INPUT-OUTPUT APPROACH: STABILITY
Definition (\(\mathcal{L}\) stability):

A causal operator \(H : \mathcal{L}_e \to \mathcal{L}_e\) is \(\mathcal{L}\) - stable if \(H(\mathcal{L}) \subseteq \mathcal{L}\), that is

\[
H(u(\cdot)) \in \mathcal{L}, \quad \forall u(\cdot) \in \mathcal{L}
\]
INPUT-OUTPUT STABILITY

Definition ($\mathcal{L}$ stability):

A causal operator $H : \mathcal{L}_e \rightarrow \mathcal{L}_e$ is $\mathcal{L}$-stable if $H(\mathcal{L}) \subseteq \mathcal{L}$, that is

$$H(u(\cdot)) \in \mathcal{L}, \quad \forall u(\cdot) \in \mathcal{L}$$

- If $\mathcal{L} = L_\infty \rightarrow$ BIBO (bounded input bounded output) stability
**INPUT-OUTPUT STABILITY**

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**Remarks**

- It is a property of the system
- It applies to both static and dynamic systems
- It depends on $\mathcal{L}$
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Theorem
A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is $\mathcal{L}$-stable if and only if there exist
\begin{itemize}
  \item a continuous increasing function $\sigma(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ with $\sigma(0) = 0$
  \item a constant $\beta \in \mathbb{R}^+$
\end{itemize}
such that
\[ \|H(u(\cdot))\| \leq \sigma(\|u(\cdot)\|) + \beta, \quad \forall u(\cdot) \in \mathcal{L} \]
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Proof.
$\leftarrow$ straightforward
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such that

$$\|H(u(\cdot))\| \leq \sigma(\|u(\cdot)\|) + \beta, \quad \forall u(\cdot) \in \mathcal{L}$$

Proof. ($\Rightarrow$)
If $H$ is $\mathcal{L}$-stable, then for any $v \in \mathbb{R}^+$ $\zeta(v) := \sup_{\|u(\cdot)\| \leq v} \sup_{u(\cdot) \in \mathcal{L}} \|H(u(\cdot))\|$
is well-defined and finite, from which we get

$$\|H(u(\cdot))\| \leq \zeta(\|u(\cdot)\|), \quad \forall u(\cdot) \in \mathcal{L}$$
INPUT-OUTPUT STABILITY

Definition ($L$ stability):
A causal operator $H : L_e \rightarrow L_e$ is $L$-stable if $H(L) \subseteq L$, that is
$$H(u(\cdot)) \in L, \quad \forall u(\cdot) \in L$$

Theorem
A causal operator $H : L_e \rightarrow L_e$ is $L$-stable if and only if there exist
- a continuous increasing function $\sigma(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\sigma(0) = 0$
- a constant $\beta \in \mathbb{R}^+$
such that
$$\|H(u(\cdot))\| \leq \sigma(\|u(\cdot)\|) + \beta, \quad \forall u(\cdot) \in L$$

Proof. ($\Rightarrow$)
Since $\zeta(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nonnegative function that is non-decreasing, then, there exists a function $\sigma(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ continuous and increasing with $\sigma(0) = 0$ and $\beta \in \mathbb{R}^+$ such that
$$\zeta(v) \leq \sigma(v) + \beta, \quad \forall v \in \mathbb{R}^+$$
and, hence,
$$\|H(u(\cdot))\| \leq \zeta(\|u(\cdot)\|) \leq \sigma(\|u(\cdot)\|) + \beta, \quad \forall u(\cdot) \in L$$
INPUT-OUTPUT STABILITY

Definition ($\mathcal{L}$ stability):
A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is $\mathcal{L}$-stable if $H(\mathcal{L}) \subseteq \mathcal{L}$, that is

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Theorem
A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is $\mathcal{L}$-stable if and only if there exist

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such that

$$\|H(u(\cdot))\| \leq \sigma(\|u(\cdot)\|) + \beta, \quad \forall u(\cdot) \in \mathcal{L}$$

Corollary
A causal weakly bounded operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is $\mathcal{L}$-stable.
INPUT-OUTPUT STABILITY

Definition ($\mathcal{L}$ stability):
A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is $\mathcal{L}$ - stable if $H(\mathcal{L}) \subseteq \mathcal{L}$, that is
\[ H(u(\cdot)) \in \mathcal{L}, \quad \forall u(\cdot) \in \mathcal{L} \]

Theorem
A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is $\mathcal{L}$ - stable if and only if there exist
- a continuous increasing function $\sigma(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ with $\sigma(0) = 0$
- a constant $\beta \in \mathbb{R}^+$
\[ \text{such that } \|H(u(\cdot))\| \leq \sigma(\|u(\cdot)\|) + \beta, \quad \forall u(\cdot) \in \mathcal{L} \]

Corollary
A causal weakly bounded operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is $\mathcal{L}$ - stable.
\[ \exists \, \hat{\gamma}, \hat{\beta} \in \mathbb{R}^+ : \|H(u(\cdot))\| \leq \hat{\gamma}\|u(\cdot)\| + \hat{\beta}, \quad \forall u(\cdot) \in \mathcal{L} \]
INPUT-OUTPUT STABILITY

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A causal operator $H : \mathcal{L}_e \rightarrow \mathcal{L}_e$ is $\mathcal{L}$ - stable if $H(\mathcal{L}) \subseteq \mathcal{L}$, that is

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Corollary
A causal weakly bounded operator $H : \mathcal{L}_e \rightarrow \mathcal{L}_e$ is $\mathcal{L}$ - stable.

$$\exists \hat{\gamma}, \hat{\beta} \in \mathbb{R}^+ : \|H(u(\cdot))\| \leq \hat{\gamma}\|u(\cdot)\| + \hat{\beta}, \quad \forall u(\cdot) \in \mathcal{L}$$

‘finite gain $\mathcal{L}$ - stability’
INPUT-OUTPUT STABILITY

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A causal operator $H : \mathcal{L}_e \rightarrow \mathcal{L}_e$ is $\mathcal{L}$ - stable if $H(\mathcal{L}) \subseteq \mathcal{L}$, that is
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$$\|H(u(\cdot))\| \leq \sigma(\|u(\cdot)\|) + \beta, \quad \forall u(\cdot) \in \mathcal{L}$$

Corollary
A causal weakly bounded operator $H : \mathcal{L}_e \rightarrow \mathcal{L}_e$ is $\mathcal{L}$ - stable.

Remark:
the opposite is not true, in general (example: $\mathcal{L} = L_\infty$ and static system described by a continuous function that grows more than linearly)
Problem:
Identify connections between various kinds of I/O stability and of Lyapunov stability.
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Results are very few.

Exception: the class of linear time invariant systems.
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Proposition
Given a linear time invariant dynamical system $S$
$S$ asymptotically stable $\rightarrow$ the operator $H$ associated with $S$ is $L_p$-stable for any $p \in (0, \infty]$
INPUT-OUTPUT AND LYAPUNOV STABILITY

Problem:
Identify connections between various kinds of I/O stability and of Lyapunov stability.

Results are very few.

Exception: the class of linear time invariant systems.

Proposition
Given a linear time invariant dynamical system S
S asymptotically stable $\rightarrow$ the operator H associated with S is $L_p$-stable for any $p \in (0, \infty]$
H is $L_p$-stable, $p \in (0, \infty] \rightarrow$ S is asymptotically stable if and only if its non-observable and non-reachable parts are asymptotically stable
STABILITY OF INTERCONNECTED SYSTEMS: CASCADE

$H_i : \mathcal{L}_e \rightarrow \mathcal{L}_e, \ i = 1, 2$
STABILITY OF INTERCONNECTED SYSTEMS: CASCADE

\[ H_i : \mathcal{L}_e \rightarrow \mathcal{L}_e, \ i = 1, 2 \]

\[ u(\cdot) \in \mathcal{L}_e \rightarrow y(\cdot) = H(u(\cdot)) = H_2(H_1(u(\cdot))) \in \mathcal{L}_e \]
**Theorem**

Two causal and weakly bounded operators $H_1$ and $H_2$, interconnected in cascade, originates an operator $H$

$$u(\cdot) \in \mathcal{L}_e \rightarrow y(\cdot) = H(u(\cdot)) = H_2(H_1(u(\cdot))) \in \mathcal{L}_e$$

causal and weakly bounded with gain $\gamma(H) \leq \gamma(H_1) \gamma(H_2)$
STABILITY OF INTERCONNECTED SYSTEMS: CASCADE

Proof:

$H_1$ weakly bounded implies that

$$\exists \gamma_1, \beta_1 \in \mathbb{R}^+ : \|H_1(u(\cdot))\| \leq \gamma_1 \|u(\cdot)\| + \beta_1, \forall u(\cdot) \in \mathcal{L} \rightarrow z(\cdot) = H_1(u(\cdot)) \in \mathcal{L}$$
STABILITY OF INTERCONNECTION OF SYSTEMS: CASCADE

Proof:

$H_1$ weakly bounded implies that

$\exists \gamma_1, \beta_1 \in \mathbb{R}^+: \|H_1(u(\cdot))\| \leq \gamma_1 \|u(\cdot)\| + \beta_1, \forall u(\cdot) \in \mathcal{L} \rightarrow z(\cdot) = H_1(u(\cdot)) \in \mathcal{L}$

$H_2$ weakly bounded implies that

$\exists \gamma_2, \beta_2 \in \mathbb{R}^+: \|H_2(z(\cdot))\| \leq \gamma_2 \|z(\cdot)\| + \beta_2, \forall z(\cdot) \in \mathcal{L}$
STABILITY OF INTERCONNECTED SYSTEMS: CASCADE

Proof:

$H_1$ weakly bounded implies that
\[ \exists \gamma_1, \beta_1 \in \mathbb{R}^+ : \|H_1(u(\cdot))\| \leq \gamma_1 \|u(\cdot)\| + \beta_1, \forall u(\cdot) \in \mathcal{L} \rightarrow z(\cdot) = H_1(u(\cdot)) \in \mathcal{L} \]

$H_2$ weakly bounded implies that
\[ \exists \gamma_2, \beta_2 \in \mathbb{R}^+ : \|H_2(z(\cdot))\| \leq \gamma_2 \|z(\cdot)\| + \beta_2, \forall z(\cdot) \in \mathcal{L} \]

Then,
\[ \|H(u(\cdot))\| = \|H_2(H_1(u(\cdot)))\| \leq \gamma_2 (\gamma_1 \|u(\cdot)\| + \beta_1) + \beta_2 \]
\[ = \gamma_2 \gamma_1 \|u(\cdot)\| + \gamma_2 \beta_1 + \beta_2, \forall u(\cdot) \in \mathcal{L} \]

that is $H$ is weakly bounded and $\gamma(H) \leq \gamma(H_1) \gamma(H_2)$
STABILITY OF INTERCONNECTED SYSTEMS: CASCADE

Example:
Linear asymptotically stable time invariant dynamical systems with transfer functions $F_1(s)$ and $F_2(s)$
$\Rightarrow$ The cascade system has transfer function $F(s) = F_1(s)F_2(s)$
Let $\mathcal{L} = L_2$. Then,
$$\gamma_2(H) = F_{\text{max}} = \max_{\omega \in \mathbb{R}^+} |F(j\omega)| \leq F_{1,\text{max}}F_{2,\text{max}} = \gamma_2(H_1)\gamma_2(H_2)$$
STABILITY OF INTERCONNECTED SYSTEMS: PARALLEL

\[ H_i : \mathcal{L}_e \rightarrow \mathcal{L}_e, \quad i = 1, 2 \]
STABILITY OF INTERCONNECTED SYSTEMS: PARALLEL

\[ H_i : \mathcal{L}_e \rightarrow \mathcal{L}_e, \quad i = 1, 2 \]

\[ u(\cdot) \in \mathcal{L}_e \rightarrow y(\cdot) = H_1(u(\cdot)) + H_2(u(\cdot)) \in \mathcal{L}_e \]
Theorem

Two causal and weakly bounded operators $H_1$ and $H_2$, interconnected in parallel, originates an operator $H$

$$ u(\cdot) \in \mathcal{L}_e \rightarrow y(\cdot) = H_1(u(\cdot)) + H_2(u(\cdot)) \in \mathcal{L}_e $$

causal and weakly bounded with gain

$$ \gamma(H) \leq \gamma(H_1) + \gamma(H_2) $$

Proof: [to do as exercise]
STABILITY OF INTERCONNECTED SYSTEMS: FEEDBACK

\[ H_i : \mathcal{L}_e \rightarrow \mathcal{L}_e, \; i = 1, 2 \]
Is the operator $H$ obtained by interconnecting in feedback the causal operators $H_1$ and $H_2$ is well-posed, i.e., the pair $(y_1, y_2)$ exists and is unique for any $(u_1, u_2) \in \mathcal{L}_e \times \mathcal{L}_e$?
Is the operator $H$ obtained by interconnecting in feedback the causal operators $H_1$ and $H_2$ is well-posed, i.e., the pair $(y_1, y_2)$ exists and is unique for any $(u_1, u_2) \in \mathcal{L}_e \times \mathcal{L}_e$?

No, in general… It is well-posed if one of the two causal operators is strictly proper.
The operator $H$ has two inputs and two outputs. Let us define the operators with one input and one output:

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- The operator $H$ has two inputs and two outputs. Let us define the operators with one input and one output:

$H_{ij} : \mathcal{L}_e \rightarrow \mathcal{L}_e \ y_i(\cdot) = H_{ij}(u_j(\cdot)), \ i, j = 1, 2$
**Small gain theorem**

Let $H$ be a well-posed causal operator obtained by connecting in feedback two causal and weakly bounded operators $H_1$ and $H_2$. If

$$
\lambda := \gamma(H_1)\gamma(H_2) < 1
$$

then, $H$ is weakly bounded, that is:

$$
\exists \hat{\gamma}_{i1}, \hat{\gamma}_{i2}, \hat{\beta}_i \in \mathbb{R}^+ : \|y_i(\cdot)\| \leq \hat{\gamma}_{i1}\|u_1(\cdot)\| + \hat{\gamma}_{i2}\|u_2(\cdot)\| + \hat{\beta}_i
$$

$\forall u_1(\cdot), u_2(\cdot) \in \mathcal{L}, i = 1, 2$

Furthermore,

$$
\gamma(H_{11}) \leq \frac{\gamma(H_1)}{1 - \lambda}, \quad \gamma(H_{22}) \leq \frac{\gamma(H_2)}{1 - \lambda}, \quad \gamma(H_{12}), \gamma(H_{21}) \leq \frac{\lambda}{1 - \lambda}
$$
If $H_1$ and $H_2$ are causal weakly bounded and $H$ is well-posed, then we have that
\[
\forall u_1(\cdot), u_2(\cdot), z_1(\cdot) = u_1(\cdot) - y_2(\cdot), z_2(\cdot) = u_2(\cdot) + y_1(\cdot) \in L_e, \forall \tau \in \mathbb{R}^+
\]
we have that
\[
\|y_{1\tau}(\cdot)\| \leq \gamma_1 \|z_{1\tau}(\cdot)\| + \beta_1 \leq \gamma_1 (\|u_{1\tau}(\cdot)\| + \|y_{2\tau}(\cdot)\|) + \beta_1
\]
Proof (small gain theorem)

If $H_1$ and $H_2$ are causal weakly bounded and $H$ is well-posed, then

$$\forall u_1(\cdot), u_2(\cdot), z_1(\cdot) = u_1(\cdot) - y_2(\cdot), z_2(\cdot) = u_2(\cdot) + y_1(\cdot) \in \mathcal{L}_e, \ \forall \tau \in \mathbb{R}^+$$

we have that

$$\|y_{1\tau}(\cdot)\| \leq \gamma_1 \|z_{1\tau}(\cdot)\| + \beta_1 \leq \gamma_1 (\|u_{1\tau}(\cdot)\| + \|y_{2\tau}(\cdot)\|) + \beta_1$$

$$\leq \gamma_1 \|u_{1\tau}(\cdot)\| + \gamma_1 (\gamma_2 \|z_{2\tau}(\cdot)\| + \beta_2) + \beta_1$$
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$$\|y_{1\tau}(\cdot)\| \leq \gamma_1 \|z_{1\tau}(\cdot)\| + \beta_1 \leq \gamma_1 (\|u_{1\tau}(\cdot)\| + \|y_{2\tau}(\cdot)\|) + \beta_1$$

$$\leq \gamma_1 \|u_{1\tau}(\cdot)\| + \gamma_1 (\gamma_2 \|z_{2\tau}(\cdot)\| + \beta_2) + \beta_1$$

$$\leq \gamma_1 \|u_{1\tau}(\cdot)\| + \gamma_1 \gamma_2 (\|u_{2\tau}(\cdot)\| + \|y_{1\tau}(\cdot)\|) + \gamma_1 \beta_2 + \beta_1$$
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If $H_1$ and $H_2$ are causal weakly bounded and $H$ is well-posed, then

$\forall u_1(\cdot), u_2(\cdot), z_1(\cdot) = u_1(\cdot) - y_2(\cdot), z_2(\cdot) = u_2(\cdot) + y_1(\cdot) \in \mathcal{L}_c, \forall \tau \in \mathbb{R}^+$

we have that

$\|y_{1\tau}(\cdot)\| \leq \gamma_1 \|u_{1\tau}(\cdot)\| + \gamma_1 \gamma_2 (\|u_{2\tau}(\cdot)\| + \|y_{1\tau}(\cdot)\|) + \gamma_1 \beta_2 + \beta_1$
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we have that

$$\|y_{1\tau}(\cdot)\| \leq \gamma_1\|u_{1\tau}(\cdot)\| + \gamma_1\gamma_2(\|u_{2\tau}(\cdot)\| + \|y_{1\tau}(\cdot)\|) + \gamma_1\beta_2 + \beta_1$$

Hence, if $\gamma_1\gamma_2 < 1$ and $u_1(\cdot), u_2(\cdot) \in \mathcal{L}$

$$\|y_1(\cdot)\| \leq \frac{1}{1 - \gamma_1\gamma_2}(\gamma_1\|u_1(\cdot)\| + \gamma_1\gamma_2\|u_2(\cdot)\| + \gamma_1\beta_2 + \beta_1) \quad \forall u_1(\cdot), u_2(\cdot) \in \mathcal{L}$$
Proof (small gain theorem)

If $H_1$ e $H_2$ are causal weakly bounded and $H$ is well-posed, then

$$\forall u_1(\cdot), u_2(\cdot), z_1(\cdot) = u_1(\cdot) - y_2(\cdot), z_2(\cdot) = u_2(\cdot) + y_1(\cdot) \in \mathcal{L}_e, \forall \tau \in \mathbb{R}^+$$

we have that

$$\|y_{1\tau}(\cdot)\| \leq \gamma_1 \|u_{1\tau}(\cdot)\| + \gamma_1 \gamma_2 (\|u_{2\tau}(\cdot)\| + \|y_{1\tau}(\cdot)\|) + \gamma_1 \beta_2 + \beta_1$$

Hence, if $\gamma_1 \gamma_2 < 1$ and $u_1(\cdot), u_2(\cdot) \in \mathcal{L}$

$$\|y_1(\cdot)\| \leq \frac{1}{1 - \gamma_1 \gamma_2} (\gamma_1 \|u_1(\cdot)\| + \gamma_1 \gamma_2 \|u_2(\cdot)\| + \gamma_1 \beta_2 + \beta_1) \quad \forall u_1(\cdot), u_2(\cdot) \in \mathcal{L}$$

Similarly for $y_2(\cdot)$.
Proof (small gain theorem)
If $H_1$ e $H_2$ are causal weakly bounded and H is well-posed, then

$$\forall u_1(\cdot), u_2(\cdot), z_1(\cdot) = u_1(\cdot) - y_2(\cdot), z_2(\cdot) = u_2(\cdot) + y_1(\cdot) \in L_e, \forall \tau \in \mathbb{R}^+$$

we have that

$$\|y_{1\tau}(\cdot)\| \leq \gamma_1 \|u_{1\tau}(\cdot)\| + \gamma_1 \gamma_2 (\|u_{2\tau}(\cdot)\| + \|y_{1\tau}(\cdot)\|) + \gamma_1 \beta_2 + \beta_1$$

Hence, if $\gamma_1 \gamma_2 < 1$ and $u_1(\cdot), u_2(\cdot) \in L$

$$\|y_1(\cdot)\| \leq \frac{1}{1 - \gamma_1 \gamma_2} (\gamma_1 \|u_1(\cdot)\| + \gamma_1 \gamma_2 \|u_2(\cdot)\| + \gamma_1 \beta_2 + \beta_1) \quad \forall u_1(\cdot), u_2(\cdot) \in L$$

Similarly for $y_2(\cdot) \rightarrow H$ weakly bounded
Proof (small gain theorem)

If $H_1$ and $H_2$ are causal weakly bounded and $H$ is well-posed, then

$$\forall u_1(\cdot), u_2(\cdot), z_1(\cdot) = u_1(\cdot) - y_2(\cdot), z_2(\cdot) = u_2(\cdot) + y_1(\cdot) \in \mathcal{L}_e, \forall \tau \in \mathbb{R}^+$$

we have that

$$\|y_{1\tau}(\cdot)\| \leq \gamma_1\|u_{1\tau}(\cdot)\| + \gamma_1\gamma_2(\|u_{2\tau}(\cdot)\| + \|y_{1\tau}(\cdot)\|) + \gamma_1\beta_2 + \beta_1$$

Hence, if $\gamma_1\gamma_2 < 1$ and $u_1(\cdot), u_2(\cdot) \in \mathcal{L}$

$$\|y_1(\cdot)\| \leq \frac{1}{1 - \gamma_1\gamma_2} (\gamma_1\|u_1(\cdot)\| + \gamma_1\gamma_2\|u_2(\cdot)\| + \gamma_1\beta_2 + \beta_1) \quad \forall u_1(\cdot), u_2(\cdot) \in \mathcal{L}$$

Similarly for $y_2(\cdot) \to H$ weakly bounded

Let

$$f_1(\gamma_1, \gamma_2) := \frac{\gamma_1}{1 - \gamma_1\gamma_2}, \quad f_2(\gamma_1, \gamma_2) := \frac{\gamma_2}{1 - \gamma_1\gamma_2}, \quad f_{12}(\gamma_1, \gamma_2) := \frac{\gamma_1\gamma_2}{1 - \gamma_1\gamma_2}$$
Proof (small gain theorem)

\[ f_1(\gamma_1, \gamma_2) := \frac{\gamma_1}{1 - \gamma_1 \gamma_2}, \quad f_2(\gamma_1, \gamma_2) := \frac{\gamma_2}{1 - \gamma_1 \gamma_2}, \quad f_{12}(\gamma_1, \gamma_2) := \frac{\gamma_1 \gamma_2}{1 - \gamma_1 \gamma_2} \]

Are increasing function of \( \gamma_1 \) and \( \gamma_2 \) in the region where \( \gamma_1 \gamma_2 < 1 \)

\[ \Rightarrow \quad \gamma(H_{11}) \leq \frac{\gamma(H_1)}{1 - \lambda}, \quad \gamma(H_{22}) \leq \frac{\gamma(H_2)}{1 - \lambda}, \quad \gamma(H_{12}), \gamma(H_{21}) \leq \frac{\lambda}{1 - \lambda} \]
Small gain theorem

Let $H$ be a well-posed causal operator obtained by connecting in feedback two causal and weakly bounded operators $H_1$ and $H_2$. If

$$\lambda := \gamma(H_1)\gamma(H_2) < 1$$

then, $H$ is weakly bounded. Furthermore,

$$\gamma(H_{11}) \leq \frac{\gamma(H_1)}{1-\lambda}, \quad \gamma(H_{22}) \leq \frac{\gamma(H_2)}{1-\lambda}, \quad \gamma(H_{12}), \gamma(H_{21}) \leq \frac{\lambda}{1-\lambda}$$

Remark: it holds irrespectively of the signs at the summation nodes.
We know that the Lyapunov stability analysis for a feedback linear system can be performed by studying the Nyquist plot of $G_1(s)G_2(s)$.
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In particular: if the two interconnected systems are asymptotically stable, then, the feedback system is asymptotically stable if

$$\sup_{\omega \in \mathbb{R}^+} |G_1(j\omega)G_2(j\omega)| < 1$$

i.e., no encirclements of the Nyquist plot of $G_1(s)G_2(s)$ around (-1,0).
We knew that the Lyapunov stability analysis for a feedback linear system can be performed by studying the Nyquist plot of \( G_1(s)G_2(s) \). In particular: if the two interconnected systems are asymptotically stable, then, the feedback system is asymptotically stable if

\[
\sup_{\omega \in \mathbb{R}^+} |G_1(j\omega)G_2(j\omega)| < 1
\]
i.e., no encirclements of the Nyquist plot of \( G_1(s)G_2(s) \) around \((-1,0)\).

In turn, this condition is satisfied if

\[
\left( \sup_{\omega \in \mathbb{R}^+} |G_1(j\omega)| \right) \left( \sup_{\omega \in \mathbb{R}^+} |G_2(j\omega)| \right) < 1
\]

\[\rightarrow\text{ We have just shown a similar result for nonlinear systems.}\]
EXAMPLE: LUR’E SYSTEM

$S_1$: linear time invariant dynamical system that is asymptotically stable and strictly proper with transfer function $G(s)$

→ causal and weakly bounded in $L_p$
EXAMPLE: LUR’E SYSTEM

$S_1$: linear time invariant dynamical system that is asymptotically stable and strictly proper with transfer function $G(s)$

$\rightarrow$ causal and weakly bounded in $L_p$

$$\gamma(H_1) = \gamma^o(G_1) = \begin{cases} \max_{\omega \geq 0} |G(j\omega)| := G_{\text{max}}, & \mathcal{L} = L_2 \\ \int_0^\infty |g(t)| dt := k_1, & \mathcal{L} = L_\infty \end{cases}$$
EXAMPLE: LUR’E SYSTEM

\[ S_2: \text{static system with sector nonlinearity } \varphi(\cdot) \text{ in } [-k, k] \]

\[ |\varphi(v)| \leq k|v|, \ \forall v \in \mathbb{R} \]
EXAMPLE 1: STATIC SYSTEM

\[ S : \quad y(t) = g(u(t)), \quad \forall t \in \mathbb{R}^+ \]

where \( g : \mathbb{R} \to \mathbb{R} \) is piecewise continuous and \( g(0) \neq 0 \)

Set \( \tilde{g}(v) := g(v) - g(0) \). Suppose that there exists some finite

\[ \tilde{\gamma} := \inf \{ k \in \mathbb{R}^+ : |\tilde{g}(v)| \leq k|v|, \forall v \in \mathbb{R} \} \]

Static system whose characteristic belongs to a conic sector
**EXAMPLE: LUR’E SYSTEM**

\[ S_2: \text{static system with sector nonlinearity } \varphi(\cdot) \text{ in } [-k, k] \]

\[ |\varphi(v)| \leq k|v|, \forall v \in \mathbb{R} \]

- \( \mathcal{L} = L_\infty \rightarrow \gamma(H_2) \leq \gamma^\circ(H_2) = \tilde{\gamma} \leq k \)
EXAMPLE: LUR’E SYSTEM

\[ S_2: \text{static system with sector nonlinearity } \varphi(\cdot) \text{ in } [-k, k] \]

\[ |\varphi(v)| \leq k|v|, \forall v \in \mathcal{R} \]

- \( \mathcal{L} = L_\infty \rightarrow \gamma(H_2) \leq \gamma^o(H_2) = \tilde{\gamma} \leq k \)
- \( \mathcal{L} = L_2 \rightarrow \gamma(H_2) \leq k \)

because

\[ \|H_2(u(\cdot))\|_2^2 = \int_0^\infty \varphi^2(u(t))dt \leq \int_0^\infty k^2 u^2(t)dt = k^2 \|u(\cdot)\|_2^2, \forall u(\cdot) \in L_2 \]
System S (the associated operator H):

- is $L_2$-stable (for any sector nonlinearity $\varphi(\cdot)$ in $[-k, k]$) if

\[ kG_{\max} < 1 \]

- is $L_\infty$-stable (for any sector nonlinearity $\varphi(\cdot)$ in $[-k, k]$) if

\[ kk_1 < 1 \quad (k_1 := \|g(\cdot)\|_1) \]
CIRCLE CRITERION (IN LYAPUNOV FORM)

Autonomous Lur’e system: absolute stability in sector $[-k, k]$

Necessary condition: $S_1$ asymptotically stable
CIRCLE CRITERION (IN LYAPUNOV FORM)

Autonomous Lur’e system: absolute stability in sector \([-k, k]\)
Necessary condition: \(S_1\) asymptotically stable
Sufficient condition (circle criterion):

\[
\Re \frac{G(j\omega)}{1/k} \quad \frac{1}{k}
\]
CIRCLE CRITERION (IN LYAPUNOV FORM)

Autonomous Lur’e system: absolute stability in sector [-k, k]

Necessary condition: $S_1$ asymptotically stable

Sufficient condition (circle criterion):

$$G_{\text{max}} < \frac{1}{k} \iff kG_{\text{max}} < 1$$
LUR’E SYSTEM: $L_2$ VERSUS ABSOLUTE STABILITY

- The connection between $L_2$-stability of a time-invariant Lur’e system and absolute stability of the same system with inputs sets to zero can be further strengthened by considering a generic sector $[k_1, k_2]$, $k_1 < k_2$ and formulating a Circle criterion for $L_2$ stability
L₂ STABILITY IN SECTOR \([k₁, k₂]\)

\[
S : \quad y^o \quad e \quad \varphi(\cdot) \quad u \quad G(s) \quad y
\]
**L₂ STABILITY IN SECTOR [k₁,k₂]**

System $S$ is $L₂$-stable for any $\varphi(\cdot) \in \Phi_{[k₁,k₂]}$ if the number of encirclements of $G(s)$ Nyquist plot around $O(k₁,k₂)$ is equal to the number of poles of $G(s)$ with positive real part.
L₂ STABILITY IN SECTOR [k₁,k₂]

Theorem (Circle criterion for L₂ stability of a Lur’ë system)

System S is L₂-stable for any \( \varphi(\cdot) \in \Phi[k₁,k₂] \) if the number of encirclements of G(s) Nyquist plot around O(k₁,k₂) is equal to the number of poles of G(s) with positive real part.
$L_2$ STABILITY IN SECTOR $[k_1, k_2]$
L₂ STABILITY IN SECTOR \([k₁,k₂]\)

\[ \begin{align*}
S : & \quad y^o \\ & \quad e \\ & \quad \varphi(\cdot) \\ & \quad u \\ & \quad G(s) \\ & \quad y \\
S^* : & \quad y^o \\ & \quad e \\ & \quad \eta(\cdot) \\ & \quad z \\ & \quad F(s) \\ & \quad y
\end{align*} \]

\[ \eta(e) := \varphi(e) - he, \quad F(s) := \frac{G(s)}{1 + hG(s)} \]
**L₂ STABILITY IN SECTOR \([k₁, k₂]\)**

**S:**

\[ S : \]

\[ y^o \rightarrow e \rightarrow \varphi(\cdot) \rightarrow u \rightarrow G(s) \rightarrow y \]

**S*:**

\[ S^* : \]

\[ y^o \rightarrow e \rightarrow \eta(\cdot) \rightarrow z \rightarrow F(s) \rightarrow y \]

\[ \eta(e) := \varphi(e) - he, \quad F(s) := \frac{G(s)}{1 + hG(s)} \]

\[ h := \frac{k₁ + k₂}{2} \Rightarrow \varphi(\cdot) \in \Phi_{[k₁, k₂]} \Leftrightarrow \eta(\cdot) \in \Phi_{[-k, k]}, \quad k := \frac{k₂ - k₁}{2} \]
L₂ STABILITY IN SECTOR \([k₁, k₂]\)

\[ S : \]
\[ e \rightarrow \varphi(\cdot) \rightarrow u \rightarrow G(s) \rightarrow y \]

\[ S^* : \]
\[ e \rightarrow \eta(\cdot) \rightarrow z \rightarrow F(s) \rightarrow y \]

\[ \eta(e) := \varphi(e) - he, \quad F(s) := \frac{G(s)}{1 + hG(s)} \]

\[ h := \frac{k₁ + k₂}{2} \quad \Rightarrow \quad \varphi(\cdot) \in \Phi_{[k₁, k₂]} \quad \Leftrightarrow \quad \eta(\cdot) \in \Phi_{[-k, k]}, \quad k := \frac{k₂ - k₁}{2} \]

Remark: System S is L₂-stable in sector \([k₁, k₂]\) if and only if system \(S^*\) is L₂-stable in sector \([-k, k]\)
L₂ STABILITY IN SECTOR \([k_1, k_2]\)

\[ S : \quad y^o \rightarrow e \rightarrow \varphi(\cdot) \rightarrow u \rightarrow G(s) \rightarrow y \]

\[ S^* : \quad y^o \rightarrow e \rightarrow \eta(\cdot) \rightarrow z \rightarrow F(s) \rightarrow y \]

\[ \eta(e) := \varphi(e) - he, \quad F(s) := \frac{G(s)}{1 + hG(s)} \]

\[ h := \frac{k_1 + k_2}{2} \quad \Rightarrow \quad \varphi(\cdot) \in \Phi_{[k_1, k_2]} \iff \eta(\cdot) \in \Phi_{[-k, k]}, \quad k := \frac{k_2 - k_1}{2} \]

Remark: System S is \(L_2\)-stable in sector \([k_1, k_2]\) if and only if system \(S^*\) is \(L_2\)-stable in sector \([-k, k]\)

\(\rightarrow\) system with \(F(s)\) asymptotically stable and \(F_{\text{max}} < \frac{1}{k}\)
L₂ STABILITY IN SECTOR \([k₁,k₂]\)

The poles of \(F(s)\) have negative real part since \(h = \frac{k₁ + k₂}{2} \in [k₁, k₂]\) and \(G(s)\) Nyquist plot encircles \(I(k₁,k₂)\) as many times as the number of poles of \(G(s)\) with positive real part.

\[
\text{0 ≤ } k₁ < k₂ \\
\text{0 < } k₁ < k₂ \\
k₁ < k₂ ≤ 0
\]
$L_2$ STABILITY IN SECTOR $[k_1, k_2]$  

$O(k_1, k_2)$ is the image through the mapping  

$$F(s) \rightarrow G(s) = \frac{F(s)}{1 - h F(s)}$$  

of the region external to the circle of radius $1/k$ and center in the origin.
L₂ STABILITY IN SECTOR [k₁,k₂]

O(k₁, k₂) is the image through the mapping

\[ F(s) \rightarrow G(s) = \frac{F(s)}{1 - h F(s)} \]

de the region external to the circle of radius 1/k and center in the origin.

Then, if G(s) Nyquist plot does not intersect O(k₁, k₂), F(s) Nyquist plot is
within that circle, i.e.,

\[ F_{\text{max}} < \frac{1}{k} \]

which concludes the proof.