Warnings:

- This file consists of 8 pages (including cover).
- During the exam you are not allowed to exit the room for any other reason than handing your work or withdrawing from the exam.
- You are not allowed to withdraw from the exam during the first 30 minutes.
- During the exam you are not allowed to consult books or any kind of notes.
- You are not allowed to use calculators with graphic display.
- Solutions and answers can be given either in English or in Italian.
- Solutions and answers must be given exclusively in the reserved space. Only in the case of corrections, or if the space is not sufficient, use the back of the front cover.
- The clarity and the order of the answers will be considered in the evaluation.
- At the end of the test you have to hand this file only. Every other sheet you may hand will not be taken into consideration.
EXERCISE 1

1. Consider the manipulator sketched in the picture:

Find the expression of the inertia matrix $B(q)$ of the manipulator.

Denavit-Hartenberg frames can be defined as sketched in this picture:

Computations of the Jacobians:

**Link 1**

$$ J_P^{(l_1)} = \begin{bmatrix} J_{P_1}^{(l_1)} & 0 \end{bmatrix} = \begin{bmatrix} z_0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} $$

**Link 2**

$$ J_P^{(l_2)} = \begin{bmatrix} J_{P_1}^{(l_2)} & J_{P_2}^{(l_2)} \end{bmatrix} = \begin{bmatrix} z_0 & \mathbf{z}_1 \times (\mathbf{p}_{l_2} - \mathbf{p}_1) \end{bmatrix} = \begin{bmatrix} 0 & -l_2 s_2 \\ 0 & 0 \\ 1 & l_2 c_2 \end{bmatrix} $$

\(^1\text{The cross product between vector } \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ is } \mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \)
\[ J_O^{(l_2)} = \begin{bmatrix} J_{O_1}^{(l_2)} & J_{O_2}^{(l_2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \]

For the above computations, we can make reference to the following picture:

\[
\begin{align*}
J & = \begin{bmatrix} l & 0 \\ 0 & z_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \\
& \begin{bmatrix} a_1 + l_2 c_2 \\ 0 \\ d_1 + l_2 s_2 \end{bmatrix}
\]

and to the following auxiliary vectors:

\[ p_{l_2} = \begin{bmatrix} a_1 + l_2 c_2 \\ 0 \\ d_1 + l_2 s_2 \end{bmatrix}, \quad p_1 = \begin{bmatrix} a_1 \\ d_1 \\ 0 \end{bmatrix}, \quad z_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \]

The inertia matrix can be computed now:

\[
B(q) = m_1 J_P^{(l_1)T} J_P^{(l_1)} + m_2 J_P^{(l_2)T} J_P^{(l_2)} + I_2 J_O^{(l_2)T} J_O^{(l_2)}
\]

\[
= m_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + m_2 \begin{bmatrix} 1 & l_2 c_2 \\ l_2 c_2 & l_2^2 \end{bmatrix} + I_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}
\]

where:

\[ b_{11} = m_1 + m_2 \]
\[ b_{12} = m_2 l_2 c_2 \]
\[ b_{22} = m_2 l_2^2 + I_2 \]

2. Ignoring the Coriolis and centrifugal terms, write the dynamic model of the manipulator.

Since the vertical axis is the \( x_0 \) axis pointing upwards, the gravity acceleration vector is:

\[ g_0 = \begin{bmatrix} -g \\ 0 \\ 0 \end{bmatrix} \]

The gravitational torques are thus:

\[ g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, \]

where:

\[ g_1 = -m_1 g_0^T J_{P_1}^{(l_1)} - m_2 g_0^T J_{P_1}^{(l_2)} = -m_1 g_0^T \begin{bmatrix} 0 \\ 0 \end{bmatrix} - m_2 g_0^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \]

\[ g_2 = -m_1 g_0^T J_{P_2}^{(l_1)} - m_2 g_0^T J_{P_2}^{(l_2)} = -m_2 g_0^T \begin{bmatrix} -l_2 s_2 \\ 0 \end{bmatrix} = -m_2 gl_2 s_2 \]

Neglecting Coriolis and centrifugal terms, the dynamic model can be written as:
\[ B(q)\ddot{q} + g(q) = \tau \]

The two equations that form the model are:

\[
\begin{align*}
(m_1 + m_2) \ddot{d}_1 + m_2l_2c_2\ddot{\varphi}_2 &= \tau_1 \\
m_2l_2c_2\ddot{d}_1 + (m_2l_2^2 + I_2) \ddot{\varphi}_2 - m_2gl_2s_2 &= \tau_2
\end{align*}
\]

3. Show that the dynamic model is linear with respect to a certain set of dynamic parameters.

The model can be written in the following form which is linear in the dynamic parameters:

\[ Y(q, \dot{q}, \ddot{q}) \Pi = \tau \]

where the vector of dynamic parameters is expressed as:

\[ \Pi = \begin{bmatrix}
    m_1 + m_2 \\
m_2l_2 \\
m_2l_2^2 + I_2
\end{bmatrix} \]

while the regressor matrix is:

\[ Y = \begin{bmatrix}
    \ddot{d}_1 & c_2\ddot{\varphi}_2 & 0 \\
    0 & c_2\ddot{d}_1 - g s_2 & \ddot{\varphi}_2
\end{bmatrix} \]

4. Write the expression of a “PD + gravity compensation” control law in the joint space for this specific manipulator.

The vector equation of the control law is:

\[ \tau = K_P (q_d - q) - K_P \dot{q} + g(q) \]

and corresponds, for the given manipulator, to the following two equations:

\[
\begin{align*}
\tau_1 &= K_{P_1} (q_{d1} - q_1) - K_{D_1} \dot{q}_1 \\
\tau_2 &= K_{P_2} (q_{d2} - q_2) - K_{D_2} \dot{q}_2 - m_2gl_2s_2
\end{align*}
\]

**EXERCISE 2**

1. Explain what is the difference between the kinematic and the dynamic scaling of a trajectory.

In the kinematic scaling the trajectory has to satisfy the constraints on the maximum velocity and acceleration, while in the dynamic scaling the constraints are on the maximum torques.

2. The parametric form of a cycloidal trajectory for kinematic scaling is given by:

\[ \sigma(\tau) = \tau - \frac{1}{2\pi} \sin(2\pi\tau) \]

Find the expressions of the maximum velocity and maximum acceleration for such trajectory in terms of the positioning time \( T \) and the total displacement \( h \).
The first and second derivatives of the parametric form are given by:

\[ \sigma'(\tau) = 1 - \cos(2\pi \tau) \]
\[ \sigma''(\tau) = 2\pi \cos(2\pi \tau) \]

The expressions of the maximum velocity and acceleration can be obtained as:

\[ \dot{q}_{\text{max}} = \frac{h}{T} \sigma'_{\text{max}}(\tau) = \frac{h}{T} \sigma'(0.5) = \frac{2h}{T} \]
\[ \ddot{q}_{\text{max}} = \frac{h}{T^2} \sigma''_{\text{max}}(\tau) = \frac{h}{T^2} \sigma''(0.25) = 2\pi \frac{h}{T^2} \]

3. Consider the design of a cycloidal trajectory from \( q_i = 10 \) to \( q_f = 30 \), with \( \dot{q}_{\text{max}} = 20 \) and \( \ddot{q}_{\text{max}} = 10 \). Find the minimum positioning time.

The total displacement is \( h = 20 \). Therefore:

\[ \frac{2h}{T} < 20 \quad \Rightarrow \quad T > 2 \]
\[ 2\pi \frac{h}{T^2} < 10 \quad \Rightarrow \quad T > \sqrt{4\pi} = 3.54 \]

The minimum positioning time is then \( T_{\text{min}} = 3.54 \)

4. In the process of the dynamic scaling, the following relation is used, for each joint of the robot:

\[ \tau_i(t) = \alpha_i(\sigma(t)) \dot{\sigma}(t) + \beta_i(\sigma(t)) \ddot{\sigma}(t) + \gamma_i(\sigma(t)), \quad i = 1, \ldots, n, \quad t \in [0, T] \]

Explain the meaning of symbol \( \sigma \) in this equation. Out of the three terms in the right hand side, which ones scale with time?

In the dynamic scaling it is assumed that all joint variables depend on time in the same way, through the scalar parameter \( \sigma(t) \): \( q = q(\sigma(t)) \)

Only the first and second terms scale with time (the third one is the gravitational term which depends only on position).

**EXERCISE 3**

1. Consider an interaction task of a manipulator, with a frictionless and rigid surface, as in this picture:

Assume a point contact and draw a contact frame directly on the picture. Based on this frame and neglecting angular velocities and moments, express the natural and the artificial constraints for this problem, and specify the selection matrix.
The contact frame can be conveniently chosen as in the following picture:

The natural constraints and artificial constraints can be easily identified:

<table>
<thead>
<tr>
<th>Natural constraints</th>
<th>Artificial constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f^c_x )</td>
<td>( \dot{p}^c_x )</td>
</tr>
<tr>
<td>( f^c_y )</td>
<td>( \dot{p}^c_y )</td>
</tr>
<tr>
<td>( \dot{p}^c_z )</td>
<td>( f^c_z )</td>
</tr>
</tbody>
</table>

The selection matrix is thus:

\[
\Sigma = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

2. Explain what an implicit force controller is and why it might be convenient with respect to an explicit solution.

An implicit force control is closed around the position control loops. This is usually the only viable solution to implement force control, since the reliable and industrially safe position controllers cannot be bypassed.

3. Suppose now that along the force controlled direction an explicit force controller has to be designed. Sketch the block diagram of such controller and design it taking a bandwidth of 30 rad/s.

The block diagram of an explicit force controller in case of rigid surface is sketched in the picture:

\[
\text{where the transfer function } G_f(s) \text{ is practically a unitary gain. We can then consider as a force controller an integrator:}
\]

\[
R_f(s) = \frac{k_{if}}{s}
\]

and the gain can be set equal to the desired bandwidth: \( k_{if} = 30 \).

4. Repeat the process in case an implicit force controller, for the same bandwidth, has to be designed.

The block diagram of an implicit force controller in case of rigid surface is sketched in the picture:
where $R(s)$ is the transfer function of the position controller. If we assume a PID position controller:

$$R(s) = \frac{K_D s^2 + K_P s + K_I}{s}$$

The partial compensator of such controller is:

$$C(s) = \frac{1}{K_D s^2 + K_P s + K_I}$$

If we select a PI controller on the force error:

$$R_f(s) = k_{pf} + \frac{k_{if}}{s}$$

the loop transfer function becomes:

$$L_f(s) = \frac{sk_{pf} + k_{if}}{s^2}$$

Since the high frequency approximation of such transfer function is $k_{pf}/s$ we can set $k_{pf} = 30$ (equal to the required bandwidth. The zero of the controller can be set at a lower frequency range, for example $k_{if}/k_{pf} = 3$, which yields $k_{if} = 90$. 

$$\begin{array}{c}
\begin{array}{ccccccc}
& f_d & + & f_c & \downarrow & R(s) & \downarrow C(s) & x_d & \downarrow R(s) & u & \equiv 1 & \Rightarrow & f
\end{array}
\end{array}$$