Boxcar Averagers

Alessandro Spinelli
Phone: (02 2399) 4001
alessandro.spinelli@polimi.it home.deib.polimi.it/spinelli
Slides are supplementary material and are NOT a replacement for textbooks and/or lecture notes.
• It’s time to begin a discussion on the techniques for improving $S/N$.
• Noise-reduction techniques obviously depend on the type of signal and of noise:

<table>
<thead>
<tr>
<th>Signal</th>
<th>Noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>LF (constant)</td>
<td>HF (White)</td>
</tr>
<tr>
<td>HF (pulse)</td>
<td>previous lesson</td>
</tr>
<tr>
<td></td>
<td>this lesson</td>
</tr>
</tbody>
</table>
Outline

- Boxcar averagers
- Ratemeters
• Short $T_F \Rightarrow$ larger signal, but larger noise
• Long $T_F \Rightarrow$ smaller signal, but smaller noise
• GIs do not increase appreciably $S/N$ if $T_P$ is small
• Can we retain the advantage of a long $T_F$ without sacrificing the signal?
• If the capacitor is **not** discharged after each pulse, we can gain signal

• $S/N$ improves because we are now averaging over many pulses

• Still a time-variant filter!
Boxcar averager

- Switch closed $\Rightarrow$ LPF
- Switch open $\Rightarrow$ the voltage stored on the capacitor cannot discharge
Weighting function

(Actual behavior may not be periodic)
Filter behavior

• In the «equivalent time» $\tau'$ we have

$$w(t, \tau') = \frac{1}{T_F} e^{-\frac{t-\tau'}{T_F}} u(t - \tau')$$

• For white input noise we have:

$$\overline{n_y^2} = \lambda \int w^2(t, \tau) d\tau = \lambda \int w^2(t, \tau') d\tau'$$

• The filter behaves exactly as an LPF and has the same $S/N$ in this case
Improvement of $S/N$

- The improvement is the same as the LPF:

$$\left(\frac{S}{N}\right)_{out} = \left(\frac{S}{N}\right)_{in} \sqrt{\frac{2T_F}{T_n}}$$

- The output depends on $x(\tau')$ only, i.e., on the input signal when the switch is closed

- The time to reach steady-state depends on $T_C$ and $T_O$ (i.e., on the sampling rate) but the performance does not
• We consider $T_C \ll T_F$ so that we can neglect the discharge over the time interval $T_C$

• A single-pulse boxcar is then equivalent to a GI with gain $K = 1/T_F$:

$$\bar{n}_y^2 = \lambda \frac{T_C}{T_F^2}$$

$$y = A \frac{T_C}{T_F}$$

$$\left( \frac{S}{N} \right)_{sp} = \left( \frac{S}{N} \right)_{in} \sqrt{\frac{T_C}{T_n}}$$
Equivalent number of samples

\[
\left( \frac{S}{N} \right)_{BA} = \left( \frac{S}{N} \right)_{sp} \sqrt{\frac{2T_F}{T_C}} = \left( \frac{S}{N} \right)_{sp} \sqrt{N_{eq}}
\]

\[N_{eq} = \frac{2T_F}{T_C}\]

\(N_{eq}\) represent the improvement in \(S/N\) due to the exponential average and is called the equivalent number of samples.
We address the case in which $T_C \ll T_F$ does not hold. The weighting function of a single-pulse boxcar is now

$$w(t, \tau) = \frac{1}{T_F} e^{-\frac{t-\tau}{T_F}} \quad t - T_C \leq \tau \leq t$$

For an input signal nearly constant over $T_C$ we have then

$$y = A \int_{t-T_C}^{t} \frac{1}{T_F} e^{-\frac{t-\tau}{T_F}} d\tau = A \left(1 - e^{-\frac{T_C}{T_F}}\right)$$
General case – noise

For a white stationary input noise we have

\[ n_y^2 = \lambda k_{w_{tt}}(0) = \lambda \int w^2(t, \tau) d\tau = \frac{\lambda}{T_F^2} \int_{t-T_C}^{t} e^{-2(t-\tau)/T_F} d\tau \]

\[ = \frac{\lambda}{T_F^2} \int_{0}^{T_C} e^{-2\gamma/T_F} d\gamma = \frac{\lambda}{2T_F} \left(1 - e^{-\frac{2T_C}{T_F}}\right) \]
General case – $S/N$ and $N_{eq}$

$$\left( \frac{S}{N} \right)_{sp} = \frac{A \left( 1 - e^{-\frac{T_C}{T_F}} \right)}{\sqrt{\frac{\lambda}{2T_F} \left( 1 - e^{-\frac{2T_C}{T_F}} \right)}} = \left( \frac{S}{N} \right)_{BA} \frac{1 - e^{-\frac{T_C}{T_F}}}{\sqrt{1 - e^{-\frac{2T_C}{T_F}}}}$$

$$N_{eq} = \frac{1 - e^{-\frac{2T_C}{T_F}}}{\left( 1 - e^{-\frac{T_C}{T_F}} \right)^2} = \frac{1 + e^{-\frac{T_C}{T_F}}}{1 - e^{-\frac{T_C}{T_F}}} \quad \text{(the previous expression is recovered if } T_C \ll T_F)$$
Correlation function (periodic case)

\[ T_C = 20 \text{ ns}, \quad T_O = 1 \text{ µs}, \quad T_F = 0.5 \text{ µs} \]

Time constant of envelope

\[ \approx T_F(T_C + T_O)/T_C \]

\[ W_{LP} = \frac{1}{2T_F} e^{-|\gamma|/T_F} \]
The envelope is nearly the single-pulse transform \( \approx \text{sinc}^2 (\pi f T_C) \)

\[
W_{LP} = \frac{1}{1 + (2\pi f T_F)^2}
\]

\( T_C = 20 \, \text{ns}, \ T_O = 1 \, \mu\text{s}, \ T_F = 0.5 \, \mu\text{s} \)
\[
\overline{n_{\gamma}^2(t)} = \int R_{xx}(\gamma) k_{\omega_{tt}}(\gamma) d\gamma = \int S_x(f) |W(t, f)|^2 df
\]
Effect on correlated noise

• If the input noise is non-white and correlated over a time $T_C$ – but not significantly so over $T_C + T_O$ – BA can provide a much better $S/N$ than LPF

• For white noise, the two filters give the same output noise (value in $\tau = 0$ of the WF autocorrelations is the same)
Main typical parameters

- Gate width (typ. from 1 – 2 ns to 20 – 30 μs; can be even shorter in fast samplers)
- Number of samples (from 1 to several 1000s)
- Delay (intrinsically 10 – 15 ns; then from 3 to 300 ns typ. – higher values are possible by custom modification)
- Trigger rate (typ. lower than 100 kHz)
Waveform recovery mode

- Trigger delay is not fixed but rather is incremented so that it sweeps between an initial and a final value.
- Each “point” is repeated $N$ times to improve $S/N$ and the data is sent to output.
- In this way, the input waveform is reconstructed (equivalent-time sampling).
Waveform recovery mode

From [1]

Average N times
Outline

- Boxcar averagers
- Ratemeters
$S_1$ blocks the input signal but does not stop the capacitor discharge

⇒ this is not a boxcar
Weighting function

\( w(t, \tau) \)

(actual behavior may not be periodic)

capacitor discharge during \( T_O \)
Output signal

- The contribution of the n-th pulse is \( t = 0 \)

\[
y^n = A \int_{-n(T_C+T_O)-T_C}^{-n(T_C+T_O)} e^{\frac{\tau}{T_F}} \frac{d\tau}{T_F} = A \left(1 - e^{-T_C/T_F}\right)e^{-n(T_C+T_O)/T_F}
\]

- The output becomes

\[
y = \sum_{n=0}^{\infty} y^n = A \left(1 - e^{-T_C/T_F}\right) \sum_{n=0}^{\infty} \left(e^{-T_C+T_O}/T_F\right)^n = A \frac{1 - e^{-T_C/T_F}}{1 - e^{-(T_C+T_O)/T_F}}
\]
Output noise

\[
k^n_{wtt}(0) = \int_{-n(T_C+T_O)}^{-n(T_C+T_O)-T_C} \frac{e^{2\tau/T_F}}{T_F^2} d\tau = \frac{1 - e^{-2T_C/T_F}}{2T_F} e^{-2n(T_C+T_O)/T_F}
\]

\[
k_{wtt}(0) = \sum_{n=0}^{\infty} k^n_{wtt}(0) = \frac{1}{2T_F} \frac{1 - e^{-2T_C/T_F}}{1 - e^{-2(T_C+T_O)/T_F}}
\]

\[
n_{out}^2 = \lambda k_{wtt}(0)
\]
\[
\left( \frac{S}{N} \right)_{\text{out}} = A \sqrt{\frac{2T_F}{\lambda}} \frac{1 - e^{-T_C/T_F}}{\sqrt{1 - e^{-2T_C/T_F}}} \sqrt{1 - e^{-2(T_C+T_O)/T_F}} \frac{1 - e^{-(T_C+T_O)/T_F}}{1 - e^{-(T_C+T_O)/T_F}}
\]
The equivalent number of samples and the whole filter performance (signal and noise) depend on $T_O$. $S/N$ is always smaller than the boxcar’s.
BA and RI: passive circuit comp.

**RATEMETER INTEGRATOR**
- Switch S acts as gate on the input source
- Switch S is decoupled from the RC passive filter by the voltage buffer
- The RC integrator is unaffected by S, it has constant parameters, it does NOT have a HOLD state
- The sample average is done on a given time, defined by the RC value

**BOXCAR INTEGRATOR**
- Switch S acts as gate on the input source
- Switch S acts also on the RC passive filter (changes the resistor value)
- The time constant $T_F$ of the integrator filter is switched from finite RC (S-down) to infinite (S-up, HOLD state)
- The sample average is done on a given number of samples, defined by the $T_F/T_G$ value

From [2]
BA and RI: active circuit comp.

**RateMeter Integrator**
- Switch $S_1$ acts as gate on the input
- Switch $S_1$ is decoupled from the active RC integrator by the buffer action of the OP-AMP virtual ground
- The $R_iC_F$ integrator is unaffected by $S_1$; it has constant parameters, it does NOT have a HOLD state
- The sample average is done on a given time, defined by the $R_iC_F$ value

**BoxCar Integrator**
- Switch $S_2$ acts as gate on the input
- Switch $S_2$ is decoupled from the active RC integrator by the buffer action of the OP-AMP virtual ground
- A second switch $S_2$ is required for switching the time constant $T_f$ of the integrator from finite $R_iC_F$ ($S_2$-down) to infinite ($S_2$-up, HOLD state)
- The sample average is done on a given number of samples, defined by the $T_f/T_S$ value

From [2]