Memory Extraction From Dynamic Scattering Junctions in Wave Digital Structures

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***Abstract*—In this letter, we show that a computable tree-like interconnection of parallel/series wave digital (WD) adaptors with memory (i.e., characterized by reflection filters instead of reflection coefficients) is equivalent to a like interconnection of standard (memoryless) adaptors whose peripheral ports are connected to mutators (two-port adaptors with memory). In proving all this, we provide a methodology for "extracting the memory" from a macro-adaptor, which can be fruitfully employed to simplify the implementation of WD structures.**

***Index Terms*—Physical modeling, scattering, sound modeling, wave digital filters (WDF).**

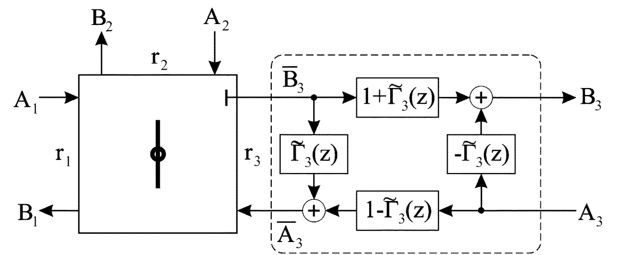

Fig. 1. Any three-port series (parallel) adaptor can always be implemented as a standard series (parallel) WDF adaptor, whose reflection-free port is connected to a WD mutator.

I. Introduction

THE importance of wave digital filters [1] (WDF) for the physical modeling of musical instruments [2] is widely established by a variety of applications, ranging from the numerical integration of physical systems described by nonlinear partial differential equations [3] to the modeling of nonlinear interactions between individually designed physical models [4], [5]. Particularly interesting for this type of applications are nonlinear wave digital (WD) structures [6], which are characterized by WD adaptors that are able to perform not just changes of port resistances but also changes of port impedances. These adaptors are thus inherently "dynamic" as they are based on reflection and transmission filters instead of simple coefficients. Dynamic adaptors allow us to accommodate a wide range of nonlinear elements (NLE) with memory (nonlinear reactances, algebraic nonlinearities, etc.) in a WD structure and model them as instantaneous NLEs. WD structures are thus suitable for the modeling of a wide range of nonlinear interactions between physical blocks.

WD structures can be generally seen as a set of WD elements connected to each other through a macro-adaptor (MA), which is a multiport adaptor with one reflection-free port, obtained by interconnecting basic (series or parallel) dynamic adaptors in a tree-like fashion. If we want to develop a strategy for automatically implementating such WD structures, then we must deal with a very large number of possible building blocks in comparison with traditional WDFs [1]. In order to overcome this difficulty, we would like to define some structural transformation that can be used for significantly reducing the number of potential building blocks. One way to do so is to construct a WD structure that is functionally equivalent to a MA but is made of an interconnection of standard (memoryless) two-port and/or three-port adaptors [1] whose peripheral ports are possibly connected to WD mutators [6] (two-port scattering junctions with memory). This can be used as a preliminary step for the applicability of novel automatic implementation solutions for WD structures, such as the wave tableau method [4] and the binary connection tree method [7].

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II. Memory Extraction From Adaptors

In this section, we show that any dynamic three-port series (parallel) adaptor can always be implemented as a standard series (parallel) WDF adaptor, whose reflection-free port is connected to the WD mutator [6] shown in Fig. 1, where $\tilde{\Gamma}_3(z)$ is obtained from the reflection filter of the reflection-free port by removing the instantaneous I/O connection. A WD mutator is a special case of the two-port dynamic adaptor that allows us to implement a wide range of dynamic nonlinearities using memoryless nonlinearities.

A. Dynamic Series Adaptor

Let us consider a three-port dynamic series adaptor with the following rational, causal, and stable port impedance (a digital filter):

$$R_k(z) = r_k + \sum_{i=1}^{N_k} c_{ik} z^{-i} 1 + \sum_{i=1}^{M_k} d_{ik} z^{-i} = r_k + \frac{\sum\limits_{i=1}^{N_k} r_{ik} z^{-i}}{1 + \sum\limits_{i=1}^{M_k} d_{ik} z^{-i}}$$

with $k = 1, 2, 3$ being the port index. The right-hand side of this equation is obtained by computing one step of the long division, where r_k is the result of the division, and $\sum_{i=1}^{N_k} r_{ik} z^{-i}$ is the reminder. The reflection transfer functions are

$$\Gamma_k(z) = \frac{2R_k(z)}{\sum R_i(z)} = \gamma_k + \tilde{\Gamma}_k(z) \quad (1)$$

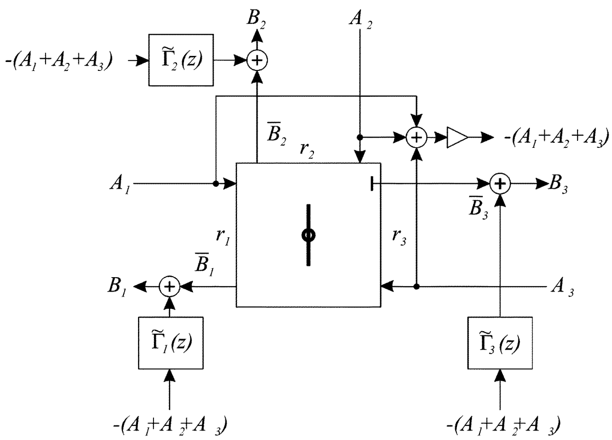

Fig. 2. Direct implementation of eq. (II-A).

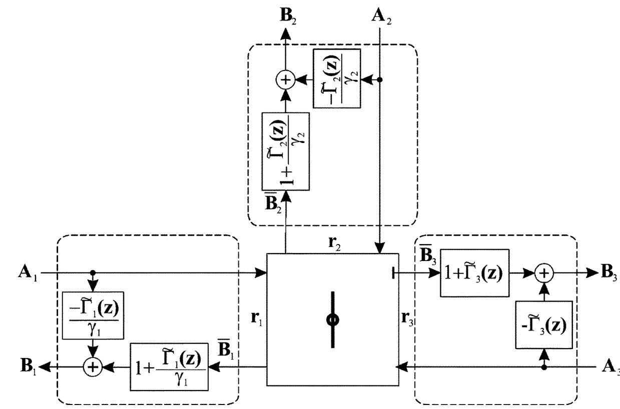

Fig. 3. Intermediate step of the memory extraction process from an adaptor. Notice that each filter accounts for just half of the scattering junction.

from which we can extract a constant γ_k, while the rest can be written as $\widetilde{\Gamma}_k(z) = z^{-1}\hat{\Gamma}_k(z)$, where $\hat{\Gamma}_k(z)$ is assumed as causal and stable. From (1), we can thus derive

$$\sum_{k=1}^{3}\Gamma_k(z) = 2 \Rightarrow \begin{cases}\gamma_1+\gamma_2+\gamma_3 = 2\\ \widetilde{\Gamma}_1(z)+\widetilde{\Gamma}_2(z)+\widetilde{\Gamma}_3(z) = 0.\end{cases} \quad (2)$$

Also the scattering matrix of this adaptor can be decomposed into the sum of a constant and a filtered term

$$\mathbf{B} = \begin{bmatrix}1-\Gamma_1 & -\Gamma_1 & -\Gamma_1\\ -\Gamma_2 & 1-\Gamma_2 & -\Gamma_2\\ -\Gamma_3 & -\Gamma_3 & 1-\Gamma_3\end{bmatrix}\mathbf{A}$$
$$= \left(\begin{bmatrix}1-\gamma_1 & -\gamma_1 & -\gamma_1\\ -\gamma_2 & 1-\gamma_2 & -\gamma_2\\ -\gamma_3 & -\gamma_3 & 1-\gamma_3\end{bmatrix} + \begin{bmatrix}-\widetilde{\Gamma}_1 & -\widetilde{\Gamma}_1 & -\widetilde{\Gamma}_1\\ -\widetilde{\Gamma}_2 & -\widetilde{\Gamma}_2 & -\widetilde{\Gamma}_2\\ -\widetilde{\Gamma}_3 & -\widetilde{\Gamma}_3 & -\widetilde{\Gamma}_3\end{bmatrix}\right)\mathbf{A}.$$

The first term represents the classical series junction. As we can see in Fig. 2, the contribution of the second term is added to the reflected waves of this junction.

This structure, however, cannot be used for constructing wave digital structures, as it is not made of elements that are connected to each other through ports. If we could express $-(A_1+A_2+A_3)$ (the input of the three filters $\widetilde{\Gamma}_k(z)$) as a linear function of just the incident and reflected waves at each one of the adaptor ports, $A_1+A_2+A_3 = \alpha A_k + \beta \overline{B}_k$, we would obtain an implementation of the $\widetilde{\Gamma}_k(z)$ as two-port junctions. We notice that each reflected wave ends up depending on just one of the three reflection coefficients, and for the instantaneous term we have

$$\begin{bmatrix}\overline{B}_1\\ \overline{B}_2\\ \overline{B}_3\end{bmatrix} = \begin{bmatrix}A_1\\ A_2\\ A_3\end{bmatrix} - \begin{bmatrix}\gamma_1(A_1+A_2+A_3)\\ \gamma_2(A_1+A_2+A_3)\\ \gamma_3(A_1+A_2+A_3)\end{bmatrix}. \quad (3)$$

For each row, we can thus write

$$-(A_1+A_2+A_3) = \frac{1}{\gamma_k}(\overline{B}_k - A_k), \quad k = 1,2,3 \quad (4)$$

which yields

$$\begin{bmatrix}B_1\\ B_2\\ B_3\end{bmatrix} = \begin{bmatrix}1-\gamma_1 & -\gamma_1 & -\gamma_1\\ -\gamma_2 & 1-\gamma_2 & -\gamma_2\\ -\gamma_3 & -\gamma_3 & 1-\gamma_3\end{bmatrix}\mathbf{A}$$
$$+ \begin{bmatrix}\frac{\widetilde{\Gamma}_1}{\gamma_1}(\overline{B}_1 - A_1) & \frac{\widetilde{\Gamma}_2}{\gamma_2}(\overline{B}_2 - A_2) & \frac{\widetilde{\Gamma}_3}{\gamma_3}(\overline{B}_3 - A_3)\end{bmatrix}^T.$$

If we rewrite this relationship while keeping the adaptation into account, $\gamma_3 = 1$, $\gamma_1+\gamma_2 = 1$, we obtain

$$\begin{bmatrix}B_1\\ B_2\\ B_3\end{bmatrix} = \begin{bmatrix}1-\gamma_1 & -\gamma_1 & -\gamma_1\\ \gamma_1-1 & \gamma_1 & \gamma_1-1\\ -1 & -1 & 0\end{bmatrix}\mathbf{A}$$
$$+ \begin{bmatrix}\frac{\widetilde{\Gamma}_1}{\gamma_1}(\overline{B}_1 - A_1) & \frac{\widetilde{\Gamma}_2}{1-\gamma_1}(\overline{B}_2 - A_2) & \widetilde{\Gamma}_3(\overline{B}_3 - A_3)\end{bmatrix}^T$$

whose implementation is shown in Fig. 3.

We have thus obtained a structure made of an instantaneous series adaptor and three junctions that account for the whole dynamics of the initial adaptor. Yet, such elements do not form scattering junctions, as we might expect; instead, they form only half such junctions (see Fig. 3). By redrawing the three filters, as shown in Fig. 2, where they all have the same input $x = -(A_1+A_2+A_3)$, it can be easily shown that the filter $\widetilde{\Gamma}_3(z)$ (whose output is summed to the wave $\overline{B}_3$) can now be moved to the inputs of the other two ports, as long as we multiply it by the corresponding γ_k, after a sign change.

At each one of the two ports that are not reflection-free, we now have two filters having the same input signal x. The output of the first one, $\widetilde{\Gamma}_m(z)$, $m = 1, 2$, is summed to the reflected wave $\overline{B}_m$, while the output of the second, $\gamma_m\widetilde{\Gamma}_3(z)$, obtained by moving the filter that was at the reflection-free port, is summed to the incident wave A_m. In order for such filters to become a scattering junction, they need to be equal to each other, i.e.,

$$\widetilde{\Gamma}_1(z) = -\gamma_1\widetilde{\Gamma}_3(z), \quad \widetilde{\Gamma}_2(z) = -\gamma_2\widetilde{\Gamma}_3(z) \quad (5)$$

which, through (2) and the conditions of instantaneous adaptation $\gamma_1+\gamma_2 = 1$ and $\gamma_3 = 1$, become

$$\widetilde{\Gamma}_1(z)+\widetilde{\Gamma}_2(z)+\widetilde{\Gamma}_3(z) = (-\gamma_1-\gamma_2+1)\widetilde{\Gamma}_3(z) = 0.$$

In conclusion, by moving the filter that was connected to the reflection-free port, we obtained a structure made of a static adaptor and two dynamic scattering junctions placed at the ports that are not reflection free (see Fig. 4).

Notice that if we had two filters of the form $-\gamma_1 K(z)$ and $-\gamma_2 K(z)$ at the outputs $\overline{B}_1$ and $\overline{B}_2$, respectively, then we could

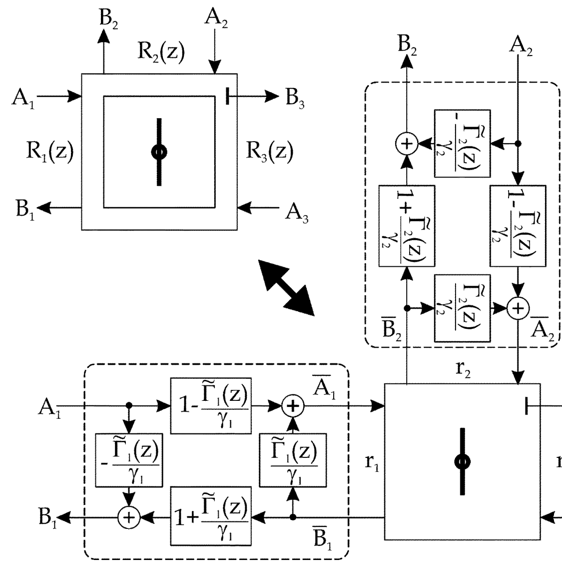

Fig. 4. Equivalence between a dynamic series adaptor and a static series adaptor with two WD mutators.

move such filters at the input a_3 to form a single filter $K(z)$. As a matter of fact, this is exactly our situation, as we have $-\gamma_m K(z) = -\gamma_m \widetilde{\Gamma}_3(z)$ at the ports that are not reflection-free. This can be easily verified by replacing $\widetilde{\Gamma}_m$ with $-\gamma_m \widetilde{\Gamma}_3$. If we move such filters at the input port 3, then, together with the filter $\widetilde{\Gamma}_3$ that we extracted before, we obtain a single scattering junction that accounts for the whole dynamics of the adaptor.

A dynamic series adaptor is equivalent to its instantaneous counterpart whose reflection-free port is connected to a WD mutator. The reflection filter of the mutator is that of the reflection-free port of the initial adaptor, up to a scaling factor.

It is important to notice that the scattering junction that we obtain is always computable, as $\widetilde{\Gamma}_k(z) = z^{-1}\widehat{\Gamma}_k(z)$, as defined in (1), with $\widehat{\Gamma}_k(z)$ causal and stable. A first consequence of this result is that, in the case of total adaptation on a dynamic series adaptor, we end up with a memoryless junction. In fact, $\Gamma_3(z) = 1$ implies $\widetilde{\Gamma}_3(z) = 0$; therefore, the scattering junction becomes a direct input/output connection.

B. Dynamic Parallel Adaptor

As far as the parallel adaptor is concerned, similar results can be achieved by exploiting the property of the gyrator [6] to transform a parallel adaptor into its dual (series adaptor). A parallel dynamic adaptor whose port admittances are

$$G_k(z) = \frac{g_k + \sum_{i=1}^{N_k} e_{ik} z^{-i}}{1 + \sum_{i=1}^{M_k} d_{ik} z^{-i}} = g_k + \frac{\sum_{i=1}^{N_k} g_{ik} z^{-i}}{1 + \sum_{i=1}^{M_k} d_{ik} z^{-i}}$$

is equivalent to a series dynamic adaptor (with the same port impedances as before), whose ports are connected to dynamic gyrators with unit gyration resistance. In the wave domain, a unit-resistance gyrator connected to a port impedance $R(z)$ is implemented as a block that individually filters the incident wave with $R(z)$, and the reflected wave with $-1/R(z)$

$$\mathbf{B} = \mathbf{M_p A} = -\mathbf{R}^{-1}\mathbf{M_s R A} \qquad (6)$$

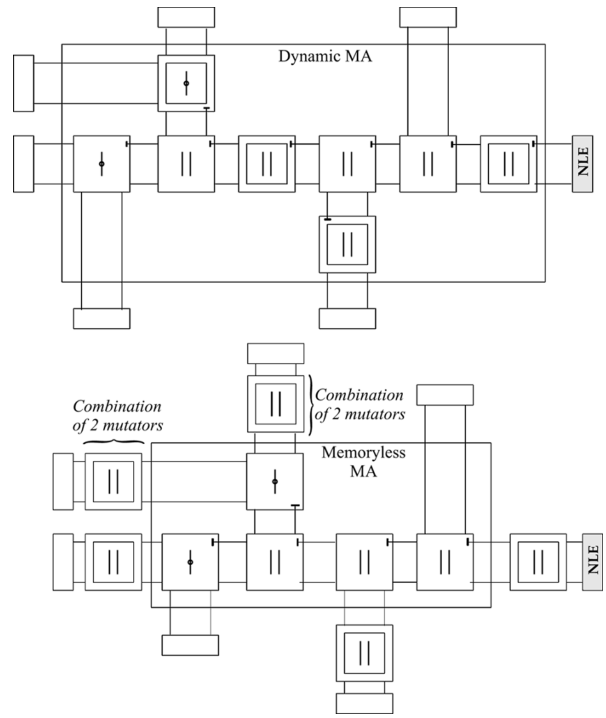

Fig. 5. Extracting the "memory" from a macro-adaptor.

where $\mathbf{R}$ is the matrix of gyration resistances. By solving for the constant part of the scattering matrix and by applying again to that the equivalence principle, using as gyration coefficients the constant part of the port impedance, we obtain the instantaneous part $\overline{\mathbf{M}}_p$ of the parallel adaptor. We thus obtain a structure that is similar to that of a series adaptor; therefore, we can extract the dynamics from it. Unlike the series case, however, in this case, we have a *dynamic transformer* at each port of the adaptor. This element can be interpreted in the wave domain as a wave filter that preserves the energy that flows through the port.

III. Memory Extraction From a Macro-Adaptor

It is now possible to solve the more general problem of "extracting the memory" from an MA. Given an adaptor, we want to find an equivalent structure made of a memoryless MA and a number of WD mutators connected to some of (or all) the ports.

Structural equivalencies—An MA made of the interconnection of parallel or series three-port dynamic adaptors can always be transformed into a new structure made of a memoryless multiport MA surrounded by WD mutators as shown in Fig. 5. This can be achieved by first extracting the memory from the dynamic adaptors of the MA and then having all the WD mutators "slide through" the inner adaptors according to specific rules, until they reach the periphery of the MA.

Our problem is to characterize the "sliding rules" that enable the "extraction of the memory" from inside the MA. In practice, we need to find the equivalence that exists between a three-port memoryless adaptor that has one port connected to a WD mutator and a like three-port memoryless adaptor, whose ports that are reflection-free are connected to WD mutators.

Let us consider a memoryless series adaptor whose reflection-free port is connected to a WD mutator having $K(z)$ as a reflection filter. The first step is to decompose the mutator into two filters with the same transfer function, driven by $X(z) = -(A_1(z) + A_2(z) + A_3(z)) = (1/\gamma_3)(B_3 - A_3) = B_3 - A_3$. Such a filter can be moved onto the other two ports (see previous section), provided that we multiply them by $-\gamma$ and that we change the sign of the port's reflection coefficient. The input of these filters can thus be written as a function of the port waves

$$\begin{aligned} X(z)K(z) &= -\gamma_m \left(-\left(A_1(z) + A_2(z) + A_3(z)\right)\right) K(z) \\ &= K(z) \left(A_m(z) - B_m(z)\right), \quad m = 1, 2. \end{aligned}$$

This corresponds to the same mutator that we started with, whose ports are now swapped (changing the sign to a mutator's port impedances corresponds to swapping its ports).

In conclusion: *a structure made of a memoryless series adaptor and a WD mutator connected to its reflection-free port is equivalent to the same adaptor whose ports that are not reflection-free are now connected to similar WD mutators.*

This rule holds true in the opposite direction: two identical WD mutators connected to the ports of an instantaneous series adaptor that are not reflection-free make a structure that is equivalent to the same adaptor connected through its reflection-free port to the same WD mutators. If only one of the two ports that are not reflection-free is connected to a WD mutator, then we can always connect the other port that is not reflection-free to the cascade of the same mutator with another one that has a port impedance of opposite sign. In fact, two mutators with opposite port impedances (and same initial conditions) cancel each other out. The pair of two-port mutators can now be moved to the reflection-free port. In conclusion: *a structure made of a memoryless series adaptor and a WD mutator connected to any of its ports is equivalent to the same adaptor whose other two ports are connected to similar WD mutators.*

Similar results can be obtained for the parallel adaptor by exploiting the property of gyrators to transform a parallel adaptor into a series one. This sliding rule is illustrated in a self-explanatory procedural fashion in Fig. 6.

Initial conditions—When we replace a structure having a WD mutator with another one having two like mutators, such elements with memory cannot be seen as independent of one another as this would correspond to increasing the number of state variables (and of initial conditions). The initialization of both mutators, in fact, depends on that of the original one. In order to derive this dependency, we can write the reflected wave of any of the ports for both structures, and then we can equate them. If we express the scattering filter in the form $X(z) = -(A_1(z) + A_2(z) + A_3(z))$ and take the adaptation condition $K(z) = z^{-1}\widehat{K}(z)$ into account, at the port 3 of the one-mutator structure, we will have $B_3' = -A_1 - A_2 + K(z)X(z)$, while for the two-mutator structure, we have

$$\begin{aligned} B_3'' &= -\left\{A_1 + \left[-\gamma_1 K(z)\right] X(z)\right\} \\ &\quad - \left\{A_2 + \left[-\gamma_2 K(z)\right] X(z)\right\} \\ &= -\left\{A_1 + \left[-K(z)\right] \gamma_1 X(z)\right\} \\ &\quad - \left\{A_2 + \left[-K(z)\right] \gamma_2 X(z)\right\}. \end{aligned}$$

B_3' and B_3'' are the same, as the sign change of the filter $K(z)$ in the expression of B_3'' depends on the mutator's orientation and $\gamma_1 + \gamma_2 = 1$. The coefficients γ_1 and γ_2 can thus be interpreted as the weights that decide how to partition the initial conditions between the two mutators in the equivalent structure.

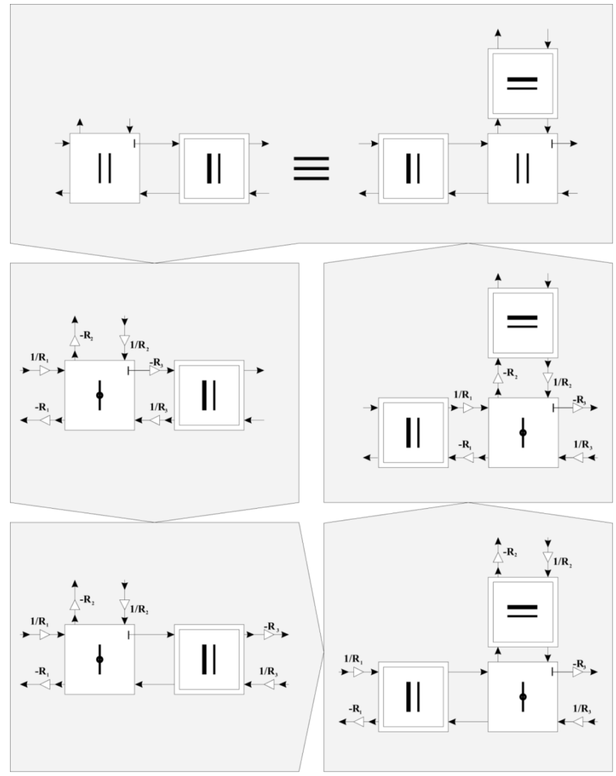

Fig. 6. All properties derived for the series adaptor remain valid for moving a mutator onto the ports that are not reflection-free in the case of parallel adaptor.

IV. Conclusion

In this letter, we showed that a tree-like network of parallel and series adaptors with memory in WD structures can always be replaced with a like network of memoryless WD adaptors whose outer ports are possibly connected to WD mutators or WD transformers. This process of memory extraction plays a key role in the automatic implementation of WD structures in a wide range of applications [4], [7]. The proof that we proposed for this result is operative, in that it provides a procedure for automating the memory extraction process. An interesting further development would be to derive similar results in the case of power-normalized waves.

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