A simplified model for probability of error
in DS CDMA systems with adaptive antenna arrays

Umberto Spagnolini
Dip. di Elettronica e Informazione, Politecnico di Milano  P.zza L. da Vinci, 32 I-20133 Milano (Italy)
e-mail: Umberto.Spagnolini@elet.polimi.it

Abstract—The multiaccess interference and the array gain has been modelled to evaluate the average probability of error in DS CDMA receivers with adaptive antenna arrays. The angular gain of the spatial filter (array beampattern) is modelled by a piece-line function that approximate the pass-band (or in-beam) with a fixed beamwidth and the stop-band (or out-beam) with an equivalent attenuation. The interferers are partitioned into in-beam and out-beam users and are counted differently for the evaluation of the probability of error. The in-beam users are counted as interference while the out-beam users increase the additive Gaussian noise level. This simple model adapted to 2D RAKE receiver structure and to space-time multiuser detection shows good agreements with simulation results.

I. INTRODUCTION

The promising approach to increase of spectrum efficiency in 3rd generation cellular systems is the use of spread-spectrum code-division multiple access (CDMA). The increase in system capacity by using antenna arrays in CDMA comes from the reducing of co-channel interference from other users within its own cell and neighboring cells. The capacity improvement with antenna arrays has been investigated in the past by evaluating the probability of outage [1] or the excess signal to noise ratio (SNR) due to array processing [2], [3]. Simultaneous demodulation of more users in multiuser detection (MUD) has been proposed to combat intra-cell interference and multipath channel distortion [4]. However, combined MUD and antenna array processing has been addressed recently [5], [6].

The exact analytical evaluation of the probability of error in CDMA system is still an open subject. Bounds are proposed [7] while Gaussian approximations have been derived in the literature for AWGN channels [8] and for MMSE MUD [9]. In this paper it is proposed to adapt these approximations to the antenna array systems by manipulating in the probability of error formulas for single-antenna receivers those terms that account for the noise and the multiple access interference (MAI). This is carried out by an equivalent model of the angular gain of the adaptive antenna array. The adaptive antenna array implies that the beamforming of the receiving system is designed to optimize the received power on the basis of the knowledge of the direction of arrivals for all the K users.

Performances in terms of average probability of error of a CDMA receiver based on adaptive antenna array calls for some simplification of the interference reduction capability of the beamforming. Spatial filter weights can be differently designed to cope with MAI, obviously the receiver performances are strictly related on the efficiency of MAI reduction. In systems characterized by users with direction of arrivals (DOAs) uniformly distributed within a specified sector the performances of a receiver with an adaptive array cannot be ascribed to the average level of interference [10] but are mainly ruled by the worst-cases. These arise when the user of interest has the same DOA of one or more interfering users and the beamforming is no-way effective for interference reduction. The average performance depends on the probabilities of each of this DOA alignments. The parameter that rules the efficiency of the array system is the ratio between the array beamwidth and the angular sector of the impinging DOAs. In this paper it is proposed to consider a simplified model of the spatial filter (beampattern) that allows to describe the angular gain function with a simple piece-line model: the pass-band (or in-beam) is approximated with a fixed beamwidth and the stop-band (or out-beam) with an equivalent attenuation. For a specified user of interest all the $K-1$ multiple access interferers are partitioned into in-beam and out-beam MAI, the simplified model proposed here handles the in-beam users as interferers while for the out-beam users the Gaussian noise approximation is assumed.

In Section 2 we present the signal model. In Section 3 is the description of the equivalent spatial filter used to derive (in Section 4) the probability of errors for CDMA systems with adaptive antenna arrays. In Section 5 approximations of the probability of error for 2D RAKE receiver structure and MMSE-MUD are derived and compared with simulations for perfect power control. Multipath Rayleigh fading is not dealt with here, see ref.[11] for the extension to this case.

II. SIGNAL MODEL

The $M \times 1$ vector of the received signal in a DS-CDMA system with $K$ users employing different normalized signa- tures $s_1(t), s_2(t), ..., s_K(t)$ and transmitting sequences of binary phase-shift keying (BPSK) symbols is modelled as

$$r(t) = \sum_{k=1}^{K} A_k a(\theta_k) \sum_l b_k[l] s_k(t - lT - \tau_k) + \eta(t).$$ (1)

The channel is modelled with one path per user. For the $k$-th user: $a(\theta_k)$ is the $M \times 1$ vector that describes the array response to the DOA $\theta_k$, the $n$-th component for half-wavelength antenna spacing is $\exp(j(m-1)\pi \sin \theta_k)$ for $1 \leq m \leq M$; $A_k$ is the received amplitude (in general complex valued), $b_k[l] \in \{-1, +1\}$ is the $l$-th transmitted symbol, $\tau_k$ is the relative delay uniformly distributed on $[0, T)$, $T$ is the symbol period. Noise is white, spatially uncorrelated ($\eta(t) \sim \mathcal{N}(0, I)$). 

---

0-7803-7097-1/01/$10.00 ©2001 IEEE 2271
and independent of \( b_k[n] \). For sake of simplicity it is assumed that the support of \( s_k(t) \) is on the interval \([0, T]\) and that it has unit energy (the symbol energy is \(|A_k|^2\)). After the beamforming with the spatial filter \( w_\ell \), the output of the \( \ell \)-th filter matched to \( s_\ell(t) \) is

\[
y_\ell[i] = \int s_\ell(t - iT - \tau_k)w_\ell^Hv(t)dt =
\sum_{k=1}^{K} A_k \sum_{l} b_k[l]G(\theta_k|\theta_\ell)\rho_{lk}(\tau_k - \tau_l) + \sigma n_\ell[i]
\]

where \( n_\ell(t) \sim \mathcal{N}(0, 1/\tau) \) as \( w_\ell^Hw_\ell = 1/\tau \), \( G(\theta_k|\theta_\ell) \) is the spatial gain of the uniform linear array beamformer designed for the specified angle \( \theta_\ell \) so that \( |G(\theta_k|\theta_\ell)| \leq |G(\theta_\ell|\theta_\ell)| = 1 \), \( \rho_{lk}(\tau_k - \tau_l) = \int s_\ell(t - iT - \tau_k)s_k(t - iT - \tau_l)dt \) is the asynchronous crosscorrelation between signatures [4]. In this paper we limit the analysis to the case of users with \( \theta_k \) uniformly distributed on \([\pm \Delta \theta/2, \pm \Delta \theta/2]\) and with the same amplitudes \( |A_1| = |A_2| = \ldots = |A_K| = A \) (i.e., perfect power control).

### III. Equivalent spatial filter

Let \( G(\theta|\bar{\theta}) \) be the gain of the spatial filter designed for \( \bar{\theta} \), the equivalent spatial filter gain \( G_{eq}(\theta|\bar{\theta}) \) is a piece-line approximation that decouples the in-beam (support \( \Theta(\bar{\theta}) = [\bar{\theta} - \theta_{BW}(\bar{\theta}), \bar{\theta} + \theta_{BW}(\bar{\theta})] \)) from the out-beam (support \( \Theta(\bar{\theta}) \)) sections:

\[
|G_{eq}(\theta|\bar{\theta})|^2 = \begin{cases} 
1 - \frac{1}{2} \frac{\theta_{BW}(\bar{\theta})}{\theta_{BW}(\bar{\theta})} & \text{for } \theta \in \Theta(\bar{\theta}) \\
\alpha_o & \text{for } \theta \notin \Theta(\bar{\theta}) 
\end{cases}
\]

This partition depends on the array selectivity, the angular aperture of the in-beam section \( 2\theta_{BW}(\bar{\theta}) \) is related to the number of antennas \( M \) and depends on \( \bar{\theta} \). A simple relationship that describes the beamwidth stretching vs. \( \bar{\theta} \) valid for small deviations from broadside (for \( \bar{\theta} \approx 0 \) deg) is [12]

\[
\theta_{BW}(\bar{\theta}) \approx \frac{\theta_{BW}}{\cos \bar{\theta}}.
\]

\( \Theta = \Theta(\bar{\theta}) \cup \Theta(\bar{\theta}) = [-\Delta \theta/2, \Delta \theta/2] \) covers all the admissible DOAs (e.g., in a mobile system with 3-cell sectorization the angles ranges within \( \pm 60 \) deg) and the users are partitioned accordingly (see figure 1). The parameters of \( G_{eq}(\theta|\bar{\theta}) \) can be found by solving numerically for the following equality

\[
\int_{\phi \in \Theta} |G(\phi|0)|^2d\phi = \int_{\phi \in \Theta} |G_{eq}(\phi|0)|^2d\phi
\]

with respect to \( \theta_{BW} \) and \( \alpha_o \). The optimization (5) carried out for conventional beamforming (i.e., \( w_\ell = a(\theta_\ell)/M \)) with \( M = 16 \) antennas leads to \( \theta_{BW} \approx 4 \) deg and \( \alpha_o \approx -18 \) dB. If needed, parameters for different \( M \) can be evaluated by optimization or by scaling the beamwidth with respect to the array aperture [11], i.e. \( \theta_{BW} = (M/M)\theta_{BW} \).

---

![Fig. 1. Equivalent beamforming \( G_{eq}(\theta|\bar{\theta}) \) (shaded area) and the in-beam/out-beam partitioning of the multiple access interference.](image)

### IV. Probability of error with adaptive array

The users can be partitioned into spatial equivalence classes similarly as proposed in [13] for the reduction of the detection complexity in CDMA receivers. The equivalence classes are interference-driven and depend on the users that have in-beam and out-beam DOAs with respect to the user of interest. The probability of error can be obtained by counting differently the effects of the in-beam from the out-beam users. To analyze the bit-error-rate we will consider, without any loss of generality, the user characterized by the DOA \( \theta_1 \). The remaining users counted as interferers are partitioned into the two disjoint subsets \( \{2, 3, \ldots, K\} = B(\theta_1) \cup \bar{B}(\theta_1) \)

such that \( \theta_j \in \Theta(\theta_1) \) if \( j \in B(\theta_1) \). Let \( |B(\theta_1)| \) be the cardinality of the set \( B(\theta_1) \) (i.e., the number of elements of \( B(\theta_1) \)), for \( |B(\theta_1)| \) in-beam users that contribute to the overall level of interference, the probability of error can be evaluated according the relationships \( P_e(\cdot) \) derived for CDMA receivers without the array of antennas (or \( M = 1 \)). The remaining \( |\bar{B}(\theta_1)| = K - 1 - |B(\theta_1)| \) users can be assimilated to Gaussian noise and thus contribute to modify the decision variable accordingly (see Sections V). For the evaluation of the average bit-error-rate all the cases should be considered according to a specified distribution of the DOAs \( p(\theta_1) \):

\[
P(A^2, \sigma^2, K) = \int_{\theta_1 \in \Theta} \left\{ \sum_{B(\theta_1) = 0}^{K-1} P_e(A^2, \sigma^2/M, |B(\theta_1)|) \right\} p(\theta_1) \ d\theta_1
\]

Note that since the noise is received at each antenna, the array gain \( 1/M \) has been included when evaluating \( P_e(A^2, \sigma^2/M, |B(\theta_1)|) \). For a perfect power control all the in-beam interferers after beamforming have (almost) the same power, a slight correction for the average energy of the in-beam interferers is needed to take into account for the shape of \( G(\theta|\bar{\theta}) \), this is described below. The probability of error due to the in-beam interferers depends on their number \( K \) and do not depend on \( \theta_1 \) as this is taken into account in the partition
(6), therefore \( P_e(A^2, \sigma^2 / M, |B(\theta_1)|) \approx P_e(A^2, \sigma^2 / M, K_I) \) and thus

\[
P(A^2, \sigma^2, K) \approx \sum_{K_I=0}^{K-1} P_e(A^2, \sigma^2 / M, K_I) \int_{\Theta \in \Theta} p(|B(\theta_1)|) p(\theta) d\theta_1
\]

Since DOAs are independent and uniformly distributed it is straightforward to evaluate the probability

\[
p(|B(\theta_1)|) = (\frac{K-1}{K_I}) p_{K_I}(\theta \in \Theta(\tilde{\theta}))(1 - p(\theta \in \Theta(\tilde{\theta})))^{K-K_I-1}
\]

(9)

\[
p(\theta \in \Theta(\tilde{\theta})) \text{ depends on the beamwidth } \theta_{BW}(\tilde{\theta}) \text{ compared to the overall support } \Theta
\]

\[
p(\theta \in \Theta(\tilde{\theta})) = \frac{\eta}{\cos \tilde{\theta}},
\]

(10)

\[\eta := 2\theta_{BW}/\Delta\theta\]

depends on the beamwidth criteria exploited. The error probability becomes:

\[
P(A^2, \sigma^2, K) \approx \sum_{K_I=0}^{K-1} \eta^{K-I} K_I \chi(\eta, K, K_I) P_e(A^2, \sigma^2 / M, K_I)
\]

(11)

where

\[
\chi(\eta, K, K_I) = \int_{-\Delta\theta/2}^{\Delta\theta/2} \frac{(\cos \tilde{\theta} - \eta) K-K_I-1}{(\cos \tilde{\theta}) K-I} d\tilde{\theta}
\]

(12)

depends on the beam stretching (4) and it can be evaluated numerically or, if needed, it can be approximated. However, if \( \Delta\theta < 60 \div 70 \deg \) the beam stretching is negligible and \( \chi(\eta, K, K_I) \approx (1 - \eta)^{K-K_I-1} \).

Because of the spatial filter approximation (3), the energy of each in-beam interferer uniformly distributed in \( \Theta(\theta_1) \) is a random variable uniformly distributed within \([A^2 / 2, A^2]\). At the first order, the error probability averaged over the energy distribution of the \( K_I \) in-beam interferers is the error probability for the average energy value \( \frac{1}{2} K_I A^2 \), this is the value considered here. It could be convenient, as for the 2D RAKE receiver discussed below, to virtually assume that all the in-beam users have the same energy; in this case the effective number of in-beam users can be corrected as \( \frac{1}{2} K_I \). The performance (11) can be adapted to different receivers depending on the probability of error of the scalar receiver \( P_e(A^2, \sigma^2 / M, K_I) \) as discussed in the next sections.

V. ADAPTIVE ANTENNA ARRAY RECEIVERS

This section adapts the approximate spatial filtering model described above to CDMA receivers. The purpose is two-folds: identify the effects of the in-beam and out-beam users and validate the approximation proposed with simulations. For all the users the DOAs are assumed known and spatial filter is based on the conventional beamforming. In practice, the results shown here remain valid until the error in DOA estimation is a fraction of the beamwidth \( \theta_{BW}(\tilde{\theta}) \).

A. RAKE receiver

In a multipath environment, the decision variable of the 2D RAKE receiver structure can be obtained by filtering each path of each user’s signal with the corresponding beamforming and by combining the matched filter outputs [14]. The interference rejection of 2D RAKE receiver relies only on the correlation properties of the spreading signatures, adaptive antenna arrays is known to provide an excess SNR for the reduction of the interference [2]. According to the one-ray model in Section 4 it is needed to evaluate the increasing of the noise level related to the number of out-beam interference modelled as Gaussian noise. The corresponding increasing of noise level at the decision variable depends on the spreading-length \( N \). For (chip and phase) asynchronous CDMA and random spreading sequences the standard Gaussian approximation holds true (see e.g., [8]):

\[
\sigma^2_{eq} = \frac{\alpha_e A^2 (K - K_I - 1)}{3N}.
\]

(13)

The probability of error for the 2D RAKE receiver is derived from (11):

\[
P_{2D-RAKE}(A^2, \sigma^2, K) = \sum_{K_I=0}^{K-1} \eta^{K-I} K_I \chi(\eta, K, K_I) \sum_{K_I=0}^{K-1} \frac{\alpha_e A^2 (K - K_I - 1)}{3N}.
\]

(14)

The error probability for chip and phase asynchronous CDMA with single-antenna has been widely investigated in the past. A simple but accurate approximation has been derived by Holtzman [8] for random spreading sequences of length \( N \). The error probability, reproposed here for sake of completeness, is composed of tree terms:

\[
P_{RAKE}(A^2, \sigma^2, K_I) = \frac{1}{2} Q \left[ \left( \frac{K_I N}{A^2} + \frac{\sigma^2}{A^2} \right)^{-1/2} \right] + \frac{1}{2} Q \left[ \left( \frac{K_I N}{A^2} - \frac{\sigma^2}{A^2} \right)^{-1/2} \right] + \frac{1}{2} Q \left[ \left( \frac{K_I N}{A^2} \right)^{-1/2} \right]
\]

(15)

\[\gamma^2 = 3K_I \left[ \frac{3\alpha_e}{300} N^2 + (N - 1)(1/20 + (K_I - 1)/36) \right] \text{ is the Q-function.} \text{ It is understood from (14) that all the users have the same energy except for the 3/4 correction of the in-beam users.}

Simulations for asynchronous CDMA have been carried out to validate the simplified model (14) with respect to the approximation by Song and Kwon [10] for phase synchronous CDMA. Here we consider the case of \( M = 16 \) antennas, random spreading sequence \( N = 31 \), \( K = 8 \) and 16 users. In figure 2 the signal to noise ratio (SNR) has been scaled with respect to the array gain \( 1/M \) so that from the performances it can be appreciated the spatial diversity gain with respect to the array gain \( 1/M \). The approximation of Song and Kwon [10] (dashed line) under-estimates the probability of error for large SNR while \( P_{2D-RAKE}(A^2, \sigma^2, K) \) (solid line) is tight to the simulated results (marks). In addition, the spatial diversity gain with respect to RAKE receiver for \( M = 1 \) demonstrates the efficacy of adaptive array systems in reducing the interference.
level. The results shown here demonstrates that the key parameters in 2D RAKE receivers is the ratio $\eta = 2\theta_{BW}/\Delta\theta$ that accounts for the probability of having the in-beam interferer when the users are uniformly distributed in $\Delta\theta$.

Remark 1: The relation (14) can be easily adapted to model the effects of intra- and inter-cell interferences in a cellular system. In this case the angular distribution of the interferers has to be re-defined according to the cell sectorizations.

\[ y_i = \sum_{k=1}^{K} A_k G(\theta_k | \theta_i) \rho_{ik} b_k[i] + \sigma n_i[i] \]  \hspace{1cm} (16)

where $\rho_{ik} = \rho_{ik}(0)$. The probability of errors for MMSE-MUD is derived in Poor-Verdu [9] for synchronous CDMA systems, the received CDMA signals after spatial filtering and de-spreading can be described by the matrix relationship:

\[ y = R A b + \sigma n' \]  \hspace{1cm} (17)

where the normalized correlation matrix is modified with respect to the scalar receiver to include the array gain $G(\theta_i | \theta_j)$:

\[ [R]_{ij} = G(\theta_i | \theta_j) \rho_{ij} + (1 - G(\theta_i | \theta_j) \rho_{ij}) \delta_{i,j} \]  \hspace{1cm} (18)

$\rho_{ij} \sim \mathcal{N}(0, R/M)$ and $\delta_{i,j}$ is the Kronecker delta. For the $k$-th user the decision variable for the MMSE-MUD is $[R + (\sigma^2/M) A^{-2}]^{-1} y_k$. Let the signals be equi-correlated so that $\rho_{ij} = \rho$ (for instance, when spreading signatures are obtained by shifting the same $m$-sequence) and with the same amplitudes $A = |A_1|$. According to the spatial model discussed in section IV the $K_j$ in-beam users experience a negligible attenuation and thus $[R]_{ij} \approx \rho + (1 - \rho) \delta_{i,j}$ for $i, j \in B(\theta_1)$. All the out-beam users are attenuated and contribute, at most, to the increasing of the noise level. However, the MMSE-MUD makes the contribution of the attenuated out-beam users decouple from the in-beam users and thus for a first order analysis it can be neglected. For single-antenna system ($M=1$) the probability of errors of MMSE-MUD when $K_j$ interferers have the same amplitude as the user 1 is [9]

\[ P_{MUD}(A^2, \sigma^2, K_1) = Q \left( \frac{A^2}{\sigma^2} - \frac{\rho^2 K_1}{1 + \rho^2 (K_1 - 1)} \right)^{1/2} \]  \hspace{1cm} (19)

This is useful to derive the average probability of error (11) by counting as interferers only the in-beam users:

\[ P_{\text{ST-MUD}}(A^2, \sigma^2, K) = \sum_{K_i=0}^{K-1} \eta^{K_1} \chi(\eta, K, K_i) (K_i - 1) P_{MUD}(A^2, \frac{\sigma^2}{K_1}, \frac{2}{K_1} K_1). \]  \hspace{1cm} (20)

It can be noticed that the calculation is as simple as for the 2D RAKE receiver (14).

Probability of error for synchronous CDMA with equi-correlated spreading sequences ($\rho = 0.25$) and $M = 16$ antennas are in figures 3-4. Simulations (marks) show a tight correspondence with the approximate model proposed here (20). Similarly as for RAKE receiver, the performance of MUD without the array of antennas ($M = 1$) are scaled with respect to the array gain $1/M$. The performance vs. signal to noise ratio for $K = 8$ users (figure 3) shows that there is only a slight additional advantage on the MAI reduction capability from adaptive
antenna arrays. However, compared to the 2D RAKE receiver (14) with \( N = 16 \) the MAI cancellation of MUD outperforms the 2D RAKE for high SNR. The performance vs. the increasing number of users \( K \) for SNR=10dB is shown figure 4 and it confirms that when the level of MAI increases the adaptive antenna array cannot exploit the MAI reduction for spatial diversity and asymptotically the gain approaches \( 1/M \).

Remark 2: The in-beam interference is over-estimated as the MUD is restricted to the in-beam users only and it is assumed \( \rho_{ij} = \rho \) for \( i \neq j \) and \( i,j \in B(\theta_1) \). In practice some users are not correlated with user 1 but still being correlated with other users in \( B(\theta_1) \), this would imply a revision of the model that assumes equicorrelated in-beam interferences. A complete solution of this problem (not covered here) would require the evaluations of the interference from the leakage coefficients of the MMSE decorrelating matrix \( (R + \sigma^2 A^T A)^{-1} \) for all the possible partitions of the matrix \( R \).

Remark 3: The negligible advantage of MMSE MUD after the adaptive array confirms that the MUD is effective in reducing the level of MAI. However, most of the benefits of MUD after spatial filtering arises in a cellular system with low reuse distance, in this case the adaptive antenna array is effective in reducing the inter-cell interference.

VI. CONCLUSIONS

The simple model proposed in this paper can be used to accurately evaluate the performances in term of probability of error in DS CDMA receivers with adaptive antenna array. The partition of the users into in-beam and out-beam interferers can largely simplify the analysis and it leads to a model that has been shown to be adequate for 2D RAKE and MMSE multiuser detection after adaptive beamforming. The proposed model can be easily extended to realistic multipath propagation environments such as Rayleigh faded channels [11]. The research in progress shows that the simplified model for beamforming and angular partitioning of in-beam/out-beam interference is not very sensitive to the spatial filter design criteria (e.g., MVDR beamforming).

REFERENCES