \( f_q(0) = 0, \forall q \in Q \)
SWITCHED SYSTEMS: EQUILIBRIUM

\[ \dot{x} = f_\sigma(x) \]

family of systems

\[ \dot{x} = f_q(x), \quad q \in Q = \{1, 2, \ldots, m\} \]

with \( f_q(0) = 0, \forall q \in Q \)

\[ \rightarrow \quad x = 0 \text{ is an equilibrium of the switched system} \]

Stability of the equilibrium \( x=0? \)
SWITCHING BETWEEN AS. STABLE LINEAR SYSTEMS

\[ \dot{x} = A_\sigma x \]

\[ \dot{x} = A_1 x \]

\[ \dot{x} = A_2 x \]
SWITCHING BETWEEN AS. STABLE LINEAR SYSTEMS

\[ \dot{x} = A_\sigma x \]

\[ \dot{x} = A_1 x \]
\[ \dot{x} = A_2 x \]

\( x = 0 \) is unstable!
Problem
find conditions that guarantee asymptotic stability under arbitrary switching
SWITCHING BETWEEN AS. STABLE LINEAR SYSTEMS

\[ \dot{x} = A_\sigma x \]

\[ \dot{x} = A_1 x \]
\[ \dot{x} = A_2 x \]

\[ x = 0 \] is as. stable!
SWITCHING BETWEEN AS. STABLE LINEAR SYSTEMS

\[ \dot{x} = A_\sigma x \]

\[ \dot{x} = A_1 x \quad \text{or} \quad \dot{x} = A_2 x \]

\[ x = 0 \text{ is as. stable!} \]

Problem
identify those switching signals that preserve asymptotic stability
SWITCHING BETWEEN UNSTABLE LINEAR SYSTEMS

\[ \dot{x} = A_\sigma x \]

\[ \dot{x} = A_1 x \]

\[ \dot{x} = A_2 x \]
SWITCHING BETWEEN UNSTABLE LINEAR SYSTEMS

\[ \dot{x} = A_{\sigma}x \]

\[ \dot{x} = A_1x \]
\[ \dot{x} = A_2x \]

x = 0 is as. stable!
SWITCHING BETWEEN UNSTABLE LINEAR SYSTEMS

\[ \dot{x} = A_\sigma x \]

\[ \dot{x} = A_1 x \]
\[ \dot{x} = A_2 x \]

\[ x = 0 \text{ is as. stable!} \]

Problem
identify those switching signals that ensure asymptotic stability
Stability for arbitrary switching

Stability for constrained switching
Stability for arbitrary switching

Stability for constrained switching
GLOBAL UNIFORM ASYMPTOTIC STABILITY

\[
\dot{x} = f_\sigma(x)
\]

\[f_q(0) = 0, \ q \in Q = \{1, 2, \ldots, m\}\]

Definition [globally uniformly asymptotically stable equilibrium]
The equilibrium \( x=0 \) is GUAS if it is GAS, uniformly with respect to the switching signals \( \sigma \)
GLOBAL UNIFORM ASYMPTOTIC STABILITY

\[ \dot{x} = f_\sigma(x) \]

\[ f_q(0) = 0, \quad q \in Q = \{1, 2, \ldots, m\} \]

Definition [globally uniformly asymptotically stable equilibrium]
The equilibrium \( x=0 \) is GUAS if it is GAS, uniformly with respect to the switching signals \( \sigma \)

Necessary condition for \( x=0 \) to be GUAS
\[ \dot{x} = f_q(x), \quad q \in Q = \{1, 2, \ldots, m\} \]

family of systems with GAS equilibrium in \( x=0 \)
COMMON LYAPUNOV FUNCTION

Definition [Common Lyapunov function]

The family of systems
\[ \dot{x} = f_q(x), \; x \in \mathbb{R}^n, \; q \in Q = \{1, 2, \ldots, m\} \]

share a common Lyapunov function at \( x=0 \) if there exists a continuously differentiable \((C^1)\) function \( V: \mathbb{R}^n \rightarrow \mathbb{R} \) such that

\[
V(x) > 0, \; \forall x \neq 0 \; V(0) = 0 \\
\frac{dV}{dx}(x) f_q(x) < 0, \; \forall x \neq 0, \; \forall q \in Q
\]
COMMON LYAPUNOV FUNCTION

Definition [Common Lyapunov function]

The family of systems
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share a common Lyapunov function at \( x=0 \) if there exists a continuously differentiable (C\(^1\)) function \( V: \mathbb{R}^n \rightarrow \mathbb{R} \) such that

\[ V(x) > 0, \quad \forall x \neq 0 \quad V(0) = 0 \]
\[ \frac{dV}{dx}(x)f_q(x) < 0, \quad \forall x \neq 0, \quad \forall q \in Q \]

A common Lyapunov function at \( x=0 \) is quadratic if it is given by

\[ V(x) = x^T P x \]

with \( P \) symmetric and positive definite
COMMON LYAPUNOV FUNCTION

\[
\dot{x} = f_\sigma(x)
\]

**Theorem**

If the family of systems

\[
\dot{x} = f_q(x), \ x \in \mathbb{R}^n, \ q \in Q = \{1, 2, \ldots, m\}
\]

share a radially unbounded common Lyapunov function V: \(\mathbb{R}^n \to \mathbb{R}\) at \(x = 0\), then, the equilibrium \(x = 0\) is GUAS.
COMMON LYAPUNOV FUNCTION

\[ \dot{V}(x) = \frac{dV(x(t))}{dt} < 0 \]

\[ V(x(t)) \]

\[ \sigma = 1 \quad \sigma = 2 \quad \sigma = 1 \quad \sigma = 2 \]
COMMON QUADRATIC LYAPUNOV FUNCTION

\[ \dot{x} = A_\sigma x \]

Theorem
If there exists \( P = P^T > 0 \) such that
\[ PA_q + A_q^T P < 0, \forall q \in Q = \{1, 2, \ldots, m\} \]
then, the equilibrium \( x = 0 \) is GUAS.

*Proof.* \( V(x) = x^T P x \) is a radially unbounded common Lyapunov function at \( x = 0 \).

Remark:
A set of LMIs to solve. This problem can be reformulated as a convex optimization problem. Efficient solvers exist.
GLOBALLY QUADRATIC LYAPUNOV FUNCTION

\[
\dot{x} = A_\sigma x
\]

The existence of a globally quadratic Lyapunov function is not necessary for \( x = 0 \) to be GUAS.

Example:

\[
A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -10 \\ 0.1 & -1 \end{bmatrix}
\]

\( x = 0 \) is GUAS but there is no common quadratic Lyapunov function.
SWITCHED LINEAR SYSTEMS WITH A SPECIAL STRUCTURE

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, \ q \in Q = \{1, 2, \ldots, m\} \)

- commute
- are upper (or lower) triangular
COMMUTING HURWITZ MATRICES $\rightarrow$ GUAS

$$\dot{x} = A_\sigma x$$

$$Q = \{1, 2\} \quad A_1 A_2 = A_2 A_1$$

$$\begin{array}{cccccc}
\sigma = 1 & \sigma = 2 & \sigma = 1 & \sigma = 2 & \cdots & \\
 s_1 & t_1 & s_2 & t_2 & \\
\end{array} \quad t$$

$$x(t) = e^{A_2 t_k} e^{A_1 s_k} \cdots e^{A_2 t_1} e^{A_1 s_1} x(0)$$

$$= e^{A_2 (t_k + \cdots + t_1)} e^{A_1 (s_k + \cdots + s_1)} x(0) \rightarrow 0$$
COMMUTING HURWITZ MATRICES $\rightarrow$ GUAS

$$\dot{x} = A_\sigma x$$

$$Q = \{1, 2\} \quad A_1 A_2 = A_2 A_1$$

$\exists$ quadratic common Lyapunov function: $V(x) = x^T P_2 x$

$$P_1 A_1 + A_1^T P_1 = -I$$

$$P_2 A_2 + A_2^T P_2 = -P_1$$
COMMUTING HURWITZ MATRICES → GUAS

\[ \dot{x} = A_\sigma x \]

\[ Q = \{1, 2\} \quad A_1 A_2 = A_2 A_1 \]

\( \exists \) quadratic common Lyapunov function: \( V(x) = x^T P_2 x \)

\[ P_1 A_1 + A_1^T P_1 = -I \]

\[ \boxed{P_2 A_2 + A_2^T P_2 = -P_1} \]

\[ P_2 = \int_0^\infty e^{A_2^T t} P_1 e^{A_2 t} dt \quad P_1 = \int_0^\infty e^{A_1^T t} e^{A_1 t} dt \]

\[ P_2 = \int_0^\infty e^{A_1^T \tau} \left[ \int_0^\infty e^{A_2^T t} e^{A_2 t} dt \right] e^{A_1 \tau} d\tau \quad P_2 A_1 + A_1^T P_2 = -Q \]
TRIANGULAR HURWITZ MATRICES $\rightarrow$ GUAS

\[ \dot{x} = A_\sigma x \]

\[ Q = \{1, 2\} \quad X = \mathbb{R}^2 \]

\[ \dot{x}_1 = \lambda_{1, \sigma} x_1 + a_\sigma x_2 \]

\[ \dot{x}_2 = \lambda_{2, \sigma} x_2 \]
TRIANGULAR HURWITZ MATRICES → GUAS

\[ \dot{x} = A_\sigma x \]

\[ Q = \{1, 2\} \quad X = \mathbb{R}^2 \]

\[ \dot{x}_1 = \lambda_{1, \sigma} x_1 + a_\sigma x_2 \]

\[ \dot{x}_2 = \lambda_{2, \sigma} x_2 \]

\[ \dot{x}_2 = \lambda_{2, \sigma} x_2 \Rightarrow |x_2(t)| \leq e^{\max_p \lambda_{2, p} t} |x_2(0)| \to 0 \]
TRIANGULAR HURWITZ MATRICES → GUAS

\[ \dot{x} = A_\sigma x \]

\[ Q = \{1, 2\} \quad X = \mathbb{R}^2 \]

\[ \begin{align*}
\dot{x}_1 &= \lambda_{1,\sigma} x_1 + a_\sigma x_2 \\
\dot{x}_2 &= \lambda_{2,\sigma} x_2
\end{align*} \]

\[ \dot{x}_2 = \lambda_{2,\sigma} x_2 \Rightarrow |x_2(t)| \leq e^{\max_p \lambda_{2,p} t} |x_2(0)| \rightarrow 0 \]

\[ \dot{x}_1 = \lambda_{1,\sigma} x_1 + a_\sigma x_2 \Rightarrow x_1(t) \rightarrow 0 \]

exponentially stable system \hspace{1cm} \text{exponentially decaying perturbation}
TRIANGULAR HURWITZ MATRICES => GUAS

\[ \dot{x} = A_\sigma x \]

\[ Q = \{1, 2\} \quad X = \mathbb{R}^2 \]

\[ \dot{x}_1 = \lambda_{1,\sigma} x_1 + b_\sigma x_2 \]
\[ \dot{x}_2 = \lambda_{2,\sigma} x_2 \]

\[ \exists \text{ quadratic common Lyapunov function} \]
\[ V(x) = x^T P x \]

with P diagonal
SWITCHED SYSTEMS WITH A SPECIAL STRUCTURE

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, \ q \in Q = \{1, 2, \ldots, m\} \)

• commute

• are upper (or lower) triangular

• can be transformed to upper (or lower) triangular form by a common similarity transformation
• Stability for arbitrary switching

• Stability for constrained switching
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_{\sigma} x \]

Hurwitz matrices \( A_q, \ q \in Q = \{1, 2\} \)

\[
\begin{array}{cccccc}
\sigma = 1 & \sigma = 2 & \sigma = 1 & \sigma = 2 & \cdots \\
s_1 & t_1 & s_2 & t_2 & \\
\end{array}
\]

The switching intervals satisfy \( t_i, s_i \geq \tau_D \) dwell time
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_{\sigma} x \]

Hurwitz matrices \( A_q, \ q \in Q = \{1, 2\} \)

\[
\begin{align*}
\sigma = 1 & \quad \sigma = 2 & \quad \sigma = 1 & \quad \sigma = 2 & \quad \ldots \\
 s_1 & \quad t_1 & \quad s_2 & \quad t_2 & \quad \\
\end{align*}
\]

The switching intervals satisfy \( t_i, s_i \geq \tau_D \)

\[
x(t) = e^{A_2 t_k} e^{A_1 s_k} \ldots e^{A_2 t_1} e^{A_1 s_1} x(0)
\]
STABILITY OF LINEAR CONTINUOUS SYSTEMS

\[ \dot{x}(t) = Ax(t) \]

Theorem (exponential stability):

Let the equilibrium \( x=0 \) be asymptotically stable. Then, the rate of convergence to \( x=0 \) is exponential:

\[ \|x(t)\| \leq \mu e^{-\lambda_0 t} \|x_0\|, \quad t \geq 0 \]

for all \( x(0) = x_0 \in \mathbb{R}^n \), where \( \lambda_0 \in (0, \min_i |\text{Re}\{\lambda_i(A)\}|) \) and \( \mu > 0 \) is an appropriate constant.

Remark:

\[ \|x(t)\| = \|e^{At}x_0\| \leq \mu e^{-\lambda_0 t} \|x_0\|, \quad t \geq 0, \forall x_0 \]

\[ \rightarrow \|e^{At}\| = \sup_{x_0 \neq 0} \frac{\|e^{At}x_0\|}{\|x_0\|} \leq \mu e^{-\lambda_0 t}, \quad t \geq 0 \]
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, \ q \in Q = \{1, 2\} \)

\[
\begin{array}{c|c|c|c|c}
\sigma = 1 & \sigma = 2 & \sigma = 1 & \sigma = 2 & \cdots \\
\hline
s_1 & t_1 & s_2 & t_2 & \cdots \\
\end{array}
\]

The switching intervals satisfy \( t_i, s_i \geq \tau_D \)

\[
x(t) = e^{A_2 t_k} e^{A_1 s_k} \cdots e^{A_2 t_1} e^{A_1 s_1} x(0)
\]

\[
\| e^{A_i \Delta t} \| \leq \mu \ e^{-\lambda_0 \Delta t} \leq \mu e^{-\lambda_0 \tau_D} = e^{-\lambda_0 \tau_D + \log \mu}
\]

slowest decay rate so that the inequality holds \( \forall i \)
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, q \in Q = \{1, 2\} \)

\[
\begin{align*}
\sigma = 1 & \quad \sigma = 2 & \quad \sigma = 1 & \quad \sigma = 2 & \quad \ldots \\
s_1 & \quad t_1 & \quad s_2 & \quad t_2 & \quad \\
\end{align*}
\]

The switching intervals satisfy \( t_i, s_i \geq \tau_D \)

\[
x(t) = e^{A_2 t_k} e^{A_1 s_k} \ldots e^{A_2 t_1} e^{A_1 s_1} x(0)
\]

\[
\| e^{A_i \Delta t} \| \leq e^{-\lambda_0 \tau_D + \log \mu}
\]
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_{\sigma} x \]

Hurwitz matrices $A_q, q \in Q = \{1, 2\}$

\[ \sigma = 1 \quad \sigma = 2 \quad \sigma = 1 \quad \sigma = 2 \quad \ldots \]

\[ s_1 \quad t_1 \quad s_2 \quad t_2 \quad \ldots \quad t \]

The switching intervals satisfy $t_i, s_i \geq \tau_D$

\[ x(t) = e^{A_2 t_k} e^{A_1 s_k} \ldots e^{A_2 t_1} e^{A_1 s_1} x(0) \]

\[ \| e^{A_i \Delta t} \| \leq e^{-\lambda_0 \tau_D + \log \mu} \leq e^{-\lambda \tau_D} \]

if there exists $\lambda \in (0, \lambda_0)$ such that $\tau_D \geq \frac{\log \mu}{\lambda_0 - \lambda}$
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, \ q \in Q = \{1, 2\} \)

\[
\begin{align*}
\sigma = 1 & \quad \sigma = 2 & \quad \sigma = 1 & \quad \sigma = 2 & \quad \ldots \\
\quad s_1 & \quad t_1 & \quad s_2 & \quad t_2 & \quad \ldots \\
\end{align*}
\]

The switching intervals satisfy \( t_i, s_i \geq \tau_D \)

\[
x(t) = e^{A_2 t_k} e^{A_1 s_k} \ldots e^{A_2 t_1} e^{A_1 s_1} x(0)
\]

\[
\| e^{A_i \Delta t} \| \leq e^{-\lambda_0 \tau_D + \log \mu} \leq e^{-\lambda \tau_D} < 1
\]

if there exists \( \lambda \in (0, \lambda_0) \) such that \( \tau_D \geq \frac{\log \mu}{\lambda_0 - \lambda} \)
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_{\sigma}x \]

Hurwitz matrices \( A_q, \ q \in Q = \{1, 2\} \)

\[ x(t) = e^{A_2 t_k} e^{A_1 s_k} \ldots e^{A_2 t_1} e^{A_1 s_1} x(0) \]

\[ \|e^{A_i \Delta t}\| \leq \mu e^{-\lambda_0 \Delta t} \]
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, \ q \in Q = \{1, 2\} \)

\[
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\hline
s_1 & t_1 & s_2 & t_2 & \\
\end{array}
\]

\[
x(t) = e^{A_2 t_k} e^{A_1 s_k} \ldots e^{A_2 t_1} e^{A_1 s_1} x(0)
\]

\[
\| e^{A_i \Delta t} \| \leq \mu e^{-\lambda_0 \Delta t} \leq e^{\tau_D (\lambda_0 - \lambda)} e^{-\lambda_0 \Delta t}
\]

\[
\tau_D \geq \frac{\log \mu}{\lambda_0 - \lambda} \quad \rightarrow \quad \mu \leq e^{\tau_D (\lambda_0 - \lambda)}
\]
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, \ q \in Q = \{1, 2\} \)

\[ \sigma = 1 \quad \sigma = 2 \quad \sigma = 1 \quad \sigma = 2 \quad \ldots \]

\[ s_1 \quad t_1 \quad s_2 \quad t_2 \quad \ldots \quad t \]

\[ x(t) = e^{A_2 t_k} e^{A_1 s_k} \ldots e^{A_2 t_1} e^{A_1 s_1} x(0) \]

\[ \| e^{A_i \Delta t} \| \leq \mu e^{-\lambda_0 \Delta t} \leq e^{\tau_D (\lambda_0 - \lambda)} e^{-\lambda_0 \Delta t} \leq e^{\Delta t (\lambda_0 - \lambda)} e^{-\lambda_0 \Delta t} = e^{-\lambda \Delta t} \]

\[ \lambda_0 > \lambda \]

\[ \Delta t \geq \tau_D \]
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_{\sigma} x \]

Hurwitz matrices \( A_q, q \in Q = \{1, 2\} \)

<table>
<thead>
<tr>
<th>( \sigma = 1 )</th>
<th>( \sigma = 2 )</th>
<th>( \sigma = 1 )</th>
<th>( \sigma = 2 )</th>
<th>( \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>( t_1 )</td>
<td>( s_2 )</td>
<td>( t_2 )</td>
<td>( t )</td>
</tr>
</tbody>
</table>

\[
x(t) = e^{A_2 t_k} e^{A_1 s_k} \ldots e^{A_2 t_1} e^{A_1 s_1} x(0)
\]

\[ \| e^{A_i \Delta t} \| \leq e^{-\lambda \Delta t} \quad \rightarrow \quad \| x(t) \| \leq e^{-\lambda t} \| x(0) \| \]
STABILITY UNDER STATE-DEPENDENT SWITCHING

\[ \sigma: X \to Q : \quad \sigma(x) = i \text{ if } x \in X_i \]
COMMON LYAPUNOV FUNCTION

Definition [Common Lyapunov function]

The family of systems

\[ \dot{x} = f_q(x), \ x \in \mathbb{R}^n, \ q \in Q = \{1, 2, \ldots, m\} \]

share a common Lyapunov function at \( x=0 \) if there exists a continuously differentiable \((C^1)\) function \( V: \mathbb{R}^n \to \mathbb{R} \) such that

\[ V(x) > 0, \ \forall x \neq 0 \ V(0) = 0 \]

\[ \frac{dV}{dx}(x) f_q(x) < 0, \ \forall x \neq 0, \ \forall q \in Q \]
STATE-DEPENDENT COMMON LYAPUNOV FUNCTIONS

Definition [state-dependent common Lyapunov function]

The family of systems
\[ \dot{x} = f_q(x), \quad x \in \mathbb{R}^n, \quad q \in Q = \{1, 2, \ldots, m\} \]

has a state-dependent common Lyapunov function at \( x=0 \) if there exists a \( C^1 \) function \( V: \mathbb{R}^n \to \mathbb{R} \) is such that

\[ V(x) > 0, \forall x \neq 0, V(0) = 0 \]

\[ \frac{\partial V}{\partial x}(x)f_{\sigma(x)}(x) < 0, \forall x \in \mathbb{R}^n \]

where \( \sigma: X \to Q \)
COMMON LYAPUNOV FUNCTION

\[ \dot{x} = f_\sigma(x), \quad \sigma : X \to Q \]

Theorem
If there exists a radially unbounded state-dependent common Lyapunov function at \( x=0 \) for the switched system, then, the equilibrium \( x = 0 \) is GAS.

Remarks:
need that \( \frac{\partial V}{\partial x}(x)f_q(x) < 0 \) only when \( \sigma \) is equal to \( q \)

in the switched linear case matrices \( A_q \) are not required to be Hurwitz
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{both unstable} \]

Assume: \( A = \alpha A_1 + (1 - \alpha) A_2 \) Hurwitz for some \( \alpha \in (0,1) \)
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad - \text{both unstable} \]

Assume: \( A = \alpha A_1 + (1 - \alpha) A_2 \) Hurwitz for some \( \alpha \in (0,1) \)

\[ A^T P + PA < 0 \]
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{both unstable} \]

Assume: \( A = \alpha A_1 + (1 - \alpha) A_2 \) Hurwitz for some \( \alpha \in (0,1) \)

\[ \alpha (A_1^T P + PA_1) + (1 - \alpha)(A_2^T P + PA_2) < 0 \]
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x , \quad \dot{x} = A_2 x \quad - \quad \text{both unstable} \]

Assume: \( A = \alpha A_1 + (1-\alpha)A_2 \) Hurwitz for some \( \alpha \in (0,1) \)

\[ \alpha (A_1^T P + PA_1) + (1-\alpha)(A_2^T P + PA_2) < 0 \]

So for each \( x \neq 0 \):

either \( x^T (A_1^T P + PA_1) x < 0 \) or \( x^T (A_2^T P + PA_2) x < 0 \)
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{both unstable} \]

Assume: \( A = \alpha A_1 + (1 - \alpha) A_2 \) Hurwitz for some \( \alpha \in (0,1) \)

\[ \alpha (A_1^T P + PA_1) + (1 - \alpha) (A_2^T P + PA_2) < 0 \]

So for each \( x \neq 0 \):

either \( x^T (A_1^T P + PA_1) x < 0 \) or \( x^T (A_2^T P + PA_2) x < 0 \)

- define region \( X_1 \) where system 1 is active
- define region \( X_2 \) where system 2 is active
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{both unstable} \]

Assume: \( A = \alpha A_1 + (1 - \alpha) A_2 \) Hurwitz for some \( \alpha \in (0,1) \)

\[ \alpha (A_1^T P + PA_1) + (1 - \alpha) (A_2^T P + PA_2) < 0 \]

So for each \( x \neq 0 \):

either \( x^T (A_1^T P + PA_1) x < 0 \) or \( x^T (A_2^T P + PA_2) x < 0 \)

Define region \( X_1 \) where system 1 is active

Define region \( X_2 \) where system 2 is active

\[ V(x) = x^T P x \] is a radially unbounded state-dependent common Lyapunov function at \( x = 0 \) for \( \dot{x} = A_{\sigma(x)} x \)

\[ x = 0 \] is GAS
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{both unstable} \]

Assume: \( A = \alpha A_1 + (1 - \alpha) A_2 \) Hurwitz for some \( \alpha \in (0,1) \)

\[ \alpha (A_1^T P + PA_1) + (1 - \alpha)(A_2^T P + PA_2) < 0 \]

So for each \( x \neq 0 \):

either \( x^T (A_1^T P + PA_1) x < 0 \) or \( x^T (A_2^T P + PA_2) x < 0 \)
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{both unstable} \]

Theorem
If the matrices \( A_1 \) and \( A_2 \) have a Hurwitz combination, then, there exists a state-dependent switching strategy such that the equilibrium \( x = 0 \) of the switching system \( \dot{x} = A_{\sigma(x)} x \) is GAS.

Extension to the case of more than 2 unstable matrices in the case when two or more matrices have a Hurwitz combination
Main source:

*Switching in Systems and Control*
Daniel Liberzon, Birkhauser, 2003

Available through SpringerLink