NUMERICAL EXAMPLES ON PASSIVITY ANALYSIS
Given the linear dynamical system
\[\begin{align*}
\dot{x} &= Ax + Bu, \quad x(0) = x_0 \\
y &=Cx
\end{align*}\]

Prove that if there exist P and Q symmetric positive definite such that
\[
A'P + PA = -Q \\
B'P = C
\]

then
1. the system is passive with storage function \(V(x) = \frac{1}{2}x'Px\)
2. the equilibrium \(x=0\) for the system with zero input is globally asymptotically stable
PASSIVE DYNAMICAL SYSTEM

\[ S : \begin{cases} \dot{x}(t) = f(x(t), u(t)), & x(0) = x_0 \in \mathbb{R}^n \\ y(t) = g(x(t), u(t)) \end{cases} \]

\[ f(0, 0) = 0, \quad g(0, 0) = 0 \]

Definition (passive dynamical system)

System S is passive if there exists a function \( V(\cdot) : \mathbb{R}^n \to \mathbb{R} \) semipositive definite and continuously differentiable, called \textit{storage function}, such that:

\[ uy \geq \frac{\partial V}{\partial x}(x)f(x, u) + \epsilon u^2 + \delta y^2 + \rho \psi(x), \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R} \]

with \( \epsilon, \delta, \rho \in \mathbb{R}^+ \) and \( \psi(\cdot) \) positive definite.
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with \( \epsilon, \delta, \rho \in \mathbb{R}^+ \) and \( \psi(\cdot) \) positive definite.

In particular, system S is

- strictly passive w.r.t. the input if \( \epsilon > 0 \)
- strictly passive w.r.t. the output if \( \delta > 0 \)
- strictly passive w.r.t. the state if \( \rho > 0 \)
SOLUTION

Given the linear dynamical system

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then the system is passive with storage function \( V(x) = \frac{1}{2}x'Px \)

Solution: \( V(x) \) is positive definite and continuously differentiable. Let us compute its derivative along the system trajectories
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Solution: \( V(x) \) is positive definite and continuously differentiable. Let us compute its derivative along the system trajectories

\[
\dot{V}(x) = \frac{1}{2}(x'Px + x'Px) = \frac{1}{2}x'(A'P + PA)x + \frac{1}{2}(uB'Px + x'PBu)
\]

\[
= \frac{1}{2}x'(A'P + PA)x + uB'Px = -\frac{1}{2}x'Qx + uCx
\]

\[
= -\varphi(x) + uy
\]
Given the linear dynamical system
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Prove that if there exist P and Q symmetric positive definite such that
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then the system is passive with storage function \( V(x) = \frac{1}{2}x'Px \)

**Solution:** \( V(x) \) is positive definite and continuously differentiable. Let us compute its derivative along the system trajectories
\[
\begin{align*}
\dot{V}(x) &= \frac{1}{2}(\dot{x}'Px + x'P\dot{x}) = \frac{1}{2}x'(A'P + PA)x + \frac{1}{2}(uB'Px + x'PBu) \\
&= \frac{1}{2}x'(A'P + PA)x + uB'Px = -\frac{1}{2}x'Qx + uCx \\
&= -\varphi(x) + uy \\
\rightarrow uy &\geq \dot{V}(x) + \varphi(x)
\end{align*}
\]
\( \rightarrow \) strictly passive w.r.t. the state
EXERCISE

Given the linear dynamical system

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\dot{x} = Ax + Bu, & x(0) = x_0 \\
y =Cx
\end{cases}
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Prove that if there exist P and Q symmetric positive definite such that

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then

1. the system is passive with storage function \( V(x) = \frac{1}{2}x'Px \)
2. the equilibrium \( x=0 \) for the system with zero input is globally asymptotically stable
PASSIVITY VERSUS (SOME FORMS OF) STABILITY

Theorem (passivity and Lyapunov stability):

If a dynamical system $S$

- is strictly passive w.r.t the state, with positive definite storage function, then, $x=0$ is a (Lyapunov) asymptotically stable equilibrium for the system with zero input.

- If the storage function is radially unbounded, then, $x=0$ is a globally asymptotically stable equilibrium for the system with zero input.
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Solution: it follows immediately from the just mentioned theorem.

Alternatively, one can observe that the system is linear and the first equation is Lyapunov equation.
EXERCISE

Given system $S$

\[
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = -x_1^3 + u \\
y = x_2
\end{cases}
\]

1. Show that $S$ is zero-state observable
EXERCISE

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1. Show that $S$ is zero-state observable

Solution:

We need to show that $x(\cdot) = 0$ is the only free evolution of the state that is compatible with $y(\cdot) = 0$
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$y(\cdot) = 0 \rightarrow x_2(\cdot) = 0 \rightarrow \dot{x}_2(\cdot) = 0$
EXERCISE

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We need to show that \( x(\cdot) = 0 \) is the only free evolution of the state that is compatible with \( y(\cdot) = 0 \).

\[
y(\cdot) = 0 \rightarrow x_2(\cdot) = 0 \rightarrow \dot{x}_2(\cdot) = 0
\]

Since \( u(\cdot) = 0 \), by the second equation we get \( x_1(\cdot) = 0 \), and, hence, S is zero-state observable.
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1. Show that $S$ is zero-state observable

2. Check that the system is passive with storage function

\[V(x) = \frac{x_1^4}{4} + \frac{x_2^2}{2}\]
EXERCISE

Given system S

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1. Show that S is zero-state observable

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\[
V(x) = \frac{x_1^4}{4} + \frac{x_2^2}{2}
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Solution:

V(x) is positive definite and continuously differentiable. We then compute its derivative along the trajectories of the system:
EXERCISE

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1. Show that S is zero-state observable

2. Check that the system is passive with storage function

\[V(x) = \frac{x_1^4}{4} + \frac{x_2^2}{2}\]

Solution:

\(V(x)\) is positive definite and continuously differentiable. We then compute its derivative along the trajectories of the system:

\[\dot{V}(x) = x_1^3 x_2 + x_2(-x_1^3 + u) = x_2 u = y u\]
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**Solution:**

\(V(x)\) is positive definite and continuously differentiable. We then compute its derivative along the trajectories of the system:
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\dot{V}(x) = x_1^3 x_2 + x_2(-x_1^3 + u) = x_2 u = y u
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→ Passive conservative system
EXERCISE

Given system $S$

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\dot{x}_1 &= x_2 \\
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\end{align*}$$

1. Show that $S$ is zero-state observable

2. Check that the system is passive with storage function

$$V(x) = \frac{x_1^4}{4} + \frac{x_2^2}{2}$$

Solution:

$V(x)$ is positive definite and continuously differentiable. We then compute its derivative along the trajectories of the system:

$$\dot{V}(x) = x_1^3 x_2 + x_2 (-x_1^3 + u) = x_2 u = yu$$

$\rightarrow$ Passive conservative system

$$u(\cdot) = 0 \rightarrow \dot{V}(x) = 0 \rightarrow V(x(t)) = V(x(0)), \forall t \geq 0$$

$\rightarrow$ The system evolves along the iso-level curves of $V(x)$
Theorem (passivity and Lyapunov stability):

If a dynamical system $S$

- is passive with positive definite storage function, then, $x=0$ is a (Lyapunov) stable equilibrium for the system with zero input
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\end{cases}$$

1. Show that $S$ is zero-state observable

2. Check that the system is passive with storage function

$$V(x) = \frac{x_1^4}{4} + \frac{x_2^2}{2}$$

3. Consider the feedback system with $u(t) = -ky(t) + v(t)$ and show that if $k > 0$ then $x=0$ is an asymptotically stable equilibrium of the zero-input feedback system
EXERCISE

Given system S
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Solution:

\( V(x) \) is positive definite and continuously differentiable. We then compute its derivative along the trajectories of the system:

\[\dot{V}(x) = yu = -ky^2 + vy\]

→ Strictly passive w.r.t. the output
EXERCISE

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V(x) is positive definite and continuously differentiable. We then compute its derivative along the trajectories of the system:
\[ \dot{V}(x) = yu = -ky^2 + yv \]
\[ \rightarrow \text{strictly passive w.r.t. the output \& zero-state observable} \]
PASSIVITY VERSUS (SOME FORMS OF) STABILITY

Theorem (passivity and Lyapunov stability):
If a dynamical system $S$

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If the storage function is radially unbounded, then, $x=0$ is a globally asymptotically stable equilibrium for the system with zero input.
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Solution:

\( V(x) \) is positive definite and continuously differentiable. We then compute its derivative along the trajectories of the system:

\[ \dot{V}(x) = yu = -ky^2 + yv \]

\( \rightarrow \) strictly passive w.r.t. the output & zero-state observable

\( \rightarrow \) \( x=0 \) GAS equilibrium for the feedback system with zero input
PASSIVITY-BASED CONTROL

We obtained $x=0$ GAS by introducing a feedback control that makes the system strictly passive w.r.t. the output.
Theorem

If the system
\[
\begin{cases}
\dot{x} = f(x, u) & f(0, 0) = 0 \quad h(0) = 0 \\
y = h(x)
\end{cases}
\]
is zero-state observable and passive with a storage function that is positive definite and radially unbounded, then \(x=0\) is a GAS equilibrium of the output feedback system obtained via

\[u = -\phi(y)\]

with \(\phi(0) = 0\) and \(y\phi(y) > 0, \forall y \neq 0\)