LUR'É PROBLEM: ABSOLUTE STABILITY

LUR'É SYSTEM

- L: time-invariant dynamic system
- N: nonlinear static system
LUR’E SYSTEM

Equivalent forms

\[ \alpha = -w \]
\[ \gamma = -u \]
\[ \eta = -y \]
\[ \beta = v \]
AUTONOMOUS LUR’E SYSTEM

\[ S: \]

\[ L : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \] (A,B,C) stabilizable

Assumption: (A,B) reachable & (A,C) observable

\[ G(s) = C(sI - A)^{-1}B \]
AUTONOMOUS LUR’E SYSTEM

\[ S: \quad e \rightarrow N \rightarrow u \rightarrow L \rightarrow y \]

\[ N: \quad u(t) = \varphi(e(t)) \]

- \( \varphi: \mathbb{R} \rightarrow \mathbb{R} \) piecewise continuous function
- \( \varphi(\cdot) \in \Phi_{[k_1,k_2]} = \{ \phi(\cdot): k_1 e \leq \phi(e) \leq k_2 e, \forall e \in \mathbb{R} \} \)

SECTOR NONLINEARITY

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\[ \Phi_{[k_1,k_2]} = \{ \phi(\cdot) : (k_2 e - u)(u - k_1 e) \geq 0, \ u = \phi(e), \ \forall e \in \mathbb{R} \} \]

AUTONOMOUS LUR’E SYSTEM

\[ S : \begin{cases} \dot{x} = Ax + B\varphi(-Cx) \\ y = Cx \end{cases} \]

\[ f(x) := Ax + B\varphi(-Cx) \]

\[ \varphi(0) = 0 \rightarrow f(0) = 0 \rightarrow \bar{x} = 0 \] is an equilibrium for \( S \), for any sector nonlinearity \( \varphi(\cdot) \in \Phi_{[k_1,k_2]} \)
ABSOLUTE STABILITY IN THE SECTOR \([k_1, k_2]\)

**Definition**

System \(S\) is absolutely stable in the sector \([k_1, k_2]\) if \(x = 0\) is a globally asymptotically stable equilibrium, for every sector nonlinearity \(\varphi(\cdot) \in \Phi_{[k_1,k_2]}\).

STABILITY OF AN EQUILIBRIUM

\[ \dot{x}(t) = f(x(t)) \]

**Definition (equilibrium):**

\(x_e \in \mathbb{R}^n\) such that \(f(x_e) = 0\)
Definition (stable equilibrium):

\[ \forall \varepsilon > 0, \exists \delta > 0 : \|x_0 - x_e\| < \delta \Rightarrow \|x(t) - x_e\| < \varepsilon, \forall t \geq 0 \]

\[ \|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \]

 execution starting from \( x(0) = x_0 \)

Graphically:

- **perturbed motion**
- **equilibrium motion**

small perturbations lead to small changes in behavior
Definition (asymptotically stable equilibrium):

\( \forall \varepsilon > 0, \exists \delta > 0 : \|x_0 - x_e\| < \delta \rightarrow \|x(t) - x_e\| < \varepsilon, \forall t \geq 0 \)

and \( \delta \) can be chosen so that \( \lim_{t \to \infty} (x(t) - x_e) = 0 \)

Graphically:

Small perturbations lead to small changes in behavior and are re-absorbed, in the long run.
Definition (asymptotically stable equilibrium):

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Graphically:

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and are re-absorbed, in the long run
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Let \( x_e \) be asymptotically stable.

Definition (domain of attraction):

The domain of attraction of \( x_e \) is the set of \( x_0 \) such that

\[ \lim_{t \to \infty} (x(t) - x_e) = 0 \]

Definition (globally asymptotically stable equilibrium):

\( x_e \) is globally asymptotically stable (GAS) if its domain of attraction is the whole state space \( \mathbb{R}^n \).
Lur'e problem

Given the transfer function $G(s)$ of the linear system $L$, determine necessary and/or sufficient conditions for the absolute stability of $S$ in the sector $[k_1, k_2]$.

Why is this problem meaningful?
A SIGNIFICANT EXAMPLE

Assumption: $g_P = g_T = 0$ and $g_R = 1$

Let $\bar{y}_0$, $\bar{d}_a$, $u_0$ be constant,

and denote with $\bar{x} = (\bar{x}_P, \bar{x}_T, \bar{x}_R)'$ the corresponding equilibrium.
Assumption: \( g_P = g_T = 0 \) and \( g_R = 1 \)

Let \( \vec{y}^o, \vec{d}_a, u_0 \) be constant,

and denote with \( \vec{x} = (\vec{x}_P, \vec{x}_T, \vec{x}_R)' \) the corresponding equilibrium.

\[
g_R = 1 \rightarrow \vec{c} = 0 \rightarrow \vec{y} = \vec{y}^o \rightarrow \vec{c} = \frac{\vec{y}^o}{\mu_T} \\
\vec{c} = \vec{m} \mu_P + \vec{d}_a \rightarrow \vec{m} = \frac{1}{\mu_P} (\vec{c} - \vec{d}_a) = \frac{1}{\mu_P} \left( \frac{\vec{y}^o}{\mu_T} - \vec{d}_a \right) \\
\vec{u} = \psi^{-1}(\vec{m}) \quad \vec{w} = \vec{u} - u_0
\]

Typical control design approach:

‘linear’ design + nonlinear analysis

(for instance, by simulation)
A SIGNIFICANT EXAMPLE

\[ \Sigma: \begin{array}{c}
\delta y \\
\delta u_0 \\
\delta u \\
\delta c
\end{array} \xrightarrow{\delta \Sigma} \begin{array}{c}
R(s) \\
\psi() \\
P(s)
\end{array} \xrightarrow{\delta \Sigma} \begin{array}{c}
d_a \\
T(s)
\end{array} \]

Linear design:
• build the system \( \delta \Sigma \) by linearizing \( \Sigma \) around the equilibrium associated with the constant inputs \( \bar{y}^0, \bar{d}_a, u_0 \)

\[ k := \frac{\partial \psi}{\partial u}(\bar{u}) \]

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\[ k := \frac{\partial \psi}{\partial u}(\bar{u}) \]
A SIGNIFICANT EXAMPLE

Linear design:
- build the system $\delta \Sigma$ by linearizing $\Sigma$ around the equilibrium associated with the constant inputs $\bar{y}^0, \bar{d}_a, u_0$
- choose $R(s)$ that makes $\delta \Sigma$ asymptotically stable

Different triples $\bar{y}^0, \bar{d}_a, u_0$ map into different equilibria for $\Sigma$.
Hence, the linear gain $k$ of the actuator is uncertain

$$k := \frac{\partial \psi}{\partial u} (\bar{u}) \in [k_{\text{min}}, k_{\text{max}}]$$
A SIGNIFICANT EXAMPLE

Different triples $\hat{y}^o$, $d_{\alpha}$, $u_0$ map into different equilibria for $\Sigma$. Hence, the linear gain $k$ of the actuator is uncertain

$$k := \frac{\partial \psi}{\partial u}(\bar{u}) \in [k_{\text{min}}, k_{\text{max}}]$$

$\Rightarrow$ robust linear control design needed to guarantee that $\delta \Sigma$ is asymptotically stable for every $k$ in the admissible range

Guarantees for $\delta \Sigma$:

every equilibrium of $\delta \Sigma$ associated with constant inputs is \textit{globally asymptotically stable} and the controlled variable will converge to the desired set-point after some suitable transient, irrespectively of the (constant) value of the disturbances
What about the nonlinear system $\Sigma$?

We need to verify that all equilibria associated with admissible constant inputs are globally asymptotically stable.
What about the nonlinear system $\Sigma$? We need to verify that all equilibria associated with admissible constant inputs are \textit{globally asymptotically stable}.

\[ \rightarrow \text{Lur'e problem} \]

Consider the constant input values $\bar{y}^\circ$, $\bar{d}_a$, $u_0$ and the corresponding equilibrium. We can then adopt the following expressions:

\[ x(t) = \bar{x} + \Delta x(t) \]
\[ e(t) = \bar{e} + \Delta e(t) \]
\[ w(t) = \bar{w} + \Delta w(t) \]
\[ u(t) = \bar{u} + \Delta u(t) \]
\[ m(t) = \bar{m} + \Delta m(t) \]
\[ c(t) = \bar{c} + \Delta c(t) \]
\[ y(t) = \bar{y} + \Delta y(t) \]
Consider the constant input values $\bar{y}^\circ$, $\bar{d}_a$, $u_0$ and the corresponding equilibrium. We can then adopt the following expressions:

\[
\begin{align*}
x(t) &= \bar{x} + \Delta x(t) & \Delta x(t) &:= x(t) - \bar{x} \\
e(t) &= \bar{e} + \Delta e(t) & \Delta e(t) &:= e(t) - \bar{e} \\
w(t) &= \bar{w} + \Delta w(t) & \Delta w(t) &:= w(t) - \bar{w} \\
u(t) &= \bar{u} + \Delta u(t) & \Delta u(t) &:= u(t) - \bar{u} \\
m(t) &= \bar{m} + \Delta m(t) & \Delta m(t) &:= m(t) - \bar{m} \\
c(t) &= \bar{c} + \Delta c(t) & \Delta c(t) &:= c(t) - \bar{c} \\
y(t) &= \bar{y} + \Delta y(t) & \Delta y(t) &:= y(t) - \bar{y}
\end{align*}
\]
A SIGNIFICANT EXAMPLE

\[ \Sigma \ast : \quad \Delta y = \Delta e \rightarrow R(s) \rightarrow \Delta w = \Delta u \rightarrow \varphi(\cdot) \rightarrow \Delta m \rightarrow P(s) \rightarrow \Delta c \]

\[ T(s) \]

\[ \Delta y^o(t) := y^o(t) - \bar{y}^o = 0 \]
\[ \Delta u_0 := u_0(t) - u_0 = 0 \]
\[ \Delta d_a(t) := d_a(t) - \bar{d}_a = 0 \]

(constant) inputs keep unchanged

A SIGNIFICANT EXAMPLE

\[ \Sigma \ast : \quad \Delta e \rightarrow R(s) \rightarrow \Delta w = \Delta u \rightarrow \varphi(\cdot) \rightarrow \Delta m \rightarrow P(s) \rightarrow \Delta c \]

\[ \Delta y \]

\[ T(s) \]

autonomous system
A SIGNIFICANT EXAMPLE

System $\Sigma^*$ in compact form:

$\Sigma^*:
\begin{align*}
&\Delta u \\
\varphi(\cdot) &\rightarrow \Delta m \\
L &\rightarrow \\
\end{align*}$

$L: G(s) = P(s)T(s) R(s)$
$\rightarrow$ Lur'e autonomous system

- Given that $x(t) = \bar{x} + \Delta x(t)$, then, the global asymptotic stability of the equilibrium $\bar{x}$ of $\Sigma$ is equivalent to that of the equilibrium $\Delta x = 0$ of $\Sigma^*$
- Function $\varphi(\cdot)$ depends on $\bar{x}$
A SIGNIFICANT EXAMPLE

\[ \Sigma^*: \quad \begin{array}{c}
\Delta u \\
\varphi(\cdot)
\end{array} \rightarrow \begin{array}{c}
\Delta m \\
L
\end{array} \]

L: \( G(s) = P(s)T(s)R(s) \)

\( \Rightarrow \) autonomous Lur’e system with \( \varphi(\cdot) \in \Phi_{[k_1,k_2]} \)

Conclusions:

If \( \Sigma^* \) is absolutely stable in the sector \([k_1, k_2]\), then, all equilibria of \( \Sigma \) are globally asymptotically stable
LUR’E PROBLEM

Lur’e problem
determine necessary and/or sufficient conditions for the absolute
stability of $S$ in some sector $[k_1, k_2]$.

A NECESSARY CONDITION

$\varphi(\cdot) \in \Phi_{[k_1, k_2]}$
A NECESSARY CONDITION

\[ S : \quad e \xrightarrow{\varphi(\cdot)} u \xrightarrow{G(s)} y \]

\[ \varphi(\cdot) \in \Phi_{[k_1,k_2]} \]

Admissible sector functions can be linear:

\[ \varphi(e) = ke \in \Phi_{[k_1,k_2]} \]

A NECESSARY CONDITION

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\[ S_L : \quad e \xrightarrow{k} u \xrightarrow{G(s)} y \]
A NECESSARY CONDITION

$$S : \begin{array}{c}
\varphi(\cdot) \\
\varphi(\cdot) \in \Phi_{[k_1, k_2]} \\
\end{array}$$

Admissible sector functions can be linear:

$$\varphi(e) = ke \in \Phi_{[k_1, k_2]}$$

$$S_L : \begin{array}{c}
k \\
\end{array}$$

If $S$ is absolutely stable in $[k_1, k_2]$, then, $S_L$ is (globally) asymptotically stable for any $k \in [k_1, k_2]$.

If $0 \in [k_1, k_2]$, then, system $L$ with t.f. $G(s)$ is asymptotically stable
For a given $k$,
System $S_L$ is asymptotically stable if and only if the Nyquist plot of $G(s)$ encircles (anti-clockwise) the point in the complex plane corresponding to the real number $-1/k$ as many times as the number of poles of $G(s)$ with positive real part
(Nyquist criterion)
ROBUST ASYMPTOTIC STABILITY OF $S_L$

If $k \in [k_1, k_2] \rightarrow$ robust stability of $S_L$

$0 \leq k_1 < k_2$

\[ G(j\omega) \]

\[ I(k_1, k_2) := \{ \alpha \in \mathbb{R} : \alpha = -\frac{1}{k}, k \in [k_1, k_2] \} \]
ROBUST ASYMPTOTIC STABILITY OF $S_L$

If $k \in [k_1, k_2] \rightarrow$ robust stability of $S_L$

$$0 \leq k_1 < k_2$$

$k_1 < 0 < k_2$

$I(k_1, k_2) := \{ \alpha \in \Re : \alpha = -\frac{1}{k}, k \in [k_1, k_2] \}$

ROBUST ASYMPTOTIC STABILITY OF $S_L$

$k$ uncertain, $k \in [k_1, k_2]$.

System $S_L$ is asymptotically stable for any $k \in [k_1, k_2]$ if and only if the Nyquist plot of $G(s)$ encircles (anti-clockwise) $I(k_1, k_2)$ as many times as the number of poles of $G(s)$ with positive real part.
A NECESSARY CONDITION

Theorem (necessary condition)

If $S$ is absolutely stable in the sector $[k_1, k_2]$, then the Nyquist plot of $G(s)$ encircles (anti-clockwise) $\mathcal{I}(k_1, k_2)$ as many times as the number of poles of $G(s)$ with positive real part. In particular, if $0 \in [k_1, k_2]$, then system $L$ with transfer function $G(s)$ is asymptotically stable.