Signals and noise

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Slides are supplementary material and are NOT a replacement for textbooks and/or lecture notes.
Outline

• Signals in time and frequency domains
• Random processes
• White noise and approximations
Signals

• Physical quantities that vary with time and contain information
• Deterministic in nature, and described by their time or frequency behavior
Fourier transform

\[ X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t} dt \]

\[ X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft} dt \]

\[ x(t) = \int_{-\infty}^{+\infty} X(\omega)e^{j\omega t} \frac{d\omega}{2\pi} = \]

\[ = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft} df \]
Initial-value theorem

\[ X(0) = \int x(t)dt \]

\[ x(0) = \int X(\omega) \frac{d\omega}{2\pi} = \int X(f)df \]
Time and frequency shift

\[ \mathcal{F}[x(t)] = X(f) \]

\[ \mathcal{F}[x(t + \tau)] = \int x(t + \tau)e^{-j2\pi ft}dt = \]

\[ = \int x(z)e^{-j2\pi f(z-\tau)}dz = e^{j2\pi f\tau}X(f) \]

Similarly:

\[ \mathcal{F}^{-1}[X(f + f_0)] = e^{-j2\pi f_0 t}x(t) \]
Scaling

\[ \mathcal{F}[x(t)] = X(f) \]

\[ \mathcal{F}[x(at)] = \frac{1}{|a|} X\left(\frac{f}{a}\right) \]

For \( a = -1 \) (time reversal) we have

\[ \mathcal{F}[x(-t)] = X(-f) = X^*(f) \]

- If \( x(t) \) is even, so is \( X(f) \)
- If \( x(t) \) is real and even, so is \( X(f) \)
Convolution

\[ \mathcal{F}[x(t)] = X(f) \]
\[ \mathcal{F}[y(t)] = Y(f) \]

\[ \mathcal{F}[x(t) \ast y(t)] = X(f)Y(f) \]
\[ \mathcal{F}[x(t)y(t)] = X(f) \ast Y(f) \]

where

\[ x(t) \ast y(t) = \int x(\tau)y(t - \tau)d\tau \]
Parseval theorem

\[ \mathcal{F}[x(t)] = X(f) \]
\[ \mathcal{F}[y(t)] = Y(f) \]
\[ \int x(t)y^*(t)dt = \int X(f)Y^*(f)df \]

and for \( x = y \)
\[ \int x^2(t)dt = \int |X(f)|^2 df \]
Uncertainty

• The Fourier transform of a Gaussian signal is also Gaussian:
  \[ \mathcal{F} \left[ e^{-\pi \sigma^2 t^2} \right] = e^{-\pi f^2 / \sigma^2} \]

• The widths of the functions are related by an uncertainty relation
  \[ \sigma_t \sigma_f = 1 \]
Extension to the general case

We consider the equivalent duration or bandwidth:

\[ \int x(t) dt = x(0)T \quad \int X(f) df = X(0)B \]
Extension to the general case

From the initial-value theorem:

\[ X(0) = \int x(t)\,dt = x(0)T \]

\[ x(0) = \int X(f)\,df = X(0)B_f = x(0)TB_f \]

\[ \Rightarrow TB_f = 1 \]

(becomes \( TB_\omega = 2\pi \) in the \( \omega \) domain)
Signal cross-correlation

- We consider real signals belonging to $L^2(\mathcal{R})$, called energy signals.
- Cross-correlation of two energy signals is defined as

$$k_{xy}(\tau) = \int x(t)y(t + \tau)\,dt$$

and measures the “similarity” between the signals, as a function of their time difference.
General properties

\[ k_{xy}(\tau) = \int x(t)y(t + \tau)dt \]

\[ = \int x(z - \tau)y(z)dz = k_{yx}(-\tau) \]

\[ |k_{xy}(\tau)| \leq \sqrt{k_{xx}(0)k_{yy}(0)} \]

\[ |k_{xy}(\tau)| \leq \frac{1}{2} (k_{xx}(0) + k_{yy}(0)) \]
Autocorrelation

• Is the correlation of a signal with itself
• It measures the “predictability” of the signal over time

\[ k_{xx}(\tau) = \int x(t)x(t + \tau)dt \]

\[ |k_{xx}(\tau)| \leq k_{xx}(0) = \int x^2(t)dt = E \]

• \( E \) is called the signal energy
Autocorrelation and convolution

\[ x(\tau + t) \]

\[ x(\tau) \]

\[ x(t - \tau) \]

\[ x(t) * x(t) \]

\[ k_{xx}(t) \]
Frequency domain

\[ k_{xx}(t) = k_{xx}(-t) = \int x(\tau)x(\tau - t) d\tau = \]

\[ = x(t) \ast x(-t) \quad \text{(Real and even)} \]

\[ \mathcal{F}[k_{xx}(t)] = X(f)X^*(f) = |X(f)|^2 \]

\[ E = k_{xx}(0) = \int |X(f)|^2 df \quad \text{Also from Parseval theorem} \]

\[ |X(f)|^2 \text{ is called the energy spectral density} \]
Power signals

• There are signals not belonging to $L^2(\mathcal{R})$, whose energy diverges.

• We consider the truncated (energy) signal

$$x_T(t) = \begin{cases} x(t) & \forall |t| \leq T \\ 0 & \forall |t| > T \end{cases}$$

for which we can define the Fourier transform $X_T(f)$ and the autocorrelation

$$k_{xx}^T(\tau) = \int x_T(t)x_T(t + \tau)dt,$$

where $\mathcal{F}[k_{xx}^T(\tau)] = |X_T(f)|^2$.
Autocorrelation and PSD

\[ K_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} k_{xx}^T(\tau) = \]

\[ = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t + \tau)dt \]

\[ \mathcal{F}[K_{xx}(\tau)] = \lim_{T \to \infty} \frac{1}{2T} |X_T(f)|^2 = S(f) \]

\[ P = K_{xx}(0) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt = \int S(f)df \]

signal power  
power spectral density
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• Signals in time and frequency domains
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Stochastic (or random) processes

• They represent the time dependence of random variables

• A few examples
  – Current flowing through a device
  – Wireless signal received by a mobile phone
  – Waiting time at a bus stop
  – Stock market
  – ...
Process realizations

• A random process is defined by the probability density function $p(x, t)$ or by the joint probabilities $p(x_1, \ldots, x_n; t_1, \ldots, t_n)$
  - At any given time, the process becomes a random variable
  - For any random variable, the process is a deterministic function of time, called realization
Stationary processes

• The joint probability distributions are independent of the time shift

\[ p(x_1, \ldots, x_n; t_1, \ldots, t_n) = \]
\[ = p(x_1, \ldots, x_n; t_1 + T, \ldots, t_n + T) \quad \forall \, n, t, T \]

• The joint pdfs become

\[ - p(x, t) = p(x, t + T) \quad \forall \, t, T \Rightarrow p(x, t) = p(x) \]
\[ - p(x_1, x_2; t_1, t_2) = p(x_1, x_2; t_1, t_1 + \tau) \quad \forall \, t_1 \Rightarrow \]
\[ p(x_1, x_2; t_1, t_2) = p(x_1, x_2; \tau) \]
Mean and variance

- **Mean value**

  \[ \bar{x} = \int x p(x) \, dx \]

- **Variance**

  \[ \sigma_x^2 = \int (x - \bar{x})^2 p(x) \, dx = \bar{x}^2 - \bar{x}^2 \]

  Independent of time for stationary processes
Moment functions

• Autocorrelation

\[ R_{xx}(\tau) = \bar{x}_1 \bar{x}_2 = \int \int x_1 x_2 p(x_1, x_2; \tau) \, dx_1 \, dx_2 \]

• Autocovariance

\[ C_{xx}(\tau) = (x_1 - \bar{x}_1)(x_2 - \bar{x}_2) = \int \int (x_1 - \bar{x}_1)(x_2 - \bar{x}_2)p(x_1, x_2; \tau) \, dx_1 \, dx_2 \]

Only dependent on \( \tau = t_2 - t_1 \)
A few notes

• In the general case, $C_{xx} = R_{xx} - x_1 \overline{x_2}$

• Values at $\tau = 0$

$$R_{xx}(0) = \overline{x^2}$$
$$C_{xx}(0) = \sigma_x^2$$

• We will mainly consider processes with zero mean value $\Rightarrow R_{xx} = C_{xx}$
Temporal averages
Ergodic processes

\[ \langle x \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt = \bar{x} \]

\[ K_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t + \tau)dt = R_{xx}(\tau) \]

• Temporal statistics converge to the ensemble ones, for all moments
• Ergodic processes are stationary
A simple counter-example

• We consider the process
  \( x(t) = A \), random variable in \([0,1]\), \( \bar{x} = \bar{A} \)

\( x(t) \) is stationary but not ergodic

• \( x(t) \) is stationary but **not** ergodic
Frequency domain

• We consider a single realization $x_i(t)$, which is a power signal, and we can define

$$S_i(f) = \lim_{T \to \infty} \frac{1}{2T} |X_{T,i}(f)|^2$$

• $S_i(f)$ is a random variable. We now take the ensemble average

$$S(f) = \overline{S_i(f)} = \lim_{T \to \infty} \frac{1}{2T} |X_{T,i}(f)|^2$$

• $S(f)$ is called the **power spectral density** of the random process
Wiener-Kintchine theorem

\[ \mathcal{F}^{-1}[S(f)] = \int \lim_{T \to \infty} \frac{1}{2T} |X_T(f)|^2 e^{j2\pi f \tau} df = \]

\[ = \lim_{T \to \infty} \frac{1}{2T} \int \frac{X_T(f)X_T^*(f)}{T} e^{j2\pi f \tau} df = \]

\[ = \lim_{T \to \infty} \frac{1}{2T} \int \int_{-T}^{T} x(t_1)e^{-j2\pi f t_1} dt_1 \int_{-T}^{T} x(t_2)e^{j2\pi f t_2} dt_2 e^{j2\pi f \tau} df = \]

\[ = \lim_{T \to \infty} \frac{1}{2T} \int \int_{-T}^{T} x(t_1)x(t_2) \int e^{j2\pi f (t_2-t_1+\tau)} df dt_1 dt_2 = \]

\[ = \lim_{T \to \infty} \frac{1}{2T} \int \int_{-T}^{T} R_{xx}(t_1, t_2) \delta(\tau + t_2 - t_1) dt_1 dt_2 = \]

\[ = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_{xx}(\tau) dt_1 = R_{xx}(\tau) \]
Uni- and bilateral spectra

- $S(f)$ is real and even, extending from $-\infty$ to $+\infty \Rightarrow$ bilateral power spectrum
- In circuit calculations, positive frequencies only are of interest $\Rightarrow$ a unilateral PSD is defined, extending from $f = 0$ to $+\infty$
- $S_u(f) = 2S_b(f) \forall f \geq 0$
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White noise

\[ R_{nn}(\tau) = \lambda \delta(\tau) \]

\[ S_n(f) = \lambda \]

The process is totally uncorrelated with itself

\[ \overline{n^2} = \infty \]
White noise approximations

• Real noises are only approximately white, i.e., they behave as such in a limited spectral (or time) range

• Correlation time and bandwidth are related by the uncertainty relation:

\[ R_{nn}(\tau) = \lambda g(\tau) \quad g(\tau) \approx 0 \quad \forall \ |\tau| > \tau_0 \]

\[ S_n(f) \approx \text{constant} \quad \forall \ |f| < 1/\tau_0 \]
Triangular approximation

\[ S_n(f) = \overline{n^2 \tau_0} \left( \frac{\sin(\pi f \tau_0)}{\pi f \tau_0} \right)^2 \]

\[ \overline{n^2} = \int S_n(f) df = S_n(0) 2 f_n \Rightarrow f_n = \frac{1}{2 \tau_0} \]
Rectangular approximation

\[ S_n(f) = \frac{n^2 \tau_0}{\pi f \tau_0} \sin(\pi f \tau_0) \]

\[ \bar{n}^2 = \int S_n(f) df = S_n(0) 2f_n \Rightarrow f_n = \frac{1}{2\tau_0} \]