Switched systems: stability

OUTLINE

Switched Systems

Stability of Switched Systems
SWITCHED SYSTEMS

• a family of systems

\[ \dot{x} = f_p(x), \ p \in \mathcal{P} = \{1, 2, \ldots, m\} \]
SWITCHED SYSTEMS

• a family of systems

\[ \dot{x} = f_p(x), \ p \in \mathcal{P} = \{1, 2, \ldots, m\} \]

• a signal that orchestrates the switching between them

![Diagram of a switched system]

GRAPH REPRESENTATION

\[ \dot{x} = f_1(x) \]
\[ \dot{x} = f_2(x) \]
\[ \dot{x} = f_\sigma(x), \ \sigma \in \mathcal{P} = \{1, 2\} \]
SWITCHED LINEAR SYSTEMS

- a family of linear systems
  \[ \dot{x} = A_p x, \quad p \in \mathcal{P} \]
- a signal that orchestrates the switching between them

SWITCHING

- time-dependent versus state-dependent switching
- autonomous versus controlled switching
SWITCHING

- time-dependent versus state-dependent switching
- autonomous versus controlled switching

TIME-DEPENDENT SWITCHING

\[ \sigma : [0, \infty) \rightarrow \mathcal{P} \]  
(exogenous) switching signal

- piecewise constant function of time
- \( \sigma(t) \) specifies the system that is active at time \( t \)
the state space $X$ is partitioned into operating regions, each one associated to a system

$\sigma : X \to \mathcal{P}$ \hspace{1cm} \text{(endogenous) switching signal}

- the state space $X$ is partitioned into operating regions, each one associated to a system
- $\sigma(x)$ specifies the system that is active when the state is $x$
SWITCHING

- time-dependent versus state-dependent switching
- autonomous versus controlled switching

AUTONOMOUS SWITCHING

- Switching events are triggered by an external mechanism over which we do not have control

Examples:
  - unpredictable environmental factors
  - component failures
CONTROLLED SWITCHING

• Switching are imposed so as to achieve a desired behavior of the resulting system \( \rightarrow \) switched control systems

SWITCHING CONTROL

- Diagram showing a bank of controllers, a supervisor, and a process, with logic that selects which controller to use.
SWITCHING CONTROL

The closed-loop system is a switched system

SWITCHING CONTROL

Reasons for switching:
• nature of the control problem (system with different operation phases)
SWITCHING CONTROL

Reasons for switching:
• nature of the control problem (system with different operation phases)
  Example: flight control system

SWITCHING CONTROL

Reasons for switching:
• large modeling uncertainty
SWITCHING CONTROL

Reasons for switching:

• large modeling uncertainty
  Example: adaptive switching control

\[ \mathcal{P} = \text{admissible model set} \]

SWITCHING CONTROL

Reasons for switching:

• sensor/actuator limitations
SWITCHING CONTROL

Reasons for switching:

• sensor/actuator limitations

Example: quantized control

![Diagram of switching control system]

EXAMPLE: THERMOSTAT

Temperature in a room controlled by a thermostat switching a heater on and off

Dynamics of the temperature $x$ (in °C):

heater on: $\dot{x} = -0.2x + 6$ (x → 30)

heater off: $\dot{x} = -0.2x$ (x → 0)

**Goal:** regulate the temperature around 20°C

**Strategy:**
- turn the heater from OFF to ON as soon as $x \leq 18$
- turn the heater from ON to OFF as soon as $x \geq 22$

![Diagram of thermostat dynamics and hysteretic behavior]
EXAMPLE: THERMOSTAT

- Continuous dynamics
  \[ \dot{x} = -0.2x \quad \dot{x} = -0.2x + 6 \]
  linear ODEs describing the temperature evolution

- Discrete dynamics
  finite automaton describing the behavior of the thermostat

\[ Q = \{\text{ON, OFF}\} \quad \text{ON} = \Phi(\text{OFF}, e_1) \quad e_1 = [x \leq 18] \]
\[ \text{OFF} = \Phi(\text{ON}, e_2) \quad e_2 = [x \geq 22] \]
EXAMPLE: THERMOSTAT

Evolution of the temperature $x$ starting from the initial condition $x(0) = 5$ with the heater ON.

$x$ generates the events causing a transition from OFF to ON $x \leq 18$ from ON to OFF $x \geq 22$.

Evolution of the heater status starting from the initial condition $x(0) = 5$ with the heater ON.

A heater transition causes a switch in the dynamics governing $x$.
EXAMPLE: THERMOSTAT

\[ \frac{dx}{dt} = -0.2 \cdot x + u \]

interface

quantized control input

(ON → heating power \( u = 6 \))

(OFF → heating power \( u = 0 \))

continuous systems controlled by a discrete logic
SWITCHING CONTROL

Reasons for switching:

- nature of the control problem (system with different operation phases)
- large modeling uncertainty
- sensor/actuator limitations
- ...

OUTLINE

Switched Systems

Stability of Switched Systems
SWITCHED SYSTEMS: EQUILIBRIUM

\[ \dot{x} = f_\sigma(x) \]

• family of systems

\[ \dot{x} = f_p(x), \ p \in \mathcal{P} = \{1, 2, \ldots, m\} \]
with \[ f_p(0) = 0, \forall p \in \mathcal{P} \]

\[ \Rightarrow \ x = 0 \text{ is an equilibrium of the switched system} \]

Stability of the equilibrium \(x=0\)?
SWITCHING BETWEEN AS. STABLE LINEAR SYSTEMS

\[ \dot{x} = A_\sigma x \]

\[ \dot{x} = A_1 x \quad \dot{x} = A_2 x \]

SWITCHING BETWEEN AS. STABLE LINEAR SYSTEMS

\[ \dot{x} = A_\sigma x \]

\[ \dot{x} = A_1 x \quad \dot{x} = A_2 x \quad x = 0 \text{ is unstable!} \]
SWITCHING BETWEEN AS. STABLE LINEAR SYSTEMS

\[ \dot{x} = A_\sigma x \]

\[ \dot{x} = A_1 x \quad \dot{x} = A_2 x \]

\[ x = 0 \text{ is unstable!} \]

\[ \|x\| \text{ overshoots sum up} \]

\[ t \]

Problem: find conditions that guarantee asymptotic stability under arbitrary switching
SWITCHING BETWEEN AS. STABLE LINEAR SYSTEMS

\[ \dot{x} = A_\sigma x \]

\[ \dot{x} = A_1 x \quad \dot{x} = A_2 x \quad x = 0 \text{ is stable!} \]

Problem: identify those switching signals that preserve asymptotic stability
Problem: identify those switching signals that ensure asymptotic stability
Stability for arbitrary switching

Stability for constrained switching

Stability for arbitrary switching

Stability for constrained switching
GLOBAL UNIFORM ASYMPTOTIC STABILITY (GUAS)

\[ \dot{x} = f_\sigma(x) \]

\[ f_q(0) = 0, \quad q \in Q = \{1, 2, \ldots, m\} \]

The equilibrium \( x_e = 0 \) is GUAS if it is globally asymptotically stable, uniformly with respect to the switching signals \( \sigma \).

Assumption:
\[ \dot{x} = f_p(x), \quad p \in P = \{1, 2, \ldots, m\} \]

family of systems with GAS equilibrium in \( x = 0 \)

Remark:
if the equilibrium \( x = 0 \) is not GAS for one of the systems, then it cannot be GUAS for the switched system.
COMMON LYAPUNOV FUNCTION

**Def.** The family of systems

\[ \dot{x} = f_p(x), \ p \in \mathcal{P} = \{1, 2, \ldots, m\} \]

share a radially unbounded common Lyapunov function at \( x = 0 \) if there exists a continuously differentiable function \( V: X \to \mathbb{R} \) such that

\[
V(x) > 0, \ \forall x \neq 0 \quad V(0) = 0
\]

\[
||x|| \to \infty \Rightarrow V(x) \to \infty
\]

\[
\frac{dV}{dx}(x)f_p(x) < 0, \ \forall x \neq 0, \ \forall p \in \mathcal{P}
\]

COMMON LYAPUNOV FUNCTION

\[ \dot{x} = f_\sigma(x) \]

If all systems in the family

\[ \dot{x} = f_p(x), \ p \in \{1, 2, \ldots, m\} \]

share a radially unbounded common Lyapunov function \( V: X \to \mathbb{R} \) at \( x = 0 \), then, the equilibrium \( x = 0 \) is GUAS.

**Proof.** Same reasoning as in standard Lyapunov theorem
COMMON LYAPUNOV FUNCTION

\[
\dot{V}(x) = \frac{dV(x(t))}{dt} < 0
\]

SWITCHED LINEAR SYSTEMS

- family of systems
  \[
  \dot{x} = A_p x, \quad p \in \mathcal{P} = \{1, 2, \ldots, m\}
  \]

- time-dependent switching rule
  \[
  \sigma : [0, \infty) \rightarrow \mathcal{P}
  \]
COMMON QUADRATIC LYAPUNOV FUNCTION

\[ \dot{x} = A_\alpha x \]

If there exists \( P = P^T > 0 \)
\[ PA_\alpha + A_\alpha^T P < 0, \quad \forall p \in \mathcal{P} = \{1, 2, \ldots, m\} \]
then, the equilibrium \( x = 0 \) is GUAS.

**Proof.** \( V(x) = x^T P x \) is a radially unbounded common Lyapunov function at \( x = 0 \).

**Remark:** A set of LMIs to solve. This problem can be reformulated as a convex optimization problem. Efficient solvers exist.

GLOBALLY QUADRATIC LYAPUNOV FUNCTION

\[ \dot{x} = A_\alpha x \]

The existence of a globally quadratic Lyapunov function is not necessary for \( x = 0 \) to be GUAS.

**Example:**
\[
A_1 = \begin{bmatrix}
-1 & -1 \\
1 & -1
\end{bmatrix} \quad A_2 = \begin{bmatrix}
-1 & -10 \\
0.1 & -1
\end{bmatrix}
\]

\( x = 0 \) is GUAS but there is no common quadratic Lyapunov function.
SWITCHED LINEAR SYSTEMS WITH A SPECIAL STRUCTURE

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_p, \ p \in \mathcal{P} = \{1, 2, \ldots, m\} \)

- commute
- are upper (or lower) triangular

COMMUTING HURWITZ MATRICES \( \rightarrow \) GUAS

\[ \dot{x} = A_\sigma x \]

\[ \mathcal{P} = \{1, 2\} \quad A_1A_2 = A_2A_1 \]

\[
\begin{array}{cccccc}
\sigma = 1 & \sigma = 2 & \sigma = 1 & \sigma = 2 & \ldots & t \\
\sigma_1 & \sigma_2 & \sigma_1 & \sigma_2 & \ldots & t \\
\end{array}
\]

\[ x(t) = e^{A_2t_k} e^{A_1s_k} \cdots e^{A_2t_1} e^{A_1s_1} x(0) \]

\[ = e^{A_2(t_k+s_k+\ldots+t_1)} e^{A_1(s_k+s_1+\ldots+s_1)} x(0) \rightarrow 0 \]
\[ \dot{x} = A_\alpha x \]
\[ P = \{1, 2\} \quad A_1 A_2 = A_2 A_1 \]

\exists \text{quadratic common Lyapunov function: } V(x) = x^T P_2 x
\[ P_1 A_1 + A_1^T P_1 = -I \]
\[ P_2 A_2 + A_2^T P_2 = -P_1 \]
\[
\dot{x} = A_\sigma x
\]

\[\mathcal{P} = \{1, 2\}, \ X = \mathbb{R}^2\]

\[
\begin{align*}
\dot{x}_1 &= \lambda_{1,\sigma} x_1 + a_{\sigma} x_2 \\
\dot{x}_2 &= \lambda_{2,\sigma} x_2
\end{align*}
\]

\[
\dot{x}_2 = \lambda_{2,\sigma} x_2 \Rightarrow |x_2(t)| \leq e^{\max_p \lambda_{2,p} t} |x_2(0)| \to 0
\]
TRIANGULAR HURWITZ MATRICES $\rightarrow$ GUAS

$$\dot{x} = A_\sigma x$$

$P = \{1, 2\}, \quad X = \mathbb{R}^2$

$$\begin{align*}
\dot{x}_1 &= \lambda_{1, \sigma} x_1 + a_\sigma x_2 \\
\dot{x}_2 &= \lambda_{2, \sigma} x_2
\end{align*}$$

$$\dot{x}_2 = \lambda_{2, \sigma} x_2 \Rightarrow |x_2(t)| \leq e^{\max_p \lambda_{2, p} t} |x_2(0)| \rightarrow 0$$

$$\dot{x}_1 + a_\sigma x_2 \Rightarrow x_1(t) \rightarrow 0$$

exponentially stable system

exponentially decaying perturbation

TRIANGULAR HURWITZ MATRICES $\Rightarrow$ GUAS

$$\dot{x} = A_\sigma x$$

$Q = \{1, 2\}, \quad X = \mathbb{R}^2$

$$\begin{align*}
\dot{x}_1 &= \lambda_{1, \sigma} x_1 + b_\sigma x_2 \\
\dot{x}_2 &= \lambda_{2, \sigma} x_2
\end{align*}$$

$\exists$ quadratic common Lyapunov function

$$V(x) = x^T P x$$

with $P$ diagonal
SWITCHED SYSTEMS WITH A SPECIAL STRUCTURE

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, \ q \in Q = \{1, 2, \ldots, m\} \)

- commute
- are upper (or lower) triangular
- can be transformed to upper (or lower) triangular form by a common similarity transformation

- Stability for arbitrary switching
- Stability for constrained switching
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_{\sigma}x \]

Hurwitz matrices \( A_q, q \in Q = \{1, 2\} \)

\[
\begin{array}{c|c|c|c|c|c}
\sigma = 1 & \sigma = 2 & \sigma = 1 & \sigma = 2 & \cdots \\
\hline
s_1 & t_1 & s_2 & t_2 & \cdots \\
\end{array}
\]

The switching intervals satisfy \( t_i, s_i \geq \tau_D \)

dwell time

\[ x(t) = e^{A_2t_k} e^{A_1s_k} \cdots e^{A_2t_1} e^{A_1s_1} x(0) \]
STABILITY OF LINEAR CONTINUOUS SYSTEMS

\[ \dot{x}(t) = Ax(t) \]

Theorem (exponential stability):

Let the equilibrium point \( x_e = 0 \) be asymptotically stable. Then, the rate of convergence to \( x_e = 0 \) is exponential:

\[ \|x(t)\| \leq \mu e^{-\lambda_0 t} \|x_0\|, \quad t \geq 0 \]

for all \( x(0) = x_0 \in \mathbb{R}^n \), where \( \lambda_0 \in (0, \min \{\text{Re}\{\lambda_i(A)\}\}) \) and \( \mu > 0 \) is an appropriate constant.

Remark:

\[ \|x(t)\| = \|e^{At}x_0\| \leq \mu e^{-\lambda_0 t} \|x_0\|, \quad t \geq 0, \quad \forall x_0 \]

\[ \rightarrow \|e^{At}\| = \sup_{x_0 \neq 0} \frac{\|e^{At}x_0\|}{\|x_0\|} \leq \mu e^{-\lambda_0 t}, \quad t \geq 0 \]

STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, \quad q \in Q = \{1, 2\} \)

\[
\begin{align*}
\sigma &= 1, & \sigma &= 1, & \sigma &= 2, & \sigma &= 2, & \cdots \\
S_1 & \quad t_1 & S_2 & \quad t_2 & \quad t
\end{align*}
\]

The switching intervals satisfy \( t_i, S_i \geq \tau_D \)

\[ x(t) = e^{A_{2t_k} e^{A_{1s_k}} \cdots e^{A_{2t_1} e^{A_{1s_1}}}} x(0) \]

\[ \|e^{A_{2t}}\| \leq \mu \leq e^{-\lambda_0 \tau_D} = e^{\frac{-\lambda_0 \tau_D + \log \mu}{2}} \]

slowest decay rate so that the inequality holds \( \forall i \)
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_\sigma x \]

Hurwitz matrices \( A_q, q \in Q = \{1, 2\} \)

\[
\begin{array}{cccccc}
\sigma = 1 & \sigma = 2 & \sigma = 1 & \sigma = 2 & \cdots & t \\
s_1 & t_1 & s_2 & t_2 & & \\
\end{array}
\]

The switching intervals satisfy \( t_i, s_i \geq \tau_D \)

\[
x(t) = e^{A_2 t_k} e^{A_1 s_k} \cdots e^{A_2 t_1} e^{A_1 s_1} x(0) \\
\left| e^{A_\Delta t} \right| \leq e^{-\lambda_0 \tau_D + \log \mu} \leq e^{-2 \tau_D} < 1 \\
\lambda \in (0, \lambda_0)
\]

\[
\tau_D \geq \frac{\log \mu}{\lambda_0 - \lambda}
\]
STABILITY UNDER SLOW SWITCHING

\[ \dot{x} = A_{\sigma}x \]

Hurwitz matrices \( A_q, \ q \in Q = \{1, 2\} \)

\[
\begin{array}{c|c|c|c|c|c}
\sigma = 1 & \sigma = 2 & \sigma = 1 & \sigma = 2 & \ldots \\
\hline
s_1 & t_1 & s_2 & t_2 & \\
\end{array}
\]

\[ x(t) = e^{A_2t_k}e^{A_1s_k} \ldots e^{A_2t_1}e^{A_1s_1}x(0) \]

\[ \|e^{A_i\Delta t}\| \leq e^{-\lambda \Delta t} \quad \Rightarrow \quad \|x(t)\| \leq e^{-\lambda t}\|x(0)\| \]

STABILITY UNDER STATE-DEPENDENT SWITCHING

\[ \sigma: X \rightarrow Q : \ \sigma(x) = i \text{ if } x \in X_i \]
STATE-DEPENDENT COMMON LYAPUNOV FUNCTIONS

If $V: \mathbb{R}^n \to \mathbb{R}$ is a $C^1$ radially unbounded function such that

\[
\begin{align*}
V(0) &= 0 \\
V(x) &> 0, \forall x \in \mathbb{R}^n \setminus \{0\} \\
\frac{\partial V}{\partial x}(x)A(x)(x) &< 0, \forall x \in \mathbb{R}^n
\end{align*}
\]

then, $x = 0$ is GAS for $\dot{x} = A_\sigma(x)x$

Remarks:

need that $\frac{\partial V}{\partial x}(x)A_q(x) < 0$ only when $\sigma = q$, i.e. on $X_q$

matrices $A_q$ are not required to be Hurwitz

STABILIZATION BY SWITCHING

$\dot{x} = A_1 x$, $\dot{x} = A_2 x$ – both unstable

Assume: $A = \alpha A_1 + (1-\alpha)A_2$ Hurwitz for some $\alpha \in (0,1)$
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{both unstable} \]

Assume: \[ A = \alpha A_1 + (1 - \alpha) A_2 \quad \text{Hurwitz for some } \alpha \in (0,1) \]

\[ A^T P + PA < 0 \]
STABILIZATION BY SWITCHING

\( \dot{x} = A_1 x, \dot{x} = A_2 x \) — both unstable

Assume: \( A = \alpha A_1 + (1-\alpha)A_2 \) Hurwitz for some \( \alpha \in (0,1) \)

\( \alpha (A_1^T P + PA_1) + (1-\alpha)(A_2^T P + PA_2) < 0 \)

So for each \( x \neq 0 \):

either \( x^T (A_1^T P + PA_1) x < 0 \) or \( x^T (A_2^T P + PA_2) x < 0 \)

Region where system 1 is active

Region where system 2 is active

\( V(x) = x^T P x \) is a Lyapunov function at \( x = 0 \)

for the system \( \dot{x} = A_{\sigma(x)} x \) \( \Rightarrow \) GAS
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{both unstable} \]

Assume: \( A = \alpha A_1 + (1-\alpha) A_2 \) Hurwitz for some \( \alpha \in (0,1) \)

\[ \alpha(A_1^T P + PA_1) + (1-\alpha)(A_2^T P + PA_2) < 0 \]

So for each \( x \neq 0 \):

either \( x^T (A_1^T P + PA_1) x < 0 \) or \( x^T (A_2^T P + PA_2) x < 0 \)

Region where system 1 is active
Region where system 2 is active
STABILIZATION BY SWITCHING

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x \quad \text{-- both unstable} \]

**Theorem:**
If the matrices $A_1$ and $A_2$ have a Hurwitz combination, then, there exists a state dependent switching strategy such that the switching system $\dot{x} = A_\sigma x$ is GAS

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Theorem:
If the matrices $A_1$ and $A_2$ have a Hurwitz combination, then, there exists a state dependent switching strategy such that the switching system $\dot{x} = A_\sigma x$ is GAS

**Extensions to the m>2 matrices case:**
- two matrices $A_i$ and $A_j$ have a Hurwitz combination
- more than 2 matrices have a Hurwitz combination
Main source:

*Switching in Systems and Control*