FEEDBACK LINEARIZATION
GOAL

Given a nonlinear system of the following form

\[
\begin{align*}
\dot{x} &= a(x) + b(x)u \\
y &= c(x)
\end{align*}
\]

design a static state feedback control law

\[u = k(x, v)\]

such that the associated feedback system is linear
GOAL

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\]

design a static state feedback control law

\[u = k(x, v)\]

such that the associated feedback system is linear

Remarks:

- Theory developed for a nonlinear system that is affine in the input, not a general system like

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= c(x)
\end{align*}
\]

but one can add an integrator and enlarge the state to get a nonlinear system which is affine in the input
GOAL

Given a nonlinear system of the following form

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y &= c(x)
\end{align*}
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design a static state feedback control law

\[ u = k(x, v) \]

such that the associated feedback system is linear

Remarks:

• Different from the approximation of a nonlinear system via linearization around some trajectory/equilibrium
EXAMPLE

Centrifuge model:

\[ J\ddot{\theta} = -k\dot{\theta}^2 \text{sgn}(\dot{\theta}) + u \]

- moment of inertia
- friction torque proportional to the square of the angular velocity
- torque control input

Goal: speed regulation
Centrifuge model: \[ J\ddot{\theta} = -k\dot{\theta}^2 \text{sgn}(\dot{\theta}) + u \]

Setting \( y = x = \dot{\theta} \) (speed control), we get

\[
\begin{cases}
\dot{x} = -\frac{k}{J} x^2 \text{sgn}(x) + \frac{1}{J} u \\
y = x
\end{cases}
\]
EXAMPLE

S: \[
\begin{cases}
\dot{x} = -\frac{k}{j} x^2 \text{sgn}(x) + \frac{1}{j} u \\
y = x
\end{cases}
\]

If we set

\[v := -\frac{k}{j} x^2 \text{sgn}(x) + \frac{1}{j} u\]

then we get the feedback system $S^*$

\[
\begin{cases}
\dot{x} = v \\
y = x
\end{cases}
\]
S: \[
\begin{align*}
\dot{x} &= -\frac{k}{J}x^2 \text{sgn}(x) + \frac{1}{J} u \\
y &= x
\end{align*}
\]

If we set
\[
v := -\frac{k}{J}x^2 \text{sgn}(x) + \frac{1}{J} u \quad \leftrightarrow \quad u = -kx^2 \text{sgn}(x) + Jv
\]
then we get the feedback system $S^*$

\[
\begin{align*}
\dot{x} &= v \\
y &= x
\end{align*}
\]

\[\begin{array}{ccc}
\text{v} & \rightarrow & \text{C} \\
\downarrow & & \downarrow \\
\text{u} & \rightarrow & \text{S} \\
\downarrow & & \downarrow \\
y = x & \rightarrow & \begin{array}{c}
\text{x} \\
\dot{x} = v
\end{array}
\end{array}\]
EXAMPLE

\[ S: \begin{cases} \dot{x} = -\frac{k}{J} x^2 \text{sgn}(x) + \frac{1}{J} u \\ y = x \end{cases} \]

If we set

\[ v := -\frac{k}{J} x^2 \text{sgn}(x) + \frac{1}{J} u \]

\[ \iff \quad u = -k x^2 \text{sgn}(x) + J v \]

then we get the feedback system \( S^* \)

\[ \begin{cases} \dot{x} = v \\ y = x \end{cases} \]

and it is now possible to design a rotational speed controller by the pole assignment method for linear systems

\[ G(s) = \frac{1}{s} \]
EXAMPLE

We can adopt a static proportional controller:

\[ v = k_p e \quad e = y^o - y \]

The resulting transfer function from \( y^o \) to \( y \) is given by

\[
F(s) = \frac{k_p \frac{1}{s}}{1 + k_p \frac{1}{s}} = \frac{1}{1 + s/k_p}
\]

\[ G(s) = \frac{1}{s} \]
Designed (nonlinear) controller:

\[ u = -k x^2 \text{sgn}(x) + J k_p (y^o - x) \]
QUESTIONS

• Under what conditions there exists a static state feedback control law that makes the feedback system linear?

• How can one design state feedback linearization?

• If a system is not fully linearizable, can we design a state feedback control law for a partial feedback linearization?
EXAMPLE 1: FULLY LINEARIZABLE SYSTEM

Let us consider a nonlinear system in the so-called normal form

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \phi(x_1, x_2, x_3, x_4) + bu \\
y &= x_1
\end{align*}
\]

where \( \phi(\cdot, \cdot, \cdot, \cdot) \) is a known nonlinear function and \( b \neq 0 \).
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where \( \phi(\cdot, \cdot, \cdot, \cdot) \) is a known nonlinear function and \( b \neq 0 \)

Design a state feedback control law \( u = k(x, v) \) such that the resulting feedback system is linear with transfer function

\[
F(s) = \frac{1}{(s + 1)^4}
\]
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\end{align*}
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where \(\phi(\cdot, \cdot, \cdot, \cdot)\) is a known nonlinear function and \(b \neq 0\)

Design a state feedback control law \(u = k(x, v)\) such that the resulting feedback system is linear with transfer function

\[
F(s) = \frac{1}{(s + 1)^4}
\]

Solution:

\[
\begin{align*}
u &= \frac{1}{b}(-\phi(x) + Kx + \alpha v) \\
&= \frac{1}{b}(-\phi(x_1, x_2, x_3, x_4) + k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4 + \alpha v)
\end{align*}
\]
We then get the linear feedback system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4 + \alpha u \\
y &= x_1
\end{align*}
\]

with transfer function

\[
F(s) = \frac{\alpha}{s^4 - k_4 s^3 - k_3 s^2 - k_2 s - k_1}
\]

We set it equal to the desired transfer function

\[
F(s) = \frac{1}{(s + 1)^4} = \frac{1}{s^4 + 4s^3 + 6s^2 + 4s + 1}
\]

thus getting

\[
\alpha = 1, \ k_1 = -1, \ k_2 = -4, \ k_3 = -6, \ k_4 = -4
\]

Remark. If \( \phi \) were linear (S linear) \( \rightarrow \) pole assignment
EXAMPLE 1: FULLY LINEARIZABLE SYSTEM

Given the nonlinear system in normal form

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\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \phi(x_1, x_2, x_3, x_4) + bu \\
y &= x_1
\end{align*}
\]

where \( \phi(\cdot, \cdot, \cdot, \cdot) \) is a known nonlinear function and \( b \neq 0 \)

we just need to set

\[
u = \frac{1}{b}(-\phi(x) + Kx + v)
\]

in order to obtain a linear feedback system.

if we can rewrite the system in the normal form via a suitable change of state variables, then the system is fully linearizable via a static state feedback \( \rightarrow \text{input/state linearization} \)
EXAMPLE 2: PARTIALLY LINEARIZABLE SYSTEM

Let us consider a nonlinear system in the normal form

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\vdots \\
\dot{\xi}_r &= a_\xi(\xi, \eta) + b(\xi, \eta)u \\
\dot{\eta} &= a_\eta(\xi, \eta) \\
y &= \xi_1
\end{align*}
\]

where \(b(\xi, \eta) \neq 0\).
EXAMPLE 2: PARTIALLY LINEARIZABLE SYSTEM

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\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
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\dot{\eta} &= a_\eta(\xi, \eta) \\
y &= \xi_1
\end{align*}
\]

where \( b(\xi, \eta) \neq 0 \). If we set

\[
u = \frac{1}{b(\xi, \eta)}(-a_\xi(\xi, \eta) + v)
\]

Then, the I/O map from \( v \) to \( y \) is linear.
EXAMPLE 2: PARTIALLY LINEARIZABLE SYSTEM

The resulting feedback system is nonlinear

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\vdots \\
\dot{\xi}_r &= v \\
\dot{\eta} &= a_\eta(\xi, \eta) \\
y &= \xi_1 
\end{align*}
\]

but the I/O map is linear and given by the differential equation

\[
\frac{d^r y}{dt^r} = v
\]

or, equivalently, by the transfer function

\[
G(s) = \frac{1}{s^r}
\]
EXAMPLE 2: PARTIALLY LINEARIZABLE SYSTEM

Let us consider a nonlinear system in the normal form

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\vdots & \quad x = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \\
\dot{\xi}_r &= a_\xi(\xi, \eta) + b(\xi, \eta)u \\
\dot{\eta} &= a_\eta(\xi, \eta) \\
y &= \xi_1
\end{align*}
\]

where \( b(\xi, \eta) \neq 0 \).

Then, the system is partially linearizable via the state feedback control law

\[
u = \frac{1}{b(\xi, \eta)}(-a_\xi(\xi, \eta) + v)
\]

The external dynamic is linearized by state feedback

\( \rightarrow \) input/output linearization
EXAMPLE 2: PARTIALLY LINEARIZABLE SYSTEM

Let us consider a nonlinear system in the normal form

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\dot{\xi}_r &= a_\xi(\xi, \eta) + b(\xi, \eta)u \\
\dot{\eta} &= a_\eta(\xi, \eta) \\
y &= \xi_1
\end{aligned}
\]

where \( b(\xi, \eta) \neq 0 \), and set

\[
u = \frac{1}{b(\xi, \eta)}(-a_\xi(\xi, \eta) + v)
\]

Is the feedback system zero-state observable?
Definition (zero-state observable dynamical system)
System S is zero-state observable if $x(\cdot) = 0$ is the only free evolution of the state compatible with identically zero output.
EXAMPLE 2: PARTIALLY LINEARIZABLE SYSTEM

Let us consider the feedback system

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_r &= v \\
\dot{\eta} &= a_\eta(\xi, \eta) \\
y &= \xi_1
\end{align*}
\]

If we set \( v(\cdot) = 0, \xi_1(0) = \xi_2(0) = \cdots = \xi_r(0) = 0 \), then \( y(\cdot) = 0 \).

Correspondingly, \( \xi_1(\cdot) = \xi_2(\cdot) = \cdots = \xi_r(\cdot) = 0 \), while \( \eta \) evolves according to the hidden internal dynamics (zero dynamics)

\[
\dot{\eta} = a_\eta(0, \eta), \quad \eta(0) = \eta_0
\]

And it is not necessarily zero, hence, the system is not zero-state observable.
EXAMPLE 2: PARTIALLY LINEARIZABLE SYSTEM

Given a nonlinear system in normal form

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\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_r &= a_\xi(\xi, \eta) + b(\xi, \eta)u \\
\dot{\eta} &= a_\eta(\xi, \eta) \\
y &= \xi_1
\end{align*}
\]

where \( b(\xi, \eta) \neq 0 \), we just need to set \( u \) in order to get a linear I/O map.

The system has a hidden dynamics.
Given a nonlinear system in normal form

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_r &= a_\xi(\xi, \eta) + b(\xi, \eta)u \\
\dot{\eta} &= a_\eta(\xi, \eta) \\
y &= \xi_1
\end{align*}
\]

where \( b(\xi, \eta) \neq 0 \), we just need to set \( u = \frac{1}{b(\xi, \eta)}(-a_\xi(\xi, \eta) + v) \) in order to get a linear I/O map.

The system has a hidden dynamics.

EXAMPLE 2: PARTIALLY LINEARIZABLE SYSTEM

if the system can be rewritten in normal form by a suitable state coordinate transformation, then, it is input-output linearizable via static state feedback.

But… one must consider the behavior of the zero dynamics!
GOAL

Given a nonlinear system of the following form
\[
\begin{aligned}
\dot{x} &= a(x) + b(x)u \\
y &= c(x)
\end{aligned}
\]

design a static state feedback control law
\[u = k(x, v)\]
such that the associated feedback system is linear
Nonlinear affine system, time-invariant, SISO:

\[
S : \begin{cases}
\dot{x} = a(x) + b(x)u \\
y = c(x)
\end{cases}
\]

Regularity assumptions on system \(S\):

\(a(\cdot), b(\cdot), c(\cdot)\) should be such that there exists a unique evolution associated to any piecewise continuous input \(u\), and continuously differentiable for any order (of class \(C^\infty\))
STATE FEEDBACK LINEARIZATION

Nonlinear affine system, time-invariant, SISO:

\[ \begin{align*}
    \dot{x} &= a(x) + b(x)u \\
    y &= c(x)
\end{align*} \]

Regularity assumptions on system \( S \):

\( a(\cdot), b(\cdot), c(\cdot) \) should be such that there exists a unique evolution associated to any piecewise continuous input \( u \), and continuously differentiable for any order (of class \( C^\infty \))

Goal:

Show that if \( S \) has a certain “relative degree” in \( x \), then there exists a static state feedback that makes the feedback system I/O map linear locally
Definition (relative degree):

The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$. 
RELATIVE DEGREE OF A SYSTEM

Definition (relative degree):

The relative degree \( r \) of a system \( S \) is given by the minimum order of the time derivative of the output \( y \) that is affected directly by the input \( u \).

In the case of a linear system, it is given by the difference between number of poles and number of zeros in the transfer function.

We show it next.
Definition (relative degree):

The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$.

Let us consider a linear time invariant SISO system:

$$S_L : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad G(s) = C(sI - A)^{-1}B$$
Definition (relative degree):

The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$

Let us consider a linear time invariant SISO system:

$$S_L : \begin{cases} \dot{x} = Ax + Bu \\ y =Cx \end{cases} \quad G(s) = C(sI - A)^{-1}B$$

Let us compute the first order time derivative of $y$:

$$\dot{y} = C\dot{x} = CAx + CBu$$
DEFINITION (RELATIVE DEGREE):

The relative degree \( r \) of a system \( S \) is given by the minimum order of the time derivative of the output \( y \) that is affected directly by the input \( u \).

Let us consider a linear time invariant SISO system:

\[
S_L: \begin{cases} 
\dot{x} = Ax + Bu \\
y = Cx 
\end{cases} \quad \quad \quad G(s) = C(sI - A)^{-1}B
\]

Let us compute the first order time derivative of \( y \):

\[
\dot{y} = C\dot{x} = CAx + CBu
\]

If \( CB \neq 0 \), \( r = 1 \), otherwise \( r > 1 \) and we compute the next derivative

\[
\ddot{y} = CA\dot{x} = CA^2x + CABu
\]
RELATIVE DEGREE OF A LINEAR SYSTEM

Definition (relative degree):
The relative degree \( r \) of a system \( S \) is given by the minimum order of the time derivative of the output \( y \) that is affected directly by the input \( u \)

Let us consider a linear time invariant SISO system:

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S_L : \begin{cases} 
  \dot{x} = Ax + Bu \\
  y = Cx 
\end{cases} \quad G(s) = C(sI - A)^{-1}B
\]

Let us compute the first order time derivative of \( y \):

\[ \dot{y} = C\dot{x} = CAx + CBu \]

if \( CB \neq 0 \), \( r = 1 \), otherwise \( r > 1 \) and we compute the next derivative

\[ \ddot{y} = CA\dot{x} = CA^2x + CABu \]

if \( CAB \neq 0 \), \( r = 2 \), otherwise \( r > 2 \) and we compute the next derivative
**Definition (relative degree):**

The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$.

Let us consider a linear time invariant SISO system:

$$ S_L : \begin{cases} \dot{x} = Ax + Bu \\ y =Cx \end{cases} \quad G(s) = C(sI - A)^{-1}B $$

Let us compute the first order time derivative of $y$:

$$ \dot{y} = C\dot{x} = CAx + CBu $$

If $CB \neq 0$, $r = 1$, otherwise $r > 1$ and we compute the next derivative

$$ \ddot{y} = CA\dot{x} = CA^2x + CABu $$

If $CAB \neq 0$, $r = 2$, otherwise $r > 2$ and we compute the next derivative

If $CB = CAB = CA^2B = \ldots = CA^{k-2}B = 0$ and $CA^{k-1}B \neq 0$, then $r = k$

$\Rightarrow r$ is the first value for $k$ such that $CA^{k-1}B \neq 0$
The relative degree of a linear system $S_L$ coincides with the difference $h$ between number of poles and number of zeros in the transfer function.

Let us consider a linear time invariant SISO system:

$$S_L : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad G(s) = C(sI - A)^{-1}B$$
RELATIVE DEGREE OF A LINEAR SYSTEM

The relative degree $r$ of a linear system $S_L$ coincides with the difference $h$ between number of poles and number of zeros in the transfer function.

Let us consider a linear time invariant SISO system:

\[
S_L : \begin{cases}
\dot{x} = Ax + Bu \\
y = Cx
\end{cases}
\]

\[
G(s) = C(sI - A)^{-1}B
\]

The impulse response of $S_L$ is given by:

\[g(t) = Ce^{At}B, \quad t \geq 0\]

since \[e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots\]

we get

\[g(t) = CB + CABt + \frac{1}{2!}CA^2Bt^2 + \cdots + \frac{1}{k!}CA^kBt^k + \cdots\]
The relative degree $r$ of a linear system $S_L$ coincides with the difference $h$ between number of poles and number of zeros in the transfer function.

Let us consider a linear time invariant SISO system:

$$S_L : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad G(s) = C(sI - A)^{-1}B$$

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$$g(t) = Ce^{At}B, \ t \geq 0$$

since $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots$ we get

$$g(t) = CB + CABt + \frac{1}{2!}CA^2Bt^2 + \cdots + \frac{1}{k!}CA^kBt^k + \cdots$$

from which it follows

$$\lim_{t \to 0} \frac{d^k g}{dt^k}(t) = CA^kB$$
The relative degree $r$ of a linear system $S_L$ coincides with the difference $h$ between number of poles and number of zeros in the transfer function.

Let us consider a linear time invariant SISO system:

$$S_L : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$$G(s) = C(sI - A)^{-1}B$$

The Laplace transform of the impulse response of $S_L$ is the transfer function $G(s)$, hence:
RELATIVE DEGREE OF A LINEAR SYSTEM

The relative degree $r$ of a linear system $S_L$ coincides with the difference $h$ between number of poles and number of zeros in the transfer function.

Let us consider a linear time invariant SISO system:

$$S_L : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad \quad \quad G(s) = C(sI - A)^{-1}B$$

The Laplace transform of the impulse response of $S_L$ is the transfer function $G(s)$, hence:

- $CB = \lim_{t \to 0} g(t) = \lim_{s \to \infty} sG(s) \begin{cases} = 0, & h > 1 \\ \neq 0, & h = 1 \end{cases}$
The relative degree $r$ of a linear system $S_L$ coincides with the difference $h$ between number of poles and number of zeros in the transfer function.

Let us consider a linear time invariant SISO system:

\[ S_L : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad G(s) = C(sI - A)^{-1}B \]

The Laplace transform of the impulse response of $S_L$ is the transfer function $G(s)$, hence:

- $CB = \lim_{t \to 0} g(t) = \lim_{s \to \infty} sG(s) \begin{cases} = 0, & h > 1 \\ \neq 0, & h = 1 \end{cases}$
- If $h > 1$, $CAB = \lim_{t \to 0} \frac{dg}{dt}(t) = \lim_{s \to \infty} s(sG(s) - g(0)) = \lim_{s \to \infty} s^2G(s) \begin{cases} = 0, & h > 2 \\ \neq 0, & h = 2 \end{cases}$
The relative degree \( r \) of a linear system \( S_L \) coincides with the difference \( h \) between number of poles and number of zeros in the transfer function.

Let us consider a linear time invariant SISO system:

\[
S_L : \begin{cases}
\dot{x} = Ax + Bu \\
y = Cx
\end{cases} \quad G(s) = C(sI - A)^{-1}B
\]

The Laplace transform of the impulse response of \( S_L \) is the transfer function \( G(s) \), hence:

- \( CB = \lim_{t \to 0} g(t) = \lim_{s \to \infty} sG(s) \begin{cases} 0, & h > 1 \\ \neq 0, & h = 1 \end{cases} \)
- if \( h > 1 \), \( CAB = \lim_{t \to 0} \frac{dg}{dt}(t) = \lim_{s \to \infty} s(sG(s) - g(0)) = \lim_{s \to \infty} s^2G(s) \begin{cases} 0, & h > 2 \\ \neq 0, & h = 2 \end{cases} \)
- if \( h > k - 1 \), \( CA^{k-1}B = \lim_{t \to 0} \frac{d^{k-1}g}{dt^{k-1}}(t) = \lim_{s \to \infty} s^kG(s) \begin{cases} 0, & h > k \\ \neq 0, & h = k \end{cases} \)
The relative degree \( r \) of a linear system \( S_L \) coincides with the difference \( h \) between number of poles and number of zeros in the transfer function.

Let us consider a linear time invariant SISO system:

\[
S_L: \begin{cases}
\dot{x} = Ax + Bu \\
y = Cx
\end{cases}
G(s) = C(sI - A)^{-1}B
\]

The Laplace transform of the impulse response of \( S_L \) is the transfer function \( G(s) \), hence:

- \( CB = \lim_{t \to 0} q(t) = \lim_{s \to \infty} sG(s) \begin{cases} = 0, & h > 1 \\ \neq 0, & h = 1 \end{cases} \]
- if \( h > 1 \), \( CAB = \lim_{t \to 0} \frac{d}{dt} g(t) = \lim_{s \to \infty} s(sG(s) - g(0)) = \lim_{s \to \infty} s^2 G(s) \begin{cases} = 0, & h > 2 \\ \neq 0, & h = 2 \end{cases} \]
- if \( h > k - 1 \), \( CA^{k-1}B = \lim_{t \to 0} \frac{d^{k-1}}{dt^{k-1}} g(t) = \lim_{s \to \infty} s^k G(s) \begin{cases} = 0, & h > k \\ \neq 0, & h = k \end{cases} \)

\( \rightarrow h \) is the first value for \( k \) such that \( CA^{k-1}B \neq 0 \) \( \rightarrow h \) is equal to \( r \).
Definition (relative degree): 

The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$

We have to compute the derivatives of the output and progressively increase the order of derivation $k$ till we get the (smallest) $k$ such that

$$y^{(k)} := \frac{d^k y}{dt^k}$$

depends directly on the input $u$.

We need first to introduce some notations and concepts.
REGULAR FUNCTIONS

Let $A$ be an open subset of $\mathbb{R}^n$ and $f$ a real function defined on $A$

$$f : A \rightarrow \mathbb{R}$$

Function $f$ is regular in $x \in A$ if it is continuously differentiable of any order in $x$: $f \in C^\infty$
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Function $f$ is regular in $A$, if it is regular in every point of $A$

The vector-value function

$$f = [f_1 \ f_2 \ldots \ f_m] : A \to \mathbb{R}^m$$

is regular if all functions in $f$ are regular

Given that it maps each $x$ into a vector $f(x) \in \mathbb{R}^m$, it is often called regular vector field defined on $A$
The Jacobian of \( f = [f_1 \ f_2 \ldots \ f_m]' \) is the following matrix of functions:

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots & \frac{\partial f_2}{\partial x_n} \\
\ldots & \ldots & \ldots & \ldots \\
\frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \ldots & \frac{\partial f_m}{\partial x_n}
\end{pmatrix}
\]

We shall denote its value at some \( x^\circ \) with \( \frac{\partial f}{\partial x}(x^\circ) \) or \( [f_x]_{x^\circ} \).
The Jacobian of $f = [f_1 \ f_2 \ldots f_m]'$ is the following matrix of functions

$$\mathbf{J}_f = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{bmatrix}$$

We shall denote its value at some $x^\circ$ with $\frac{\partial f}{\partial x}(x^\circ)$ or $[\mathbf{J}_f]_{x^\circ}$

Let $A$ and $B$ be open sets in $\mathbb{R}^n$.
Then, function $f : A \rightarrow B$ is a diffeomorphism if it is bijective (invertible) and both $f$ and $f^{-1}$ are regular functions.

**Theorem:** a regular function $f : A \rightarrow B$ is a local diffeomorphism in $x^\circ$ if its Jacobian $\mathbf{J}_f$ is non-singular at $x^\circ$
LIE DERIVATIVE

Let $A$ be an open set in $\mathbb{R}^n$ and $f : A \rightarrow \mathbb{R}^n$ a regular vector field on $A$. The operator $L_f$ defined as

$$L_f := \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}$$

is called *Lie derivative along the vector field* $f$. 
LIE DERIVATIVE

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Given a regular vector field $h : A \rightarrow \mathbb{R}^m$, the Lie derivative of $h$ along the vector field $f$ is given by:

$$L_f h = h_x f : A \rightarrow \mathbb{R}^m$$

Jacobian of $h$
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$$L_f h = h_x f : A \to \mathbb{R}^m$$

Remark: we have already seen it when computing the time derivative of a Lyapunov function $V(x)$ along the trajectories of a system

$$\dot{x} = f(x) \rightarrow \dot{V}(x) = \sum_{i=1}^{n} f_i(x) \frac{\partial V}{\partial x_i} = V_x f = L_f V$$
LIE DERIVATIVE

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Given a regular vector field $h : A \to \mathbb{R}^m$, the Lie derivative of $h$ along the vector field $f$ is given by:

$$L_f h = h_x f : A \to \mathbb{R}^m$$

If we iterate the Lie derivative along $f$ we get

$$L_f^2 h := L_f (L_f h)$$

In general

$$L_f^k h := L_f (L_f^{k-1} h), \quad L_f^0 h = h$$
LIE DERIVATIVE

Let $A$ be an open set in $\mathbb{R}^n$ and $f : A \to \mathbb{R}^n$ a regular vector field on $A$. The operator $L_f$ defined as

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is called *Lie derivative along the vector field* $f$.

Given a regular vector field $h : A \to \mathbb{R}^m$, the Lie derivative of $h$ along the vector field $f$ is given by:

$$L_f h = h_x f : A \to \mathbb{R}^m$$

Given a regular vector field $g : A \to \mathbb{R}^n$ with values in $\mathbb{R}^n$ like $f$,

$$L_g L_f h = \frac{\partial (L_f h)}{\partial x} g$$
RELATIVE DEGREE OF A NONLINEAR SYSTEM

Definition (relative degree): The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$

\[
S : \begin{cases} 
\dot{x} = a(x) + b(x)u \\
y = c(x) 
\end{cases}
\]

We next compute the relative degree of $S$ in $x^o \in \mathbb{R}^n$, by determining the derivatives of the output $y$
Definition (relative degree):

The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$

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We next compute the relative degree of $S$ in $x^o \in \mathbb{R}^n$, by determining the derivatives of the output $y$

- First order time derivative of $y$:

$$\dot{y} = c_x \dot{x} = c_x (a + b \ u) = L_a \ c + u \ L_b \ c$$
Definition (relative degree):

The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$

$$S : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases}$$

We next compute the relative degree of $S$ in $x^\circ \in \mathbb{R}^n$, by determining the derivatives of the output $y$

- First order time derivative of $y$:

$$\dot{y} = c_x \dot{x} = c_x (a + b \ u) = L_a \ c + u \ L_b \ c$$

If $[L_b \ c]|_{x^\circ} \neq 0$, then $[L_b \ c]|_{x} \neq 0$ in a neighborhood of $x^\circ$ (by the regularity assumption on $S$) and we can conclude that the relative degree of $S$ in $x^\circ$ is $r = 1$
RELATIVE DEGREE OF A NONLINEAR SYSTEM

Definition (relative degree):

The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$

$$S : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases}$$

We next compute the relative degree of $S$ in $x^\circ \in \mathbb{R}^n$, by determining the derivatives of the output $y$

- First order time derivative of $y$:
  $$\dot{y} = c_x \dot{x} = c_x (a + b u) = L_a c + u L_b c$$

  If $[L_b c]_{x^\circ} \neq 0$, then $[L_b c]_x \neq 0$ in a neighborhood of $x^\circ$ (by the regularity assumption on $S$) and we can conclude that the relative degree of $S$ in $x^\circ$ is $r = 1$

  If $[L_b c]_{x^\circ} = 0$, then, the relative degree $r$ of $S$ in $x^\circ$ is either not well-defined or is larger than 1
RELATIVE DEGREE OF A NONLINEAR SYSTEM

- If \([L_b c]_{x^\circ} = 0\) but in any neighborhood of \(x^\circ\) there is some \(x\) such that \([L_b c]_x \neq 0\), then the relative degree of \(S\) in \(x^\circ\) is not well-defined.

- If \([L_b c]_x = 0\) in some neighborhood of \(x^\circ\), then we have to take the second order derivative to determine \(r\).
RELATIVE DEGREE OF A NONLINEAR SYSTEM

Definition (relative degree): 

The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$ 

$$S: \begin{cases} 
\dot{x} = a(x) + b(x)u \\
 y = c(x) 
\end{cases}$$

We next compute the relative degree of $S$ in $x^\circ \in \mathbb{R}^n$, by determining the derivatives of the output $y$: 

- First order time derivative of $y$: 
  $$\dot{y} = c_x \dot{x} = c_x (a + b u) = L_a c + u L_b c$$

- Let $[L_b c]_{x^\circ} = 0$ in some neighborhood of $x^\circ$, that is, 
  $$\dot{y} = L_a c$$
Definition (relative degree):

The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$

$$S : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases}$$

We next compute the relative degree of $S$ in $x^* \in \mathbb{R}^n$, by determining the derivatives of the output $y$

- First order time derivative of $y$:
  $$\dot{y} = c_x \dot{x} = c_x (a + b u) = L_a c + u L_b c$$

- Let $[L_b c]_{x^*} = 0$ in some neighborhood of $x^*$, that is $\dot{y} = L_a c$

  Second order time derivative of $y$:
  $$\ddot{y} = \frac{\partial (L_a c)}{\partial x} (a + b u) = L_a^2 c + u L_b L_a c$$
RELATIVE DEGREE OF A NONLINEAR SYSTEM

Definition (relative degree):

The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$

$$S : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases}$$

We next compute the relative degree of $S$ in $x^o \in \mathbb{R}^n$, by determining the derivatives of the output $y$

- Second order time derivative of $y$:

$$\ddot{y} = \frac{\partial(L_a \ c)}{\partial x} (a + b \ u) = L_a^2 \ c + u \ L_b \ L_a \ c$$

If $[L_b \ L_a \ c]_{x^o} \neq 0$, then, $r = 2$.

If $L_b \ L_a \ c \equiv 0$ in some neighborhood of $x^o$, then $r > 2$ (otherwise $r$ is not well-defined in $x^o$), $\dddot{y} = L_a^2 c$, and we need to move on to the third order time derivative $y^{(3)}$. 
RELATIVE DEGREE OF A NONLINEAR SYSTEM

Definition (relative degree):

The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$

$$S : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases}$$

We next compute the relative degree of $S$ in $x^\circ \in \mathbb{R}^n$, by determining the derivatives of the output $y$

- By iterating this procedure, if in some neighborhood of $x^\circ$ we have
  $$L_b L_a^i c \equiv 0, \quad i = 0, 1, 2, \ldots, k-2$$
  then the time derivative of order $k$ of $y$ is given by
  $$y^{(k)} = L_a^k c + u L_b L_a^{k-1} c$$
  and if $[L_b L_a^{k-1} c]_{x^\circ} \neq 0$ then $r = k$. 
RELATIVE DEGREE OF A NONLINEAR SYSTEM

Definition (relative degree):

The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$

$$S : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases}$$

We next compute the relative degree of $S$ in $x^0 \in \mathbb{R}^n$, by determining the derivatives of the output $y$:

- By iterating this procedure, if in some neighborhood of $x^0$ we have
  $$L_b L_a^i c \equiv 0, \quad i = 0, 1, 2, \ldots, k-2$$
  then the time derivative of order $k$ of $y$ is given by
  $$y^{(k)} = L_a^k c + u L_b L_a^{k-1} c$$

  and if $[L_b L_a^{k-1} c]_{x^0} \neq 0$ then $r = k$.

  If this does not happen for any $k$, then the relative degree of $S$ in $x^0$ is not defined.
RELATIVE DEGREE OF A NONLINEAR SYSTEM

Definition (relative degree):

The relative degree \( r \) of a system \( S \) is given by the minimum order of the time derivative of the output \( y \) that is affected directly by the input \( u \)

\[
S : \begin{cases}
\dot{x} = a(x) + b(x)u \\
y = c(x)
\end{cases}
\]

Definition (relative degree of \( S \) in \( x^\circ \)):

System \( S \) has relative degree \( r \) in \( x^\circ \) if, in a neighborhood of \( x^\circ \),

\[
L_b L_a^k c \equiv 0, \quad k = 0, 1, 2, \ldots, r-2,
\]

and

\[
[L_b L_a^{r-1} c]_{x^\circ} \neq 0.
\]
RELATIVE DEGREE OF A NONLINEAR SYSTEM

Definition (relative degree):
The relative degree \( r \) of a system \( S \) is given by the minimum order of the
time derivative of the output \( y \) that is affected directly by the input \( u \)

\[
S : \begin{cases}
    \dot{x} = a(x) + b(x)u \\
y = c(x)
\end{cases}
\]

Definition (relative degree of \( S \) in \( x^0 \)):
System \( S \) has relative degree \( r \) in \( x^0 \) if, in a neighborhood of \( x^0 \),

\[
L_b L_a^k c = 0, \quad k = 0, 1, 2, \ldots, r-2,
\]

and

\[
[L_b L_a^{r-1} c]_{x^0} \neq 0.
\]

Remark: if \( S \) is linear, then

\[
L_b c = L_B(Cx) = CB
\]

\[
L_b L_a c = L_B(L_a(Cx)) = L_B(CAx) = CAB
\]

\[
L_b L_a^2 c = L_B(L_a(L_a(Cx)))) = L_B(L_a(CAx)) = L_B(CA^2x) = CA^2 B
\]
RELATIVE DEGREE OF A NONLINEAR SYSTEM

Definition (relative degree):
The relative degree $r$ of a system $S$ is given by the minimum order of the time derivative of the output $y$ that is affected directly by the input $u$

$$S : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases}$$

Definition (relative degree of $S$ in $x^0$):
System $S$ has relative degree $r$ in $x^0$ if, in a neighborhood of $x^0$,

$$L_b L_a^k c = 0, \quad k = 0, 1, 2, \ldots, r-2,$$

and

$$[L_b L_a^{r-1} c]_{x^0} \neq 0.$$

Remark: if $S$ is linear, then

$$L_b L_a^k c = CA^k B$$

and the relative degree $r$ of $S$ is the smallest $k$ such that $CA^{k-1}B \neq 0$. 
STATE FEEDBACK LINEARIZATION

Nonlinear affine system, time-invariant, SISO, regular:

\[ S : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases} \]

**Theorem (input-output state feedback linearization)**

If system S has relative degree \( r \) in \( x^\circ \), then, one can obtain a (locally) linear I/O map via state feedback.
STATE FEEDBACK LINEARIZATION

Nonlinear affine system, time-invariant, SISO, regular:

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S : \begin{cases}
\dot{x} = a(x) + b(x)u \\
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\]

**Theorem (input-output state feedback linearization)**

If system S has relative degree \(r\) in \(x^o\), then, one can obtain a (locally) linear I/O map via state feedback.

**Proof.**
STATE FEEDBACK LINEARIZATION

Nonlinear affine system, time-invariant, SISO, regular:

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S : \begin{cases}
\dot{x} = a(x) + b(x)u \\
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**Theorem (input-output state feedback linearization)**

If system S has relative degree \( r \) in \( x^\circ \), then, one can obtain a (locally) linear I/O map via state feedback.

**Proof.** If \( S \) has relative degree \( r \) in \( x^\circ \), then

\[
y^{(r)} = L_a^T c + u L_b L_a^{-1} c
\]

and \([L_b L_a^{-1} c]_{x^\circ} \neq 0\); hence, \( L_b L_a^{-1} c \) is nonzero in a neighborhood of \( x^\circ \).
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Theorem (input-output state feedback linearization)

If system S has relative degree \( r \) in \( x^o \), then, one can obtain a (locally) linear I/O map via state feedback.

Proof. If \( S \) has relative degree \( r \) in \( x^o \), then

\[
y^{(r)} = L_a^r c + uL_b L_a^{r-1} c
\]

and \( [L_b L_a^{r-1} c]_{x^o} \neq 0 \); hence, \( L_b L_a^{r-1} c \) is nonzero in a neighborhood of \( x^o \). If we then set

\[
v := L_a^r c + uL_b L_a^{r-1} c
\]

where \( v \) is the new input variable, then:

\[
y^{(r)} = \frac{d^r y}{dt^r} = v
\]
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Nonlinear affine system, time-invariant, SISO, regular:

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\[ y^{(r)} = L_a^r c + u L_b L_a^{r-1} c \]

and \([L_b L_a^{r-1} c]_{x^\circ} \neq 0\); hence, \( L_b L_a^{r-1} c \) is nonzero in a neighborhood of \( x^\circ \). If we then set

\[ v := L_a^r c + u L_b L_a^{r-1} c \]

where \( v \) is the new input variable, then:

\[ u = \frac{1}{L_b L_a^{r-1} c} (v - L_a^r c) \]

\[ y^{(r)} = \frac{d^r y}{dt^r} = v \]
Nonlinear affine system, time-invariant, SISO, regular:

\[ S : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases} \]

**Theorem (input-output state feedback linearization)**

If system S has relative degree \( r \) in \( x^\circ \), then, one can obtain a (locally) linear I/O map via state feedback.

\[ u = \frac{1}{L_b L_a^{r-1} c} (v - L_a^r c) \]
STATE FEEDBACK LINEARIZATION

Nonlinear affine system, time-invariant, SISO, regular:

\[ S : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases} \]

**Theorem (input-output state feedback linearization)**

If system S has relative degree \( r \) in \( x^o \), then, one can obtain a (locally) linear I/O map via state feedback.

\[ y^{(r)} = \frac{d^r y}{dt^r} = v \]
STATE FEEDBACK LINEARIZATION

Nonlinear affine system, time-invariant, SISO, regular:

\[ S: \begin{align*}
\dot{x} &= a(x) + b(x)u \\
y &= c(x)
\end{align*} \]

**Theorem (input-output state feedback linearization)**

If system S has relative degree \( r \) in \( x^o \), then, one can obtain a (locally) linear I/O map via state feedback.

**Remark:** If \( r < n \), there is some hidden dynamics!

- We need to isolate and analyze the hidden dynamics by using a suitable canonical form (the normal canonical form).
EXAMPLE 2: PARTIALLY LINEARIZABLE SYSTEM

Let us consider a nonlinear system in the normal form

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_r &= a_\xi(\xi, \eta) + b(\xi, \eta)u \\
\dot{\eta} &= a_\eta(\xi, \eta) \\
y &= \xi_1
\end{align*}
\]

where \( b(\xi, \eta) \neq 0 \).
EXAMPLE 2: PARTIALLY LINEARIZABLE SYSTEM

Let us consider a nonlinear system in the normal form

\[
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\vdots \\
\dot{\xi}_r &= a_{\xi}(\xi, \eta) + b(\xi, \eta)u \\
\dot{\eta} &= a_\eta(\xi, \eta) \\
y &= \xi_1 \\
x &= \begin{bmatrix} \xi \\ \eta \end{bmatrix}
\end{aligned}
\]

where \( b(\xi, \eta) \neq 0 \). If we set

\[
u = \frac{1}{b(\xi, \eta)}(-a_{\xi}(\xi, \eta) + v)
\]

Then, the I/O map from \( v \) to \( y \) is linear.
EXAMPLE 2: PARTIALLY LINEARIZABLE SYSTEM

The resulting feedback system is nonlinear

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_r &= v \\
\dot{\eta} &= a_\eta(\xi, \eta) \\
y &= \xi_1
\end{align*}
\]

but the I/O map is linear and given by the differential equation

\[
\frac{d^r y}{dt^r} = v
\]

or, equivalently, by the transfer function

\[
G(s) = \frac{1}{s^r}
\]
EXAMPLE 2: PARTIALLY LINEARIZABLE SYSTEM

Let us consider a nonlinear system in the normal form
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\vdots \\
\dot{x}_r &= a_x(x, \eta) + b(x, \eta)u \\
\dot{\eta} &= a_\eta(x, \eta) \\
y &= x_1
\end{align*}
\]

where \( b(x, \eta) \neq 0 \).

Then, the system is partially linearizable via the state feedback control law
\[
u = \frac{1}{b(x, \eta)}( - a_x(x, \eta) + v )
\]

The external dynamic is linearized by state feedback
\[\text{input/output linearization}\]
EXAMPLE 2: PARTIALLY LINEARIZABLE SYSTEM

Let us consider the feedback system

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\vdots \\
\dot{\xi}_r &= v \\
\dot{\eta} &= a_\eta(\xi, \eta) \\
y &= \xi_1
\end{align*}
\]

\[x = \begin{bmatrix} \xi \\ \eta \end{bmatrix}\]

If we set \( v(\cdot) = 0, \xi_1(0) = \xi_2(0) = \cdots = \xi_r(0) = 0 \), then \( y(\cdot) = 0 \).

Correspondingly, \( \xi_1(\cdot) = \xi_2(\cdot) = \cdots = \xi_r(\cdot) = 0 \), while \( \eta \) evolves according to the hidden internal dynamics (zero dynamics)

\[\dot{\eta} = a_\eta(0, \eta), \eta(0) = \eta_0\]

And it is not necessarily zero, hence, the system is not zero-state observable.
Given a nonlinear system in normal form

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\vdots & \\
\dot{\xi}_r &= a_\xi(\xi, \eta) + b(\xi, \eta)u \\
\dot{\eta} &= a_\eta(\xi, \eta) \\
y &= \xi_1
\end{align*}
\]

where \( b(\xi, \eta) \neq 0 \), we just need to set

\[
u = \frac{1}{b(\xi, \eta)} (-a_\xi(\xi, \eta) + v)
\]

in order to get a linear I/O map.

The resulting feedback system has a hidden dynamics.

Concluding remarks:

if the system can be rewritten in normal form by a suitable state coordinate transformation, then, it is input-output linearizable via static state feedback.

But… one must consider the behavior of the zero dynamics!