2-D phase unwrapping and phase aliasing

Umberto Spagnolini*

ABSTRACT

The phase of complex signals is measured modulo-2π (wrapped phase); continuous-phase information is obtained by adding properly chosen multiples of 2π shift to the wrapped phase. Unwrapping searches for the 2π combinations that minimize the discontinuity of the unwrapped phase as only the unwrapped phase can be analyzed and interpreted by further processing. The key problem of phase unwrapping is phase aliasing, a condition mainly caused by rapid phase variations. The extension of the one-dimensional (1-D) phase unwrapping algorithms to a two-dimensional (2-D) domain by 1-D slicing gives unsatisfactory results even in the presence of low-phase aliasing, whereas 2-D phase unwrapping deals with the complete problem, overcoming the limitations of 1-D unwrapping.

The 2-D unwrapped phase is obtained as the solution of a variational problem that minimizes the differences between the gradients of the wrapped and unwrapped phase. The Euler equation is then integrated using the boundary conditions obtained from the wrapped phase. In addition to determining a unique unwrapped phase, this approach has the advantage that it limits the influence of phase aliasing. It is also more attractive than iterative 1-D unwrapping since it limits the propagation of unwrapping errors.

Error propagation in phase unwrapping can strongly influence the result of any phase processing. Examples in this paper apply 2-D phase unwrapping to problems of refraction statics and interferometrical imaging using a remote system (SAR) and demonstrate how limited error propagation allows phase processing.

INTRODUCTION

Phase information is fundamental to signal processing (Oppenheim and Lim, 1981). Phase measured modulo-2n is called principal values (PV) and the amount of phase information is independent of any integer multiple of 2π added to the PV phase. However, to be useful for linear processing, this PV phase has to be unwrapped: this results in a continuous function, the 2π discontinuities being eliminated or, at least, reduced. The unwrapping of one- and two-dimensional (1-D and 2-D) phases requires different approaches, depending on the dimensions of the domain.

Homomorphic signal processing (Oppenheim et al., 1968) and deconvolution (Ulrych, 1971) both require unwrapped phase estimates. Two-dimensional phase unwrapping, when applied to surface-consistent deconvolution, phase-versus-offset processing of prestack data and Rytov approximation tomography (Woodward, 1989), is a valuable tool for improving processing and interpretation and is useful for interferometrical imaging using a remote system (Prati et al., 1990). The evaluation of lens aberration necessary for compensation in optical imaging requires the estimation of wavefront distortion only from phase differences; consequently, in-depth investigations have been made into the problems of phase ambiguities (see, e.g., Fried, 1977; Herrmann, 1980; Hudgin, 1977).

Conventional 1-D phase-unwrapping algorithms integrate the wrapped phase difference between two contiguous points, as described in Oppenheim and Schafer (1975). An improvement, suggested in Tribolet (1977), uses an adaptive integration of phase differences. Alternatively, phase-difference ambiguity due to sparse sampling can be overcome by taking samples at progressively closer intervals (Poggiagliolmi et al., 1982). Extending 1-D algorithms to contiguous or crossing 1-D slices of 2-D phase measurements does not yield satisfactory results. The quality of 2-D unwrapping depends on the strategy adopted for the 1-D slicing, as well as on 2-D phase sampling and noise.

In this paper I will address the problem of 1-D and 2-D phase aliasing of the sampled phase. Though the two approaches are similar, there are definite advantages to dealing
with phase in a 2-D domain. The regions where aliasing occurs are easily detected in the wrapped phase and can, if necessary, be isolated; this is not the case in a 1-D domain. Let us consider the phase in the 2-D domain: the technique for 2-D phase unwrapping presented here minimizes the discontinuities of the unwrapped phase. The objective function for 2-D phase unwrapping is the sum over the whole domain of the differences between the gradients of the wrapped (evaluated modulo-2π) and unwrapped phase. The solution of this variational problem is the 2-D unwrapped phase. The integration of the Euler equation requires the boundary conditions that are obtained from the wrapped phase, and it can be demonstrated that the computation expense for numerical integration can be reduced using the fast Fourier transform (FFT). Since the phase field obtained from the gradient of the wrapped phase can be considered as a vector field sum of rotational and n-rotational components, the 2-D aliasing and unwrapping can be analyzed from another point of view. The unwrapped phase corresponds to the n-rotational (curl-free) component while the residual phase (i.e., caused by phase aliasing) coincides with the rotational component.

Following the algorithm development, two applications of 2-D phase unwrapping are considered. In the statics study the measured phase of refracted first-arrival wavelets is unwrapped and decomposed into source, refracted and receiver components. The source and receiver components represent the estimated weathering phase distortion. The ratio between the weathering phase distortion and its estimated thickness (specific phase distortion) is, for the analyzed data, approximately constant. In the second example, two remote complex images from synthetic aperture radar (Rocca et al., 1989) are combined yielding the interferometrical phase image. Since the phase difference between two distinct points in the image is proportional to their relative heights, a terrain map is obtained after the 2-D phase unwrapping of the phase image.

PHASE ALIASING

For the complex signal \( p(x) \exp \{ j [ \theta (x)] \} \), the term \( [ \theta(x)]_p \) indicates the principal value (PV) of the unwrapped phase \( \theta(x) \) in n-dimensional space

\[
-\pi < [ \theta(x)]_p \leq \pi .
\]  

(1)

All the phases differing from the PV by multiples of \( 2\pi \) are ambiguous because they contain the same information. Algorithms for the unwrapping of the wrapped phase \( \theta_w(x) \) search for the \( 2\pi \) shift combination that produces a continuous-phase curve \( \epsilon(x) = \theta_w(x) + 2\pi n(x) \). The unwrapped phase difference \( \theta(x_2) - \theta(x_1) \) could be estimated from the wrapped phase. To avoid phase aliasing in phase-difference evaluation it is fundamental to assume that for the closely spaced points \( x_1 \) and \( x_2 \) the following relationship holds:

\[
\theta(x_2) - \theta(x_1) = [ \theta(x_2) - \theta(x_1)]_p .
\]  

(2)

Phase unwrapping techniques are generally based on estimates of PV-phase difference assuming this relationship. Since PV phase difference should be independent of the unwrapping (i.e., the congruency condition on phase difference: \( [ \theta(x_2) - \theta(x_1)]_p = [ \theta_w(x_2) - \theta_w(x_1)]_p \), the unwrapped phase can be obtained from the summation of the PV phase differences evaluated for adjacent points

\[
\theta(x_N) = \theta_w(x_1) + \sum_{k=2}^{N} [ \theta_w(x_k) - \theta_w(x_{k-1})]_p ;
\]  

(3)

any ambiguity due to multiples of \( 2\pi \) of the unwrapped phase is irrelevant. Phase aliasing is the crucial problem of phase unwrapping and is examined through the analysis of the phase difference in the 1-D and 2-D domains.

1-D phase aliasing

Figure 1 illustrates 1-D phase unwrapping. The 1-D unwrapped phase \( \theta(x) \) is obtained from the original wrapped phase, or PV phase, \( \theta_w(x) \), from the equation

\[
\theta(x) = \theta_w(x) + 2\pi \sum a_i \delta(x - x_i) ,
\]  

(4)

where \( x_i \) are the discontinuities where the unwrapped phase differs from the wrapped phase by multiples of \( 2\pi \), and the step function (\( \delta \)) has amplitude \( a_i = \pm 1, \pm 2, \ldots \) . The first derivative of (4) is:

\[
\frac{d\theta(x)}{dx} = \frac{d\theta_w(x)}{dx} + 2\pi \sum i a_i \delta(x - x_i) .
\]  

(5)

One-dimensional phase unwrapping should identify the amplitudes \( a_i \) and the positions \( x_i \) of the sequence of unknown impulses \( \sum_i a_i \delta(x - x_i) \) from all that is known of the wrapped phase \( \theta_w(x) \).

To avoid aliasing in 1-D sampled phase, certain conditions, all obtained from the relationship (2), must be fulfilled:

1) the step amplitude should be \( a_i = \pm 1 \);
2) the minimum distance between two discontinuities should be

\[
\pi \leq |x_2 - x_1| = \Delta x .
\]  

FIG. 1. The 1-D unwrapped phase \( \theta(x) \) is obtained from the wrapped phase \( \theta_w(x) \) and a combination of \( 2\pi \) steps: \( \theta(x) = \theta_w(x) + 2\pi \sum_i a_i \delta(x - x_i) \), (where \( a_i = \pm 1, \pm 2, \ldots \) ). The 1-D phase aliasing occurs when the maximum phase variation between two samples is greater than \( \pi \).
where $\Delta x$ is the sampling interval.

Both conditions require that the effective maximum-phase variation between two samples be less than $\pm \pi$. This corresponds to an upper limit in the reliable evaluation of the first derivative of the phase. Dealing with complex signals, this is the instantaneous frequency or the instantaneous time delay when considering Fourier transformed data.

Since 1-D unwrapping algorithms generally retrieve the phase from an integration of the estimated first derivative in a sampled domain (i.e., PV phase difference) (Tribolet, 1977; Poggiagliolmi, 1982), an incorrect estimation caused by phase aliasing of the amplitude and position of the impulses from the wrapped phase leads to uncontrolled propagation of phase-unwrapping errors. Unfortunately, the control or even the identification of phase aliasing in 1-D unwrapping is not always feasible.

2-D phase aliasing

Figure 2 illustrates 2-D phase unwrapping where the signal $p(x, y) \exp \{i \theta(x, y)\}$ is defined in a 2-D domain $\Omega$. The unwrapped phase $\theta(x, y)$ is now obtained from the wrapped phase $\theta_w(x, y)$ by adding a combination of 2-D steps, each step lying along a curve of the equation $\alpha_i(x, y) = 0$ in $\Omega$ as shown in Figure 2. Let $P(x, y)$ indicate the vector identifying the point $(x, y)$; then $P_i(x, y)$ is the coordinate-vector along the curve $\alpha_i(x, y) = 0$. The unwrapped phase becomes the 2-D equivalent of equation (4)

$$\theta(x, y) = \theta_w(x, y) + 2\pi \sum_i a_i \text{step}_{2-D}(P(x, y) - P_i(x, y)),$$

where $\text{step}_{2-D}(P(x, y) - P_i(x, y))$ is the 2-D step function. By applying the gradient to equation (7) the 2-D equivalent of equation (5) is obtained

$$\nabla \theta(x, y) = \nabla \theta_w(x, y) + 2\pi \sum_i a_i \delta(P(x, y) - P_i(x, y)).$$

Here, the vector $\delta(P(x, y) - P_i(x, y))$ describes along the curve $\alpha_i(x, y) = 0$, a front of impulses oriented toward the increasing 2-D step function. Whereas in 1-D the aliasing depends on the discontinuity location, in 2-D it depends on the implicit curves $\alpha_i(x, y) = 0$ in the domain $\Omega$. In 2-D phase unwrapping the curves $\alpha_i(x, y) = 0$ needs to be recovered from the wrapped phase.

Thus the conditions to avoid 2-D phase aliasing are:

1) the step amplitude should be $a_i = \pm 1$;
2) (a) $\alpha_i(x, y) = 0$ is a continuous curve in the domain $\Omega$ and can become, as a limit, an isolated point;
   (b) the curve $\alpha_i(x, y) = 0$ can be either a closed line or it should have end points coincident with the boundaries of the domain $\Omega$;
3) the minimum distance between two points $(x, y)_i$ and $(x, y)_j$ of two distinct curves $\alpha_i(x, y) = 0$ and $\alpha_j(x, y) = 0$ should be $|(x, y)_i - (x, y)_j| \geq \Delta_{\min}(x, y)$, where $\Delta_{\min}(x, y)$ is the minimum sampled distance.

The implicit curves never intersect since this would contradict condition 1. In 2-D phase unwrapping the implicit curves should be recovered from the wrapped phase, and aliasing could occur (i.e., implicit curves are mistaken or incomplete). The end points of the recovered curves $\alpha_i(x, y) = 0$ are called singular points and identify the sites of phase aliasing, as shown in the example in Figure 2. Unlike 1-D unwrapping where aliasing leads to uncontrolled error propagation, singular points can be detected from the 2-D wrapped phase.

In 2-D phase unwrapping, the incomplete retrieval of the implicit curve limits the unwrapping errors to a region, while the singular points spread the errors all over the domain $\Omega$. Strong error propagation arises whenever an algorithm similar to the one proposed from relationship (3) is applied along an arbitrary path neglecting the location, or even the existence, of the singular points (e.g., the unwrapped phase increases by $\pm 2\pi$ for each closed path around each singular point). The recommended strategy to reduce error propagation caused by phase aliasing is to try to retrieve the closed curves $\alpha_i(x, y) = 0$ [property (2b)] from only the knowledge of the incomplete open curves and their singular points. Phase-only techniques for phase unwrapping are strongly sensitive to the distribution and values of the singular points. Any attempt to unwrap the phase in the presence of singular points will fail unless the application itself provides a priori constraints on the location of the curves $\alpha_i(x, y) = 0$ (Prati et al., 1990).

The phase-aliasing problem is further complicated by a low signal-to-noise (S/N) ratio in real data, as low S/N ratio of signals increases the singular points distribution. Singular points due to phase aliasing and low S/N ratios are barely distinguishable, and any assumption about their classification is unlikely to lead to a unique general unwrapping criterion. Figure 3a shows an example of wrapped phase in the presence of phase aliasing. Figure 3b shows error propagation caused by the $2\pi$ step in the unwrapped phase, when 2-D phase unwrapping is obtained from the summation of PV phase differences along vertical paths. When the standard deviation of spatially uncorrelated Gaussian-additive phase noise is approximately $\sigma \approx 40$ degrees (Figure 3c) the error propagation leads to completely unreliable results (Figure 3d).

2-D PHASE UNWRAPPING

The algorithm

A phase-only unwrapping algorithm is hereafter considered as one that minimizes the mean square difference between the PV gradients of the wrapped phase and the gradients of the unwrapped phase [i.e., imposing the more general nonaliasing condition given by relationship (2) in the domain $\Omega$]. The technique is extended to include the distribution of singular points due to low S/N ratio.

Let us consider the PV spatial derivatives of the wrapped phase in a domain $\Omega$ sampled at regular intervals (defined as: $x = x_i$ and $y = y_j$):

$$[\Delta \theta_w(x_i, y_j)]_p = [\theta_w(x_i + 1, y_j) - \theta_w(x_i, y_j)]_p,$$

$$[\Delta \theta_w(x_i, y_j + 1)]_p = [\theta_w(x_i, y_j + 1) - \theta_w(x_i, y_j)]_p$$

and

$$[\Delta \theta_w(x_i + 1, y_j)]_p = [\theta_w(x_i + 1, y_j) - \theta_w(x_i, y_j)]_p$$

$$[\Delta \theta_w(x_i, y_j + 1)]_p = [\theta_w(x_i, y_j + 1) - \theta_w(x_i, y_j)]_p$$

where $\Delta_{\min}(x, y)$ is the minimum sampled distance.
The 2-D unwrapped phase \( \theta(x, y) \) in a domain \( \Omega \) is obtained from the wrapped phase \( \theta_w(x, y) \) which is limited \(-\pi \leq \theta_w(x, y) \leq \pi\) and a combination of 2-D \( 2\pi \) steps all represented here in gray scale [i.e. \( \theta(x, y) = \theta_w(x, y) + 2\pi \sum_i \alpha_{i, \text{step}} (P-P_i) \)]. The \( 2\pi \) steps are located along curves \( \alpha_i (x, y) = 0 \) in the domain \( \Omega \). However, mistaken or incomplete curves \( \alpha_i (x, y) = 0 \) recovered from \( \theta_w(x, y) \) underline a condition of 2-D phase aliasing as indicated by singular points in the wrapped phase (two small dots in \( \theta_w(x, y) \)).
The phase field \( \Theta(x, y) \) obtained from the principal value phase gradient \( \nabla_p \theta_w(x, y) \) for a wrapped phase \( \theta_w(x, y) \) is defined as:

\[
\Theta(x, y) = \nabla_p \theta_w(x, y) = \left[ \Delta \theta_x(x, y) \right]_p i + \left[ \Delta \theta_y(x, y) \right]_p j,
\]

(11)

where \( i \) and \( j \) are unit vectors on the \( x \) and \( y \)-axis. The mean-square (MS) error function is defined as the difference between the gradients of the wrapped (evaluated PV) and the unwrapped phase integrated in the domain \( \Omega \)

\[
\int \int_\Omega |\Theta(x, y) - \nabla \theta(x, y)|^2 \, dx \, dy.
\]

(12)

The unwrapped phase \( \theta(x, y) \) is then estimated from the minimization of this MS error function. The same relationship (12) is useful for the solution of those problems where the phase can be represented using an assigned model with few parameters (model fitting). Herrmann (1980) first introduced this approach for the estimation of optical aberration and compensation in imaging. This method of phase unwrapping is more general since no a priori assumptions are made on the model of the unwrapped phase. In a square domain \( \Omega \) consisting of \( N^2 \) points, there are \( N^2 \) unknown unwrapped phase values and \( 2(N - 1)^2 \) known phase gradient measurements. Consequently, the unwrapped phase estimation problem, as well as the model fitting, is overdetermined and can be solved using a least-mean square (LMS) algorithm.

Fig. 3. Example of 2-D phase unwrapping using the summation of PV phase difference for aliased phase model shown in Figure 2. (a) The 2-D aliased phase model. (b) The unwrapped phase using iterative 1-D algorithm (vertical top down 1-D slicing) of aliased 2-D phase propagates the error because of \( \frac{2\pi}{40} \) step as compared with the unwrapped phase in Figure 2. (c) Spatially uncorrelated Gaussian phase noise (\( \sigma \approx 40 \) degrees) added to the wrapped phase makes the iterative 1-D unwrapping useless since singular points due to phase aliasing and noise are scarcely distinguishable, as shown in (d).
The variational problem (12) leads to the Euler equation for the unwrapping:
\[ \nabla \cdot \Theta(x, y) - \nabla^2 \theta(x, y) = 0. \]  
(13)

When the phase has been properly sampled with respect to rapid phase slopes (i.e., when it is not aliased), the LMS estimation leads to a unique and true solution of the unwrapped phase. The PV phase gradient of the wrapped phase and the gradient of the unwrapped phase are equivalent since the relationship (2) holds true. From analysis of equation (13), it follows that the nonaliased phase field \( \Theta(x, y) \) is irrotational; in other words, phase unwrapping has a unique solution only when the phase field is irrotational. In this case, the simple and efficient unwrapping algorithm using the integration of PV phase gradient along any path follows:
\[ \theta(x, y) = \int_{(x_0, y_0)}^{(x, y)} \Theta(x, y) \cdot dl + \theta(x_0, y_0). \]  
(14)

Only an irrotational phase field guarantees the independence of the unwrapped phase from the integration path.

In real phase data the 2-D aliasing conditions are verified using only indirect methods on PV phase gradients. The phase field obtained from the PV phase gradient of the wrapped phase (11) can be decomposed into an irrotational \( \Theta_I(x, y) \) and a rotational \( \Theta_R(x, y) \) phase field:
\[ \Theta(x, y) = \Theta_I(x, y) + \Theta_R(x, y), \]  
(15)

where \( \nabla \times \Theta_I(x, y) = 0 \) and \( \nabla \cdot \Theta_R(x, y) = 0 \) holds. An analysis of equation (15) reveals the condition of 2-D phase aliasing. The divergence operator of phase field (15) separates the irrotational from the rotational component. The PV of the Laplace operator for the wrapped phase, defined here as
\[ \nabla^2 \theta(x, y) = -[\Delta \theta_I(x, y)]_p + [\Delta \theta_I(x, y)]_p \]
\[ -[\Delta \theta_R(x, y)]_p + [\Delta \theta_R(x, y)]_p, \]  
(16)

is useful for the unwrapping. From the Euler equation (13) the phase-unwrapping algorithm becomes
\[ \nabla^2 \theta(x, y) - \nabla^2 \theta(x, y) = 0. \]  
(17)

This relationship indicates that the phase unwrapping is limited only to the rotational phase component. When a 2-D phase is properly sampled and aliasing does not occur (i.e., \( \Theta_R(x, y) = 0 \)) the two unwrapping techniques, represented by equation (17) and the integration of PV phase gradient (14), are equivalent. The integration of the unwrapping differential equation (17) requires boundary conditions. In phase unwrapping, the PV of the normal derivative of the wrapped phase at the boundaries is examined (Neumann condition) unless the unwrapped phase is known at the boundary (Dirichlet condition). Assuming that the 2-D relationship (2) holds true at the boundary, the Neumann condition is obtained using the PV phase gradient of the wrapped phase
\[ \frac{\partial \theta(x, y)}{\partial n} = \Theta(x, y) \cdot n(x, y), \]  
(18)

where \( n(x, y) \) corresponds to the unit vector at the boundary of \( \Omega \). The unwrapped phase with the Neumann condition has a constant phase shift that is of little importance in phase unwrapping and can be determined by minimizing the difference between wrapped phase and PV unwrapped phase. Phase aliasing in the Neumann boundary conditions can influence the results of the 2-D unwrapping.

The unwrapping partial differential equation (17) is integrated using iterative techniques with the boundary conditions (e.g., Lapidus and Pinder, 1982). The most flexible and useful iterative method to solve elliptic equations is the successive over-relaxation technique that guarantees convergence in approximately 2N iterations for a square domain \( \Omega \) of \( N^2 \) nodes. The Appendix shows an alternative approach that allows the reduction of the computational time using the 2-D Fourier transform.

Comments on phase unwrapping and phase aliasing

The rotational phase field \( \Theta_R(x, y) \) represents the component due to 2-D phase aliasing. The rotational component is caused by the presence of rotational sources in the domain \( \Omega \) that correspond to the singular points of the curves \( \alpha_j(x, y) = 0 \) previously considered. The integration of the phase field along an arbitrary closed path in \( \Omega \) depends on the rotational component
\[ \oint \Theta(x, y) \cdot dl = \oint \Theta_R(x, y) \cdot dl = \sum 2\pi m_i, \]  
(19)

where \( m_i \) is the multiplicity of each singular point enclosed in the path. This relationship is useful for detecting, from the wrapped phase, isolated singular points within the domain \( \Omega \). Needless to say, the minimum closed line (4 points in the sampled domain) should be used. On the basis of the detection of singular points, Takajo and Takahashi (1988) proposed a 2-D unwrapping algorithm that estimates the \( 2\pi \) discontinuities of \( \Theta_R(x, y) \) from the wrapped phase. As the problem is underdetermined, it is necessary to make arbitrary assumptions that tend to influence the uniqueness of the unwrapped phase.

When strong phase discontinuities are unwrapped using algorithm (17), then the unwrapped phase around each singular point is smoother than true discontinuity (here \( \geq \pm \pi \)), and this gives rise to limited areas where the error \( |\Theta(x, y) - \nabla \theta(x, y)|^2 \) is not negligible compared to the overall MS error in the domain \( \Omega \) (the error decreases with the distance \( r_i \) from each singular point as \( m_i/r_i \)). In these areas, the unwrapping is considered ambiguous, and the estimated unwrapped phase is not congruent with the wrapped phase. The unwrapping may still have limited areas of ambiguous solution even when the multiplicity of all the singular points inside the domain \( \Omega \) vanishes. This condition indicates that the aliasing is likely to be limited within the domain, while the unwrapped phase obtained through the integration of the PV phase gradient along the boundary (14) is useful as the Dirichlet boundary condition.

The estimate of the residual phase \( \theta_R(x, y) \), which is caused by the rotational component considered herein, is obtained from the congruency condition on the phase:
\[ \theta_R(x, y) = [\theta(x, y)]_p + \Theta_R(x, y). \]  
(20)
Depending on the choice of the phase shift, the wrapped residual contains $2\pi$ discontinuities along arbitrary lines connecting the singular points. Whenever strong phase aliasing occurs in limited areas of the domain, the unwrapping of the residual $\theta_R(x, y)$ is one possible solution to also partially retrieve the ambiguous residual phase. However, the proposed algorithm is ineffective in unwrapping such a residual which is due to the rotational component, thus the integration of the PV phase gradient of the residual should be used. Depending on the particular application, other approaches that attempt to retrieve the rotational phase component are equally valid. The unwrapped phase for weak aliasing (a few isolated singular points on limited areas) that satisfies the congruency conditions on the phase could be obtained as a combination of the unwrapped phase and the residual phase, if necessary unwrapped: $\theta(x, y) + \theta_R(x, y)$.

Figure 4 shows the new algorithm of 2-D phase unwrapping applied to the example in Figure 3. The noise-free unwrapped phase that satisfies the congruency conditions on phase is shown in Figure 4a; the residual (Figure 4b) contains the $2\pi$ discontinuity along a line connecting the two singular points. Even the noisy wrapped phase of Figure 3c can be retrieved successfully, and the discontinuity is found in approximately the same position (Figure 4c) as in the noise free example. The residual in Figure 4d illustrates the position of the singular points due to aliasing and noise.

When considering limited S/N ratio of data, the unwrapped phase can be obtained from the minimization of the weighted residual of the phase field (12):

$$\int \int_{\Omega} w(x, y)|\Theta(x, y) - \nabla \theta(x, y)|^2 \, dx \, dy; \quad (21)$$

where the weight $w(x, y)$ depends on S/N ratio of the complex signal. The variational problem leads to the Euler equation

$$\nabla w(x, y) l [\Theta(x, y) - \nabla \theta(x, y)] + w(x, y) \times [\nabla l \Theta(x, y) - \nabla l \theta(x, y)] = 0, \quad (22)$$

that can then be integrated using numerical over-relaxation with the same boundary conditions previously examined.

APPLICATIONS

Phase decomposition of refracted first arrivals

In refraction statics the measured amplitude $A(s, r)$, phase shift $\theta(s, r)$, and time delay $T(s, r)$ of first-arrival wavelets are three full bandwidth parameters depending on the source and receiver locations $s$ and $r$ (Spagnolini, 1991). Under the assumption that the refracted path is the minimum time path, each parameter can be decomposed into three terms: one depending on source location only, one on the receiver location only, and the third depending on source and receiver:

1) amplitude decomposition: $A(s, r) = a(s)a(r)$;
2) phase decomposition: $\theta(s, r) = \phi(s) + \int k(x) \, dx + \phi(r)$; the phase $\phi(x)$ in the space location $x$ represents the weathering layer phase distortion, while $k(x)$ is the refractor specific phase distortion;
3) time decomposition: $T(s, r) = t(s) + \int \frac{dx}{V(x)} + t(r)$; the time delay $t(x)$ (linear phase delay) depends on the velocity and thickness of the weathering layer; the refracted time delay depends on the refractor slowness $1/V(x)$.

In other words the decompositions are equivalent to the cascade of three linear filters where the parameters of each filter depend on the spatial location.

LMS techniques give reliable estimates of parameter decomposition by taking advantage of the high data redundancy due to multiple coverage of the seismic gather. The time decomposition corresponds to the basic assumption used for statics computations from refracted first-arrival times $T(s, r)$. The wrapped phase shift $\theta_w(s, r)$ can be measured from common shot gathers using a coherent picking of first arrivals [see Spagnolini (1991) for details]. As phase and time decompositions involve the same relations, they share the same LMS parameter estimation technique, however the measured phase shift should first be unwrapped in the source-receiver domain.

Assuming a phase distortion varying slowly in space, the phase noise of the real data analyzed was evaluated from a histogram of the PV phase gradient $\nabla \theta(s, r)$ of the measured unwrapped phase. Figure 5 shows the histogram fit with a zero mean wrapped around Gaussian distribution of $\sigma$ = 40 degrees. This gives an approximation of the noise level in the field data. The noise-induced singular points for the 2-D phase unwrapping are approximately the same as in the example in Figure 3c. As demonstrated in the synthetic example in Figure 3d for the same estimated noise level, the unwrapping that uses the summation of PV phase difference along arbitrary paths is unreliable for phase decomposition. Moreover, the unwrapping is not unique since it depends on the chosen summation path. In practice, the 2-D unwrapping of data with a phase noise level $\sigma$ > 30 degrees is difficult to handle using the summation of PV phase differences.

The LMS decompositions of phase, unwrapped with the Neumann boundary condition and first-arrival times, give phase distortion and weathering layer thickness (obtained from time delay $t(x)$ assuming a constant weathering velocity of 600 m/s) as shown in Figure 6. The plotting of estimated weathering phase distortion versus the estimated thickness (Figure 7) allows an evaluation to be made of weathering specific phase distortion. The correlation is good for the first 200 stations along the seismic line. The slope of the fitting line measures the weathering specific phase distortion, approximately 30 rad/km. The average (over the seismic line) refractor specific phase distortion is $k(x) \approx 3.5$ rad/km. The two different values of specific phase shift demonstrate a greater phase distortion in the weathering layer than in the refractor, as expected (Angeleri and Loinger, 1984).

Interferometrical imaging using remote system

The remote sensing of the earth’s surface can be achieved using electromagnetic waves with wavelengths of 3-30 cm. Synthetic aperture radar (SAR) is a remote system that provides its own source of illumination and measures the backscattered energy (Fitch, 1988). Rocca et al. (1989) have
shown that, because of the strict correlation of seismic imaging to SAR processing, the methods for downward continuation of seismic data and those for the focusing of SAR images are, essentially, identical. However, since SAR deals with narrowband data, the preservation of the quality of amplitude and phase information depends on the accuracy of the processing.

Assume that $S_1$ and $S_2$ are two source locations that illuminate a scattering point $P$ at coordinate $(x, y)$ in the image coordinate system. Since the remote system measures the backscattered energy, the gathered image data $I_1(x, y)$ and $I_2(x, y)$ are zero offset as shown in Figure 8. The amplitude of the backscattered fields of the point $P$ depends only on the reflectivity $R(P)$ (assumed independent of the illumination). Since the distance of the point from the sources is $S_1P = S_2P$, the amplitudes are $|I_1(x, y)| = |I_2(x, y)| = R(P)$. The phase depends on the travel path:

$$I_1(x, y) = R(P) \exp \left\{ j \frac{2\omega_0}{c} S_1 P \right\},$$  \hspace{1cm} (2.3)

$$I_2(x, y) = R(P) \exp \left\{ j \frac{2\omega_0}{c} S_2 P \right\},$$  \hspace{1cm} (2.4)

where $\omega_0$ is the angular frequency of the transmitted wave and $c$ is the speed of light. The phase $\theta(x, y)$ obtained from the composite complex image $I_1(x, y)I_2^*(x, y) = |R(P)|^2 \exp \{ j \theta(x, y) \}$ (where $*$ indicates the complex conjugate) depends on the scattering point location (interferometrical phase image). However, geometrical analysis of the experiment

![Figure 4](image)

**Figure 4.** Example of the 2-D phase unwrapping using the proposed algorithm that minimizes the differences between the gradients of the wrapped and the unwrapped phase (17) using Neumann conditions at the boundary. (a) The unwrapped phase retrieves the 2\pi step in the aliased phase when compared to iterative I-D algorithm of Figure 3b and to the correctly unwrapped phase of Figure 2; (b) residual $\theta_R(x, y)$ is restricted around a straight line connecting singular points. (c) The unwrapped noisy phase shown in Figure 3c should be compared to the vertical I-D slicing in Figure 3d; (d) the residual $\theta_R(x, y)$ also shows singular points due to aliasing and noise.
shows that the phase difference between two distinct points \( \theta(P_2) - \theta(P_1) \) is proportional to their relative heights. With regard to the image axes, the processing requires that narrow-band images \( I_1(x, y) \) and \( I_2(x, y) \), obtained from two viewpoints, are first aligned using spatial correlation between the backscattered amplitude images \( |I_1(x, y)| \) and \( |I_2(x, y)| \). The unwrapped interferometrical phase image is then proportional to the terrain elevation. The phase aliasing and the singular points in the phase image are mainly caused by steep dips or edges in the illuminated topography. Since the two images are not usually gathered simultaneously, moving objects appear noisy in phase when the object displacement is greater than the field wavelength (e.g. corrugated water surfaces).

Figure 9a shows the wrapped interferometrical phase image (top left) of remote data obtained from a Brufjorden zone (Norway) of 6 x 6 km (selected amplitude image in gray scale, bottom left). The unwrapped phase (Figure 9a bottom right and Figure 6b), obtained from the algorithm with Neumann boundary conditions, is proportional to the terrain elevation, as appears from comparison with the amplitude image, and is useful for remote surface mapping. The residual due to the rotational component still contains a few open implicit lines \( \alpha(x, y) = 0 \). It is more valuable to compare the local error \( \mathbf{r}(x, y) = \nabla \theta(x, y) \) with the estimated terrain elevation to appreciate the reliability of the mapping image as shown in Figure 9 ((a) top right). The large number of singular points, indicated by an arrow in this example, are due to a small lake in the center of the image.

CONCLUSIONS

Two-dimensional phase aliasing has been examined, and a phase unwrapping algorithm has been proposed. Aliasing and noise make phase unwrapping ambiguous. An unwrapped aliased phase is always affected by errors but a careful implementation of the algorithm can limit error

---

**FIG. 5.** The histograms of PV spatial derivatives \( ([\Delta \theta_s(r, s)]_r \text{ and } [\Delta \theta_s(r, s)]_s) \) of the phase shift \( \theta_w(s, r) \) of the first arrival wavelets. Assuming a slowly varying phase, the noise level is estimated with the fit of histograms and wrapped around Gaussian distribution (solid) with standard deviation \( \sigma \approx 40 \) degrees.

**FIG. 6.** LMS decompositions of measured time and phase of refracted first arrivals. From the first-arrivals traveltimes \( T(s, r) \) the time delay \( t(x) \) that depends on the space location \( x \) is estimated. (a) The weathering thickness is obtained assuming a constant weathering velocity of 600 m/s. (b) The weathering phase distortion \( \phi(x) \) is obtained by decomposition of the first-arrivals phase shift \( \theta_w(s, r) \) after 2-D phase unwrapping in the source receiver domain.

**FIG. 7.** The measurement of the weathering specific phase distortion \( d\phi/dh = 30 \) rad/km is made from the correlation of the weathering-phase distortion and weathering thickness \( h \), both shown in Figure 6. The average refractor specific phase distortion \( (K(x) \approx 3.5 \text{ rad/km}) \) is lower than the weathering specific phase distortion. This corresponds to a lower phase distortion of the refractor, as expected (Angeleri and Loinger, 1984).
propagation. If a priori conditions are available, even an aliased phase can be properly unwrapped and the information content preserved.

The ambiguity in phase unwrapping is due to a rotational phase component that arises from the presence of the singular points. The unique unwrapped phase is obtained from a variational problem that corresponds to a minimization of the difference between gradient of the wrapped (evaluated PV) and the unwrapped phase. This corresponds to the unambiguous unwrapped phase when no singular points are present. The solution of the Euler equation is obtained by relaxation with the Neumann boundary condition. Solution with the Dirichlet condition leads to a more robust algorithm but the knowledge of the unwrapped phase at the boundary is required; in some applications this knowledge is available. The algorithm examined in this paper is less affected by error propagation than the algorithms based upon line integration. The minimization of the weighted difference between gradients allow the inclusion of signal-to-noise ratio in the evaluation of the unwrapped phase. However, the limitation of error propagation in the 2-D unwrapping when very strong phase aliasing occurs (steep phase dips or equivalently high instantaneous frequency) is a problem that still needs to be investigated.

In the unwrapping algorithm, only PV phase gradients have been considered. All phase only or complex signals that require phase unwrapping from the phase gradients only (i.e., signal crosscorrelations) are a potential field of applications. A real data example of phase unwrapping is given from the measurements of the refracted first-arrival phase values. The unwrapped phase decomposition emphasizes weathering and refractor phase distortion. Interferometrical imaging of satellite phase data also requires 2-D phase unwrapping for proper surface mapping.

Given the versatility of 2-D phase unwrapping as a standard processing tool, it can be seen that the two applications addressed here serve to underline that there is a wide scope for research in the interpretation of phase information,

![Fig. 8. Geometry of interferometrical SAR imaging](image)

![Fig. 9. Interferometrical SAR imaging from a Brutjorden zone (Norway) represented in gray scale. Since the phase difference between two points in the data is proportional to their relative heights, the terrain elevation is obtained from 2-D phase unwrapping: (a) bottom left] selected amplitude image \( \phi(x,y) \); (a) top left] wrapped interferometrical phase image \( \hat{\phi}(x,y) \); (a) bottom right or(b) 2-D unwrapping of the wrapped phase image shown in (a) top left] gives the terrain elevation. (a), top right] The local error, i.e., the difference between the gradients of the wrapped and unwrapped phases, indicates the regions where phase aliasing occurs because of steep dips or a small lake (indicated by an arrow).]
especially in the fields of surface consistent deconvolution, tomography, interferometry, and phase-versus-offset processing of prestack data.

ACKNOWLEDGMENT

The author is indebted to F. Rocca and G. Drufuca for their many valuable suggestions and criticisms and to C. Prati for helpful discussions and for providing the interferometrical SAR image. The author is also grateful to the Geophysical Research and Data Processing Department of AGIP, S.p.A. for encouragement during the applications to real data. M. J. Woodward revised the manuscript providing several suggestions indispensable to achieve this final form. AGIP S.p.A. has provided the refracted data set.

REFERENCES


APPENDIX-UNWRAPPING IN FOURIER DOMAIN

Let us consider the Laplace operator for sampled unwrapped phase \( \theta(x_i, y_j) \), relationship (17) becomes
\[
\nabla^2 \theta_u(x_i, y_j) = \theta(x_{i+1}, y_j) + \theta(x_i, y_{j+1}) - \theta(x_{i+1}, y_{j+1}) - \theta(x_i, y_j)
\]
where the PV of the Laplace operator has been computed from the wrapped phase using equation (16). Indicating with \((K_x, K_y)\) the wavenumbers of the 2-D Fourier transform of the sampled domain \((x, y)\) (where \(AX\) and \(AY\) are the sampling intervals), the Fourier transform of equation (A-1) is
\[
\nabla^2 \theta_u (K_x, K_y) = 2 \theta(K_x, K_y) \times [\cos(K_x \Delta x) + \cos(K_y \Delta y) - 2].
\]
In the Fourier domain, the unwrapping corresponds to a filtering, provided that the constant phase shift (i.e., the wavenumber \(K_x = 0\) and \(K_y = 0\)) has been arbitrarily chosen. Once the PV Laplace operator for the wrapped phase has been computed, the unwrapping in the Fourier domain requires two 2-D Fourier transforms and a filtering.

For a square domain \(N \times N\), each iteration of the successive over-relaxation requires \(2N^2\) multiplications and converges to \(2N\) iterations, the numerical relaxation technique globally requires \(4N^3\) multiplications. However, Fourier domain unwrapping using 2-D FFT involves \(N^2(6 + 8 \log_2 N)\) only.