TUTORIAL V:
Continuation of homoclinic orbits with MATCONT

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This session is devoted to location and continuation of orbits homoclinic to hyperbolic equilibria in autonomous systems of ODEs depending on two parameters

\[ \dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}^2, \]

and to detection of their codim 2 bifurcations.

1 Traveling pulses in the FitzHugh-Nagumo model

The following system of partial differential equations is the FitzHugh-Nagumo model of the nerve impulse propagation along an axon:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - f_a(u) - v, \\
\frac{\partial v}{\partial t} &= bu,
\end{align*}
\]

where \( u = u(x, t) \) represents the membrane potential; \( v = v(x, t) \) is a phenomenological “recovery” variable; \( f_a(u) = u(u - a)(u - 1) \), \( 1 > a > 0 \), \( b > 0 \), \( -\infty < x < +\infty \), \( t > 0 \).

Traveling waves are solutions to these equations of the form

\[ u(x, t) = U(\xi), \quad v(x, t) = V(\xi), \quad \xi = x + ct, \]

where \( c \) is an a priori unknown wave propagation speed. The functions \( U(\xi) \) and \( V(\xi) \) satisfy the system of three ordinary differential equations

\[
\begin{align*}
\dot{U} &= W, \\
\dot{W} &= cW + f_a(U) + V, \\
\dot{V} &= bU/c,
\end{align*}
\]

where the dot means differentiation with respect to “time” \( \xi \). System (2) is called a wave system. It depends on three positive parameters \( (a, b, c) \). Any bounded orbit of (2) corresponds to a traveling wave solution of the FitzHugh-Nagumo system (1) at parameter values \( (a, b) \) propagating with velocity \( c \).

For all \( c > 0 \) the wave system has a unique equilibrium \( 0 = (0, 0, 0) \) with one positive eigenvalue \( \lambda_1 \) and two eigenvalues \( \lambda_{2,3} \) with negative real parts. The equilibrium can be either a saddle or a saddle-focus and has in both cases a one-dimensional unstable and a two-dimensional stable invariant manifolds \( W^{u,s}(0) \). The transition between saddle and saddle-focus cases is caused by the presence of a double negative eigenvalue; for fixed \( b > 0 \) this happens on the curve

\[ D_b = \{(a, c) : c^4(4b - a^2) + 2ac^2(9b - 2a^2) + 27b^2 = 0\}. \]

A branch \( W^u(0) \) of the unstable manifold leaving the origin into the positive octant can return back to the equilibrium, forming a homoclinic orbit \( \Gamma_0 \) at some parameter values.

For \( b > 0 \), these parameter values form a curve \( P_b^{(1)} \) in the \( (a, c) \)-plane that can only be found numerically. As we shall see, this curve passes through the saddle-focus region delimited by \( D_b \). Any homoclinic orbit defines a traveling impulse. The shape of the impulse depends on the type of the corresponding equilibrium: It has a monotone “tail” in the saddle case and an oscillating “tail” in the saddle-focus case.

The saddle quantity \( \sigma_0 = \lambda_1 + \Re \lambda_{2,3} \) is always positive for \( c > 0 \). Therefore, the phase portraits of (2) near the homoclinic curve \( P_b^{(1)} \) are described by Shilnikov’s Theorems. In particular, near the homoclinic bifurcation curve \( P_b^{(1)} \) in the saddle-focus region, system (2) has an infinite number of saddle cycles. These cycles correspond to periodic wave trains in the FitzHugh-Nagumo model (1). Secondary homoclinic orbits existing in (2) near the primary saddle-focus homoclinic
bifurcation correspond to double traveling impulses in (1). An infinite number of the corresponding secondary homoclinic bifurcation curves \( P^{(2)}_{b,j} \) in (2) originate at each point \( A_{1,2} \), where \( P^{(1)}_b \) intersects \( D_b \).

We will locate a critical value of \( c \) for \( a = 0.15 \) and \( b = 0.0025 \), at which (2) has a homoclinic orbit, continue this homoclinic orbit with respect to the parameters \( (a,c) \), and detect the codim 2 bifurcations points \( A_{1,2} \) in \( P^{(1)}_b \).

2 System specification

Start a version of MATCONT that supports homoclinic continuation, and specify a new ODE system with the coordinates \((U,W,V)\) and time \(t^1\):

\[
U' = W \\
W' = cc*W + U*(U-aa)*(U-1.0) + V \\
V' = bb*U/cc
\]

The parameters \( a, b, c \) are denoted by \( aa, bb, cc \), respectively. Generate the derivatives of order 1, 2, and 3 symbolically.

3 Location of a homoclinic orbit by homotopy

This consists of several steps, each presented in a separate subsection.

3.1 Approximating the unstable manifold by integration

Select Type|Initial point|Equilibrium and Type|Curve|Connection|Saddle.

In the appearing Integrator window, increase the integration Interval to 20 (see the right panel of Figure 1).

Via the Starter window, input the initial values of the system parameters

\(^1\)Due to MATLAB restrictions, the name \( xi \) cannot be used here!

Figure 1: Starter and Integrator windows for the integration of the unstable manifold.
as well as

\begin{align*}
\text{Up}aram_{1} & = -1 \\
\text{eps0} & = 0.01
\end{align*}

that specify direction and distance of the displacement from the saddle

\begin{align*}
x_{0, U} & = 0 \\
x_{0, W} & = 0 \\
x_{0, V} & = 0
\end{align*}

along the unstable eigenvector\textsuperscript{2}. The \textbf{Starter} window should look like in left panel of Figure 1.

Open a \texttt{2Dplot} window with \texttt{Window|Graphic|2Dplot}. Select \texttt{U} and \texttt{V} as variables along the corresponding axes and input the following plotting region

\begin{align*}
\text{Abscissa:} & \quad -0.2 \quad 0.5 \\
\text{Ordinate:} & \quad -0.05 \quad 0.1
\end{align*}

Start \texttt{Compute|Forward}. You will get an orbit approximating the unstable manifold that departs from the saddle in a nonmonotone way, see Figure 2. This orbit does not resemble a homoclinic orbit.

\textsuperscript{2}\text{Up}aram_{2} is only used when dim$W^{u} = 2$. 

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Figure 2: A segment of the unstable manifold of the saddle at the initial parameter values.
Press **Select Connection** button in the **Starter** window. **MATCONT** will search for a point in the computed orbit where the distance to the *stable* eigenspace of the Jacobian matrix of the saddle is stopped decreasing for the last time. This point is selected as the end-point of the initial connecting orbit (as we shall see, it corresponds to the time-interval $T=8.40218$. The program will ask to choose the BVP-discretization parameters $n_{\text{st}}$ and $n_{\text{col}}$ that will be used in all further continuations. Set $n_{\text{st}}$ equal to 50 and keep $n_{\text{col}}$ equal to 4 (Figure 3). Press **OK**.

![Choose n_{\text{st}} and n_{\text{col}}](image)

Figure 3: The discretization parameters for homotopy BVPs.

### 3.2 Homotopy towards the stable eigenspace

In the new **Starter** window, activate the parameters $cc$, $S_{\text{Param1}}$, and $\epsilon_{\text{1}}$ (see Figure 4), and **Compute|Backward**. A family of curves will be produced by continuation (see Figure 5) and the message

$S_{\text{Param}}$ equal to zero

will indicate that the end-point has arrived at the stable eigenspace of the saddle (i.e. reached the plane tangent to the stable invariant manifold at the saddle and given by the condition $S_{\text{Param1}}=0$). The corresponding orbit segment is labeled $\text{HTHom}$. Stop the continuation there.

![Starter window for the homotopy towards the stable eigenspace.](image)

Figure 4: **Starter** window for the homotopy towards the stable eigenspace.
Figure 5: The unstable manifold with the end-point in the stable eigenspace of the saddle.
3.3 Homotopy of the end-point towards the saddle

The obtained segment is still far from the homoclinic orbit but can be selected as the initial point for a homotopy of the end-point towards the saddle. Select

2) HTHom: SParam equal to zero

via Select|Initial point menu.

In the Continuer window, set MaxStepsize to 0.5, see in the right panel of Figure 6.

![Figure 6: Starter and Integrator windows for the homotopy towards the saddle.](image)

In the Starter window, SParam1 now equals to zero, while the parameter cc is adjusted. Activate parameters cc, T, and eps1 there. Set eps1tol equal to 0.01; this will be used as the target distance eps1 from the end-point to the saddle.

Open a Numeric window to monitor the values of the active parameters. Clean the 2DPlot window and Compute | Forward. You should get Figure 7, where the last computed segment is again labeled by HTHom. The message

eps1 small enough

appears in the main window and indicates that a good approximation of the homoclinic orbit is found. The begin- and the end-points are now both located near the saddle (at distance 0.01).

The Numeric window at the last computed point is presented in Figure 8. It can be seen that the eps1 became 0.01, while the time-interval T increased to 36.6206. Stop the continuation.

3.4 Continuation of the homoclinic orbit

Select just computed

2) HTHom: eps1 small enough

via Select|Initial point menu as the initial data. Select Type|Curve|Homoclinic to Saddle and check that the curve type is Hom, while the initial point is of type HTHom.

In the new Starter window, activate two system parameters: aa and cc as well as the homoclinic parameter T (see Figure 9). These parameters will vary along the homoclinic curve, while
Figure 7: The homotopy results in the manifold segment with both the begin- and the end-points near the saddle.

Figure 8: **Numeric** window at the last point of the homotopy towards the saddle.
Figure 9: **Starter** window for the two-parameter homoclinic continuation.
both \( \text{eps0} \) and \( \text{eps1} \) (the begin- and end-distances to the saddle) will be fixed, see Figure 9. Also, choose \text{Yes} \) to detect the singularity \textbf{Double SL-eigenvalue} (double stable leading eigenvalue) along the homoclinic curve.

In the \textbf{Continuer} window, increase the \textbf{MaxStepsize} to 1.

Change the attributes of the \textbf{2Dplot} window: Select \( \text{aa} \) and \( \text{cc} \) as the abscissa and ordinate with the visibility limits

\begin{align*}
\text{Abscissa:} & \quad 0 \quad 0.3 \\
\text{Ordinate:} & \quad 0 \quad 0.8 \\
\end{align*}

Now you are ready to start the continuation. \textbf{Compute|Forward} and \textbf{Backward}, resume computations at special points, and terminate them when the computed points leave the positive quadrant of the \((a,c)\)-plane. Two special points will be detected, where the equilibrium undergoes the saddle-to-saddle-focus transition. These are codim 2 bifurcation points \( A_{1,2} \) introduced in Section 1.

Delete all previously computed curves except the last two, namely

\( \text{HTHom\_Hom}(1) \)
\( \text{HTHom\_Hom}(1) \)

and \textbf{Plot|Redraw diagram}. This should produce Figure 10.

![Figure 10: The homoclinic bifurcation curve in the \((a,c)\)-plane. The saddle to saddle-focus transitions \( A_{1,2} \) are labeled by DRS.](image)
Figure 11: The family of homoclinic homoclinic orbits in the phase space of system (2) for $b = 0.0025$. 
To verify that all computed points indeed correspond to homoclinic orbits, open a 3Dplot window and select \( U, W \) and \( V \) as variables along the coordinate axes with the visibility limits

\[
\begin{align*}
\text{Abscissa:} & \quad -0.4 \quad 1.2 \\
\text{Ordinate:} & \quad -0.15 \quad 0.15 \\
\text{Applicate:} & \quad -0.01 \quad 0.2
\end{align*}
\]

respectively. Plot | Redraw diagram in this new window should produce Figure 11 after an appropriate rotation.

4 Additional Problems

A. Consider the famous Lorenz system

\[
\begin{align*}
\dot{x} &= \sigma(-x + y), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= -bz + xy,
\end{align*}
\]

with the standard parameter value \( b = \frac{8}{3} \). Use MATCONT to analyse its homoclinic bifurcations:

1. Locate at \( \sigma = 10 \) the bifurcation parameter value \( r_{\text{Hom}} \) corresponding to the primary orbit homoclinic to the origin. Hint: Use homotopy starting from \( r = 15.5 \).

2. Compute the primary homoclinic bifurcation curve in the \((r, \sigma)\)-plane for \( b = \frac{8}{3} \). Try to reach \( r = 100 \) and \( \sigma = 100 \).

3. Locate and continue in the same \((r, \sigma)\)-plane several secondary homoclinic to the origin orbits in the Lorenz system. Hint: These orbits make turns around both nontrivial equilibria. The simplest one can be found starting from \((\sigma, r) = (10, 55)\).

B. Study with MATCONT homoclinic bifurcations in the adaptive control system of Lur’e type:

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= -\alpha z - \beta y - x + x^2,
\end{align*}
\]

where \( \alpha \) and \( \beta \) are parameters.