Exercises on random variables

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Exercises

Exercise 1

The (normalized) temperature of an engine is a random variable whose probability density function is:

$$f(x) = n(1-x)^{n-1} \tag{1}$$

if $0 \le x \le 1$, 0 otherwise.

Show that $f(x)$ is a valid PDF.

Solution

It's enough to check that the integral of the PDF is equal to 1:

$$\int_0^1 n(1-x)^{n-1}dx = (1-x)^n|_0^1 = 1 \tag{2}$$

Exercise 2

A random variable X had pdf:

$$f(x) = cx^2(1-x) \tag{3}$$

if $0 \le x \le 1$, 0 otherwise.

1. Determine c so that $f(x)$ is a valid PDF
2. Find the expected value and variance of X

Solution

1. $$\int_0^1 cx^2(1-x) = 1 \Rightarrow c(\frac{x^3}{3} - \frac{x^4}{4})|_0^1 \Rightarrow c = 12 \tag{4}$$

2. $$E[X] = 12\int_0^1 x^3(1-x) = 3/5 \tag{5}$$

$$Var[X] = E[X^2] - E[X]^2 \tag{6}$$

$$E[X^2] = 12\int_0^1 x^4(1-x) = 2/5 \tag{7}$$

$$Var[X] = 2/5 - 9/25 = 1/25 \tag{8}$$

Exercise 3

The lifetime of an automobile battery is described by a r.v. X having negative exponential distribution with parameter $\lambda = \frac{1}{3}$.

1. Determine the expected lifetime of the battery and the variation around this mean.
2. Calculate the probability that the lifetime will be between 2 and 4 time units.
3. If the battery has lasted for 3 time units, what is the probability that it will last for at least an additional time unit?

Solution

1. For the negative exponential r.v. we have that $E[X] = \frac{1}{\lambda}$ and $Var[X] = \frac{1}{\lambda^2}$. Therefore $E[X] = 3$, $Var[X] = \sigma_X^2 = 9$ and $\sigma_X = \sqrt{\sigma_X^2} = 3$.
2. The cumulative density function of x is:
$$F(x) = \int_0^x \lambda e^{-\lambda x} = -e^{-\lambda x}|_0^x = 1 - e^{-\frac{x}{3}} \tag{9}$$
therefore:
$P(2 < X < 4) = P(X \leq 4) - P(X \leq 2) = F(4) - F(2) = (1 - e^{-\frac{4}{3}}) - (1 - e^{-\frac{2}{3}}) = 0.252$
3. The required (conditional) probability is $P(X > 4|X > 3)$. This is exactly equal as $P(X > 1)$, by the memoryless property of the exponential distribution. Therefore we have:
$P(X > 4|X > 3) = P(X > 1) = 1 - P(X \leq 1) = 1 - F(1) = e^{-\frac{1}{3}} = 0.716$

Exercise 4

A roulette wheel consists of 18 black slots, 18 red slots and 2 green slots. If a gambler bets 10 USD on red, what is the gambler expected gain or loss?

Solution

Define the rv. X as: $X = 10$ with probability 18/38 and $X = -10$ with probability 20/38. Therefore, $E[X] = 10 \times 18/38 - 10 \times 20/38 =$ -0.526. Therefore a gambler loses 53 cents every 10 dollars played.

Exercise 5

Suppose that 15 people, chosen at random from a target population, are asked if they are in favour of a certain proposal. If 43.75% of the target population are in favour the proposal, calculate the probability that at least 5 of the 15 polled favour the proposal

Solution

Let X bet he number of those favouring the proposal, then X is a binomial random variable with $p = 0.4375$. Therefore: $P(X \geq 5) = 1 - P(X \leq 4) = 1 - (P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)) =$
$= 1 - (\binom{15}{0} p^0 (1-p)^{15} + \binom{15}{1} p^1 (1-p)^{14} + \binom{15}{2} p^2 (1-p)^{13} + \binom{15}{3} p^3 (1-p)^{12} + \binom{15}{4} p^4 (1-p)^{11}) = 0.859$

Exercise 6

The r.v. X has p.d.f f given by:

$$f(X) = \begin{cases} 0 & \text{if } x < 4 \\ 0.1 & \text{if } x = 4 \\ 0.3 & \text{if } x = 5 \\ 0.3 & \text{if } x = 6 \\ 0.2 & \text{if } x = 8 \\ 0.1 & \text{if } x = 9 \\ 0 & \text{if } x > 9 \end{cases}. \tag{10}$$

1. Calculate the probabilities $P(X \leq 6.5), P(X > 8.1), P(5 < X < 8)$.
2. Calculate the c.d.f $F(X)$
3. Calculate $E[X]$ and $Var[X]$.

Solution

1. $P(X \leq 6.5) = 0.7, P(X > 8.1) = 0.1, P(5 < X < 8) = 0.3$.
2. $$F(X) = \begin{cases} 0 & \text{if } x < 4 \\ 0.1 & \text{if } 4 \leq X < 5 \\ 0.4 & \text{if } 5 \leq X < 6 \\ 0.7 & \text{if } 6 \leq X < 8 \\ 0.9 & \text{if } 8 \leq X < 9 \\ 1.0 & \text{if } x \geq 9 \end{cases}. \tag{11}$$
3. $E[X] = 0.1 * 4 + 0.3 * 5 + 0.3 * 6 + 0.2 * 8 + 0.1 * 9 = 6.2$
4. $E[X^2] = 0.1 * 16 + 0.3 * 25 + 0.3 * 36 + 0.2 * 64 + 0.1 * 81 = 40.8$, $Var[X] = 40.8 - 6.2^2 = 2.36$