A Nonlinear Dynamical Model for the Dynastic Cycle*

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The intervals between the fall of one dynasty and the rise of another are . . . accompanied by a time of chaos during which population is reduced considerably.

Usher [1, p. 1033]

Abstract—A three-class model of society (farmers, bandits and rulers) is considered in order to explain alternation between despotism and anarchy in ancient China. In the absence of authority, the dynamics of farmers and bandits are governed by the well-known prey–predator interactions. Rulers impose taxes on farmers and punish bandits by execution. Thus, farmers are a sort of renewable resource which is exploited both by bandits and by rulers. Assuming that the dynamics of rulers is slow compared with those of farmers and bandits, slow–fast limit cycles can be identified through a singular perturbation approach. This provides a possible explanation for the accomplishment of an endogenously generated dynastic cycle, i.e. a periodic switching of society between despotism and anarchy. Moreover, there is numerical evidence for the occurrence of a cascade of period doubling bifurcations leading to chaotic behaviour.

1. INTRODUCTION

According to Usher [1] dynastic cycle is a periodic alternation of the society between despotism and anarchy. As illustration, he mentions the Chinese history: a new line of emperors creates a regime of peace and prosperity. When the dynasty becomes old and feeble, a downswing takes place which is accompanied by civil war, misery, and population decline.

Usher [1] explores possible explanations of the formation and disintegration of dynasties. He considers three social classes: farmers who produce a product, bandits who steal this
product, and rulers fighting against banditry and taxing farmers. Two states of the society are identified: anarchy, when an equilibrium develops between bandits who steal and farmers who defend their crops in the absence of any authority, and despotism, when rulers tax farmers and execute bandits to protect their tax revenue. Violence acts as a determinant of the distribution of income: indeed, the most fundamental departure in Usher's model of anarchy and despotism from the usual economic analysis lies in the usage of mortality rates which are associated with violence.

Based on Usher's three-class society, Feichtinger and Novak [2] consider a one-dimensional differential game between bandits and rulers competing for the stock of farmers, where the latter can be interpreted as a renewable resource. In this paper, on the contrary, no explicit decision-making is taken into account, but a three-dimensional model is considered instead, in order to explore the possibility of an endogenously generated switching of the society between periods of despotism and anarchy. More precisely, the state variables of the model represent the number of farmers, bandits and rulers. In the absence of the latter, it is supposed that farmers and bandits interact dynamically according to the usual prey–predator relationship. Moreover, rulers tax farmers and punish bandits by execution.

The paper is organized as follows. Section 2 states the model, which has some unusual ingredients. In Section 3, the interaction of farmers and bandits is studied by assuming a constant number of rulers. A two-dimensional bifurcation analysis is provided by using the size of the rulers’ class and the efficiency of the bandits as parameters. Next, in Section 4 we get some insights into the behaviour of the full model by assuming that the dynamics of the rulers in slow compared with that of farmers and bandits. This assumption is not unrealistic in actual historical situations and provides a powerful tool of analysis (singular perturbation theory). In Section 5, the slow-fast assumption is relaxed, and a numerical bifurcation analysis is carried out on the three-dimensional model. It turns out that there exist parameter values for which the system displays a chaotic behaviour. The results are interpreted in the concluding Section 6, where some ideas for future research are also discussed.

2. THE MODEL

As in Usher [1] a society consisting of three classes is considered: farmers, bandits, and rulers. Farmers are assumed to grow logistically due to the limitation of natural resources. They may be conceived as a renewable resource exploited by bandits as well as by rulers, who tax them. Moreover, bandits are hunted by the rulers who execute them when caught. The growth of the rulers is governed both by the number of farmers to protect and by the bandits to fight. Both bandits and rulers exhibit natural mortality or retirement.

Denoting by \( X(t) \), \( Y(t) \), and \( Z(t) \) the number of farmers, bandits, and rulers, respectively, the dynamics of the system is described by the following system of ordinary differential equations:

\[
\begin{align*}
\dot{X} &= rX \left( 1 - \frac{X}{k} \right) - \frac{aXY}{b + X} - hXZ \quad \text{(la)} \\
\dot{Y} &= e \frac{aXY}{b + X} - mY - \frac{cYZ}{d + Y} \quad \text{(1b)} \\
\dot{Z} &= f \frac{aXY}{b + X} - gZ \quad \text{(1c)}
\end{align*}
\]
where all the parameters are positive. It can easily be checked that if \( X(0), Y(0), Z(0) \geq 0 \) then \( X(t), Y(t), Z(t) \geq 0 \) for all \( t \geq 0 \), i.e. system (1) is positive.

The equation for \( X(t) \) points out that, in the absence of bandits and rulers \((Y = Z = 0)\), the farmers have a logistic growth, \( k \) being their carrying capacity, while \(-aXY/(b + X)\) is a mortality rate due to predation by bandits: the form adopted, well known among ecologists [3], reflects the existence of a non-zero searching time spent by bandits in locating the most convenient objective to attack. The term \(-hXZ\) is a mortality rate due to subtraction of resources by the rulers via taxation. In (1b) the predation \( aXY/(b + X) \) is transformed into a growth rate by a conversion factor \( e \), while \(-mY\) represents the rate of bandits naturally leaving the system. The term \(-cYZ/(d + Y)\) is a mortality rate due to the action of the rulers: again, its saturating form reflects the difficulty that rulers have in finding bandits when these are not too many. In the equation for \( Z(t) \) the hiring rate is proportional, by a factor \( f \), to the rate of predation \( aXY/(b + X) \) of the farmers by the bandits, which is a measure of the rate of criminality existing in the system at time \( t \), while the term \(-gZ\) is the rate of rulers naturally leaving their job. Notice that bandits are not considered as a cause of mortality for rulers, since it is assumed that training and equipment of the latter are so high-level that they have a negligible probability of dying in a fight.

It is worthwhile to note that system (1) has relevant analogies with what in ecology is called a food-chain, i.e. a system describing the interactions among plant and/or animal species where each species gets its resources by predation of lower levels and, in turn, is a resource for higher levels (see Rosenzweig [4] and Hastings and Powell [5] for an early and a recent contribution on the subject). In this sense, farmers, bandits, and rulers can be considered, respectively, as prey, predators, and superpredators. Notice that the latter exploit both the other species. The main peculiarity of system (1) with respect to standard food-chains lies in the form of (1c) which, as already pointed out, relates the hiring rate of the rulers (fixed by the authority) to the criminality rate.

System (1) can be rescaled in order to reduce the number of parameters. By defining a new time variable \( t \rightarrow rt \) and

\[
\begin{align*}
x &= \frac{X}{k} \\
y &= \frac{aY}{kr} \\
z &= \frac{acZ}{kmr} \\
B &= \frac{b}{k} \\
H &= \frac{hkm}{ac} \\
Q &= \frac{m}{r} \\
E &= \frac{ac}{m} \\
D &= \frac{ad}{kr} \\
R &= \frac{acf}{mr} \\
G &= \frac{gm}{acf}
\end{align*}
\]

system (1) reduces to

\[
\begin{align*}
\dot{x} &= x\left[1 - x - \frac{y}{B + x} - Hz\right] = x\phi(x, y, z) \\
\dot{y} &= Qy\left[E - \frac{x}{B + x} - 1 - \frac{z}{D + y}\right] = Qy\psi(x, y, z) \\
\dot{z} &= R\left[\frac{xy}{B + x} - Gz\right] = R\varphi(x, y, z)
\end{align*}
\]

A deep understanding of the behaviour of system (2) can be obtained if the subsystem \((x, y)\) is first analysed by considering \( z \) as a parameter. This will be done in the next section.
3. FARMERS–BANDITS INTERACTION UNDER CONSTANT AUTHORITY

In this section the subsystem (2a, b) will be analysed by assuming that the variable \( z \) is constant in time. As discussed in detail in the following, this will be of great help in understanding the behaviour of system (2) when \( R \to 0 \), namely when (2) is assumed to be composed by a fast \( (x, y) \) and a slow \( (z) \) subsystem.

The (nonnegative) equilibria of (2a, b) are given by

\[
\begin{align*}
    x &= 0, \quad y = 0 \quad (3a) \\
    x &= 1 - Hz, \quad y = 0 \quad (3b) \\
    \phi(x, y, z) &= 0, \quad \psi(x, y, z) = 0. \quad (3c)
\end{align*}
\]

The trivial equilibria (3a) and (3b) collide at \( z = 1/H \) (for \( z > 1/H \) the variable \( x \) in (3b) is negative and hence meaningless), while the collision of a solution of (3c) with (3b) can be found by letting \( \phi(1 - Hz, 0, z) = 0, \psi(1 - Hz, 0, z) = 0 \), obtaining

\[
E = \frac{Hz^2 + (DH - B - 1)z - D(B + 1)}{D(Hz - 1)} = E_{cr}(z). \quad (4)
\]

By evaluating the Jacobian matrix

\[
J = \begin{bmatrix}
    \phi + x \phi_x & x \phi_y \\
    Qy \psi_x & Q(\psi + y \psi_y)
\end{bmatrix}
\]

of system (2a–b) in correspondence of (3a) and (3b), it can easily be checked that (3a) is unstable for \( z < 1/H \) and stable for \( z > 1/H \), while (3b) is stable for \( E < E_{cr}(z) \) and unstable for \( E > E_{cr}(z) \). On the contrary, the discussion of the number of (nonnegative) solutions of (3c) and of their stability turns out to be extremely complex, so that a numerical analysis has been performed for a fixed set of parameters in order to gain some insights into the system behaviour. The following parameter values have been fixed:

\[
B = 0.17 \quad H = 0.1 \quad Q = 0.4 \quad D = 0.42 \quad (5)
\]

while \( E \) is left free and will be used as a bifurcation parameter. These values turned out to be particularly suited to point out the relevant phenomena characterizing the behaviour of the system.

In Fig. 1 the bifurcation diagram of the nontrivial equilibria of system (2a, b) is represented in the parameter space \((z, E)\). The curves (with the exception of HO, see below) have been derived by means of a continuation procedure implemented in an interactive PC program (LocBif) [6]. The diagram contains:

i. The curve HF, marking a Hopf bifurcation: for \((z, E)\) on this curve, the Jacobian \( J \) has eigenvalues \( \lambda_{1,2} = \pm i\omega, \omega > 0 \). By crossing HF from below a nontrivial equilibrium of (2a, b) becomes unstable and a stable limit cycle is born.

ii. The curve SN, marking a saddle-node bifurcation: for \((z, E)\) on this curve, \( J \) has an eigenvalue \( \lambda_1 = 0 \). The second eigenvalue \( \lambda_2 \) is negative on the branch SN– and positive on SN+. By crossing SN from above two nontrivial equilibria of (2a, b) (a stable node and a saddle on SN–, an unstable node and a saddle on SN+) collide and disappear.

iii. The point DZ, marking a double-zero codimension-2 bifurcation: for \((z, E)\) on this point, \( J \) has eigenvalues \( \lambda_1 = \lambda_2 = 0 \). The analysis of the normal form of this bifurcation (e.g. Wiggins) [7] predicts the existence of a curve HO, rooted at point DZ, marking a homoclinic bifurcation: for \((z, E)\) on this curve, system (2a, b) has a homoclinic orbit. Since HO cannot be derived through local bifurcation analysis, its
location in the \((z, E)\) plane has been approximately derived on the basis of repeated simulations.

iv. The curve TC, marking a transcritical bifurcation, and corresponding to \(E = E_{cr}(z)\): for \((z, E)\) on this curve, a nontrivial equilibria of (2a, b) collide with (3b).

The analysis of Fig. 1 allows one to derive a bifurcation diagram of system (2a–b) in the space \((x, y, z)\) (remember that \(z\) is a parameter) for any fixed value of \(E\). In Fig. 2, four such diagrams are depicted, corresponding to the values \(E_i, i = 1, 2, 3, 4\), evidenced in Fig. 1. They are qualitative diagrams, because the shape of the limit cycle cannot be ascertained from Fig. 1. Notice that all diagrams of Fig. 2 have a limit cycle at \(z = 0\), since \(E_i > E_{HF}\) for all \(i\) (the case \(E < E_{HF}\) is less interesting and can easily be discussed by the reader). In Fig. 2(a) \((E = E_1)\) the stable limit cycle existing at \(z = 0\) shrinks at \(z = z_{HF}\) to a stable equilibrium, which then collides with an unstable one at \(z = z_{SN}^-\). If \(z\) is decreased, this unstable equilibrium collides at \(z = z_{TC}\) (on the plane \(y = 0\)) with the trivial one (3b). The case of Fig. 2(b) \((E = E_2)\) differs from Fig. 2(a) only because \(z_{TC} < z_{HF}\). In Fig. 2(c), \((E = E_3)\) system (2a, b) has homoclinic orbits for two different values \(z = z_{HQ}^-\) and \(z = z_{HO}^-\). Finally, in Fig. 2(d), \((E = E_4)\) system (2a–b) has only one homoclinic orbit, and at \(z = z_{SN}^+\) there is a saddle-node collision of two unstable equilibria.

As already pointed out, the bifurcation diagram of Fig. 2 is of great help in understanding the behaviour of the three-dimensional system (2) in a special but very interesting case, namely when the parameter \(R\) in 2(c) is very small. This will be discussed in the next section.

4. SYSTEM BEHAVIOUR UNDER THE FAST–SLOW HYPOTHESIS

Let us assume that the parameter \(R\) in (2c) is very small. By remembering that \(R = acf/(mr)\) this means, for instance, that the hiring rate \(f\) of the rulers is very small (see (1c)), namely that the authority reacts very weakly to the criminality observed in the society. Or, that the predation of farmers by bandits gives a small contribution to the
Fig. 2. Bifurcation diagrams of system (2a, b) in the state-parameter space \((x, y, z)\), for the values \(E_i \) \((i = 1, 2, 3, 4)\) indicated in Fig. 1.
mortality of the former with respect to their natural increasing rate (see \( a \) and \( r \) in (1a, b)). Or, that the action of rulers on bandits slightly modifies the mortality of the latter (see \( c \) and \( m \) in (1b, c)).

If \( R \) is very small then \( \dot{z}(t) \) is very small too, so that \( z(t) \) is almost constant in time or, to be more precise, varies at a much slower rate than \( x(t) \) and \( y(t) \). Consider a generic initial state, \((x(0), y(0), z(0))\), and assume that the subsystem \((x, y)\) has one or more attractors (equilibria or limit cycles) for \( z = z(0) \) frozen as a parameter. Thus, in the three-dimensional system, the fast subsystem \((x, y)\) will evolve very fast according to (2a, b) toward that attractor in the basin of attraction of which the initial state, \((x(0), y(0))\), lies, while \( z \) can be considered to remain almost constant at \( z = z(0) \). Then, as \( z(t) \) (slowly) varies according to (2c), the state \((x, y)\) of the fast subsystem remains trapped by the attractor and follows its modifications.

Let us now also assume that the manifold \( g(x, y, z) = 0 \) (see (2c)) separates, for any \( z \), the nontrivial attractors of the subsystem \((x, y)\) from the trivial ones (see Fig. 3). Thus, \( z(t) \) will be increasing above the manifold \( g(x, y, z) = 0 \), and decreasing below it. Notice that this condition (called the separation principle in Muratori and Rinaldi [8]) can always be satisfied by selecting \( G \) sufficiently small.

Under these two assumptions \((R \text{ and } G \text{ small})\), it is easy to show that system (2) has a limit cycle with peculiar geometrical characteristics. The cycle is composed by four parts, two of them corresponding to slow transitions, and two to fast transitions. In the former, \( z(t) \) evolves very slowly while \((x(t), y(t))\) are trapped by the attractor of the two-dimensional system (2a, b) and follow its evolution, while the latter are catastrophic transitions from an attractor to another of system (2a, b), at almost constant \( z(t) \). In Fig. 3, the four cases of Fig. 2 are reconsidered in order to point out the existence of the limit cycle. In Fig. 3(a) the cycle is composed by the fast branch \( 1 \rightarrow 2 \), followed by the slow branch \( 2 \rightarrow SN- \) where \((x, y)\) are trapped by the nontrivial stable equilibrium, then by another fast branch \( SN- \rightarrow 3 \), and finally by the slow branch \( 3 \rightarrow 1 \) corresponding to the trivial equilibrium. In Fig. 3(b), the upper branch is characterized by oscillations of \( x \) and \( y \) at relatively high frequency, whose amplitude slowly decreases to zero as \( z(t) \) approaches \( z_{HF} \). In Fig. 3(c) and 3(d), the amplitude of the high-frequency oscillations remains rather constant, while the period increases as \( z(t) \) approaches the value \( z = z_{HO} \) (\( z = z_{HO} \) in (3c)) corresponding to the homoclinic orbit of the two-dimensional system (2a, b). In Fig. 4, the results of some simulations for the cases (a), (b), and (c) of Fig. 3 are reported. In Fig. 5 the time patterns of \( x(t) \) and \( y(t) \) in case (c) are shown in a time interval around the catastrophic transition \((z = z_{HO})\) where bandits are almost eliminated: peaks corresponding to an explosion in the number of bandits become more and more rare and finally disappear.

Two observations are worthwhile at this point. First, it must be pointed out that one cannot prove the existence and uniqueness of the stable limit cycle by means of the slow–fast argument, as done above, but only the existence of a stable trajectory lying in a tube whose radius is infinitesimal with \( R \). For \( R \) sufficiently small, this trajectory can obviously be considered as a limit cycle from any practical point of view. Second, as evidenced in Fig. 3, the catastrophic transition that moves the system from the trivial equilibrium \((y \equiv 0)\) up to the nontrivial attractor of the \((x, y)\) subsystem (for example, transition \( 1 \rightarrow 2 \) in Fig. 3(a)) takes place at a value \( z \) which is notably less than the value \( z_{TC} \) of the transcritical bifurcation. A detailed justification of this phenomenon (which is typical for slowly varying bifurcation parameters), as well as the exact computation of the delay, is however beyond the scope of this work, as it requires the use of the so-called Dynamic Bifurcation theory [9] (see also Muratori and Rinaldi [10] for a similar application to food-chain systems).
Fig. 3. Slow-fast limit cycles of system (2) for the four cases of Fig. 2.


5. BIFURCATION ANALYSIS

If the two assumptions made in the previous section (R and G small) are relaxed, the behaviour of system (2) becomes different from that illustrated in Figs 3–5. A different approach is required namely a numerical study of the bifurcations of system (2).

In this section, E and R are bifurcation parameters, while B, H, Q, and D are fixed as in (5), and G = 0.09. The (nonnegative) equilibria of (2) are given by

\[ x = 0, \quad y = 0, \quad z = 0 \]  

(6a)
Fig. 5. Time patterns of $x(t)$ and $y(t)$ in the case of Fig. 3(c) around the catastrophic transition $z = z_{HO}$.

\[ x = 1, \quad y = 0, \quad z = 0 \]  \hspace{1cm} (6b)
\[ \phi(x, y, z) = 0, \quad \psi(x, y, z) = 0, \quad \varphi(x, y, z) = 0 \]  \hspace{1cm} (6c)

and the numerical analysis reveals that, for the above parameter values, the solution of (6c) is unique. It collides with (6b) at $E = B + 1$ (transcritical bifurcation).

In Fig. 6 the bifurcation diagram of system (2) in a region of the parameter space $(R, E)$ is presented: it has been derived by means of a continuation procedure [6]. The nontrivial equilibrium given by (6c) is stable in region 0, namely above the transcritical bifurcation curve TC (i.e., the line $E = B + 1$). By increasing $E$, the equilibrium undergoes a Hopf bifurcation on the curve HF, so that in region 1, a stable limit cycle

Fig. 6. Bifurcation diagram of the three-dimensional system (2) in the parameter space $(R, E)$. 

exists (let us refer to this cycle as the period-1 cycle), while the equilibrium has become
unstable. The period-1 cycle then goes through a period-doubling bifurcation if the curve
PD1 is crossed. Thus, in region 2, the cycle generated by the Hopf is unstable, while a
stable period-2 cycle exists: by denoting with $T$ the period of the period-1 cycle on a point
$(R, E)$ on the curve PD1, the period of the p-2 cycle tends to $2T$ as $(R, E) \rightarrow (\bar{R}, \bar{E})$
from region 2.

The bifurcation diagram of Fig. 6 also displays a second period-doubling curve PD2. By
crossing PD2 the period-2 cycle loses stability, and a stable period-4 cycle appears. PD2 is
closed and inside this curve, but very close to it, other period-doubling closed curves PD4,
PD8, ... have been detected. This indicates the existence of a Feigenbaum's cascade (e.g.
Guckenheimer and Holmes [11]): by increasing $E$, keeping $R$ constant to a suitable value,
the system experiences a sequence of period-doubling bifurcations at some values $E_1$, $E_2$,
... accumulating to $E = E_\infty$ where the system enters the region of chaotic behaviour.

Then, if one continues to increase $E$, a reverse cascade takes place, namely a sequence of
period-halvings that eventually bring the system back to a relatively regular behaviour. In
other words, chaos takes place at intermediate values of $E$ (as well as at intermediate
values of $R$, as appears from Fig. 6).

In Fig. 7 several attractors are presented for different values of $E$ with $R$ fixed at
$R = 0.1$. In Fig. 7(a) the period-1 cycle is shown, while in Fig. 7(b) and (c) the stable limit
cycles generated by the first and second period-doubling bifurcations (period-2 and
period-4, respectively) are depicted. Fig. 7(d) shows the chaotic attractor resulting form the
Feigenbaum's cascade.

6. DISCUSSION AND CONCLUSIONS

The analysis carried out in the previous sections has evidenced a great variety of
behaviour for the analysed model. The slow–fast limit cycles of Figs 3 and 4 can be
interpreted as an alternation of periods of anarchy and despotism in the society. Indeed,
the point of the limit cycle corresponding to the eradication of the bandits (e.g. point 3 in
Fig. 3(a)), just after the fast transition leading to $y \equiv 0$, is the point corresponding to the
moment of maximum power of the regime, e.g. to a well-established dynasty in the case of
ancient China. In this situation the farmers live almost undisturbed, provided they pay
taxes to the rulers. Then, since there are no threats by bandits, there is no necessity for
additional rulers, so that, due to natural retirement, the stock of rulers begins gradually to
decay (slow transition $3 \rightarrow 1$). This provides an explanation of the fall of a dynasty. As
soon as the power of the regime has fallen below a critical level, the bandit’s profession
becomes attractive again: the number of bandits suddenly increases (fast transition $1 \rightarrow 2$)
and the farmers are exploited again so that their number is reduced. Thus, a period of
anarchy is established, in which the rulers have almost no control since they are too few.
Then, the permanent activity of the bandits leads to a gradual increase of the number of
rulers (slow transition $2 \rightarrow SN-)$, until they again reach the critical level that allows them
to eliminate bandits (fast transition $SN \rightarrow 3$).

As the slow–fast hypothesis is relaxed, for instance when the authority strongly reacts to
criminality, the behaviour of the system may become even more complex, as evidenced for
instance in Fig. 7, where it is shown that different states of the society may alternate in a
chaotic and hence unpredictable way.

Some extensions of the present work are possible. For example, the bifurcation analysis
of Section 5 could be extended to other regions of the parameter space, in order to locate
other routes to chaos. On the other hand, the model could be extended by considering
additional interactions. For instance, the encounters between bandits and rulers could also
be a cause of mortality for the latter. Another extension could be to take into account that, in the periods of increasing anarchy, rulers could change their role and become bandits, a phenomenon often observed in actual historical situations. Figures 8 and 9 provide a coloured illustration for slow–fast limit cycles and attractors, respectively.

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Fig. 8. Slow–fast limit cycles. Different colours refer to different speeds (blue = slow, green = medium, yellow = fast).

Fig. 9. Slow–fast attractor (blue = slow, green = medium, yellow = fast velocity).
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