The control problem

Let’s introduce one of the most important topic of this course.

Given a plant, how can we enforce a desired behaviour of its output by acting of its input (through an actuator) and by measuring its output (through a transducer), regardless all possible disturbances and uncertainties acting on the system?
The control problem – cont’d

In order to address this problem, which is so far too general, let’s introduce some assumptions:

- all (sub)systems are LTI systems – strong assumption;
- all (sub)systems have exactly one input and one output (SISO);
- disturbances are additive;
- all transfer functions are known, although not exactly.

Let’s analyse each single component.
The control problem – cont’d

The plant (process to be controlled):

The transducer (sensor):
The control problem – cont’d

The actuator:

\[ A(s) \]

\[ d_A \]

\[ m \]

The controller (or regulator), to one be designed:

\[ R(s) \]

\[ u \]
The control problem – cont’d

Let’s have a look at the whole picture.

The system can be simplified (with some basic algebraic manipulation) to the following one:
Finally, as all (sub)systems are linear, we can change the position of the transducer as follows:

\[ G(s) = A(s)P(s)T(s) \]
The control problem – cont’d

We are now able to formalize the control problem.

Given $G(s)$ we now aiming at designing $R(s)$ such that:
- the closed-loop system is asymptotically stable;
- the output of the system is as close as possible to its reference, both:
  - during possible transients
  - at steady state (e.g. when the reference is constant)
- the control effort (energy!) is not too “high”
- the effect of disturbances on performance is not so “big”
The control problem – cont’d

The first unavoidable requirement is **closed-loop stability**. In case $G(s)$ has no poles in the open right half-plane we can make use of the Bode criterion.

Given $G(s)$ we should design a controller $R(s)$ without poles in the open right half-plane such that no cancellations occur in the right half-plane and such that the frequency response of

$$L(s) = R(s)G(s)$$

has a magnitude diagram which crosses exactly once the 0 dB axis from top and

$$\mu = \lim_{s \to 0} s^g L(s) > 0 \quad \psi_m > 0$$

What does it mean (graphically and in practice) …?
The control problem – cont’d

The crossing assumption limits the shape of the loop transfer function:

\[ |L(j\omega)| \]

Is any slope acceptable?

**Example:** consider \( L(s) = \frac{1}{s^2} \) \( \Rightarrow \) slope is -40 dB/dec

\( \omega_c = 1, \psi_m = 0^\circ \)

\( \omega_c = 3.08, \psi_m = 18^\circ \) slope is -40 dB/dec

\( \omega_c = 1.62, \psi_m = -58.3^\circ \) slope is -60 dB/dec
The control problem – cont’d

In general, the more poles are at lower frequencies with respect to the crossover frequency, the smaller phase margin (eventually negative).

We will also see how zeros can create this problem.

From the last examples, and from the speculations we have done, we can conclude that the lower slope the lower phase margin.

**Good choice:** -20 dB/dec slope (unitary negative slope) at the crossover frequency.

This might help in obtaining a **positive phase margin**. We will see, however, that this is not sufficient.
Complementary sensitivity

As a consequence, for all reasonable design, the loop transfer function should look like this:

\[ |L(j\omega)| \]

“Big” at low frequency (before crossover)

We can now focus on characterizing the tracking performance, i.e. how similar, when no disturbances apply, the output is to its reference.

\[
Y(s) = \frac{L(s)}{1 + L(s)} Y^0(s) = F(s) Y^0(s)
\]

Complementary sensitivity transfer function
Complementary sensitivity – cont’d

We have seen that $|L(j\omega)|$ should be “big” before the crossover frequency and “small” after, therefore

$$|F(j\omega)| = \frac{|L(j\omega)|}{|1 + L(j\omega)|} \approx \begin{cases} \frac{1}{2}, \omega << \omega_c \\ |L(j\omega)|, \omega \gg \omega_c \end{cases}$$
Complementary sensitivity – cont’d

Assuming the closed-loop system to be stable, the tracking performance can be analyse by applying the frequency response to the complementary sensitivity.

\[ y^0 \rightarrow F(s) \rightarrow y \]

We have seen that \( F(s) \) is basically a low-pass filter which preserves all frequencies until the crossover \( \omega_c \) therefore, we should aim at achieving the crossover frequency as high as possible.

Let’s try to understand something more about \( F(s) \).
Complementary sensitivity – cont’d

We have seen that we can approximate its behaviour far from the crossover frequency. What actually happens in the neighborhood of $\omega_c$?

$$|L(j\omega_c)| = 1 \quad \angle L(j\omega_c) = \psi_c$$

We obtain

$$|F(j\omega_c)| = \frac{|L(j\omega_c)|}{|1 + L(j\omega_c)|} = \frac{1}{|1 + e^{i\psi_c}|} = \frac{1}{|1 + \cos \psi_c + j \sin \psi_c|} =$$

$$= \frac{1}{\sqrt{(1 + \cos \psi_c)^2 + \sin^2 \psi_c}} = \frac{1}{\sqrt{2} (1 + \cos \psi_c)} = \frac{1}{\sqrt{2} (1 - \cos \psi_m)} = \frac{1}{2 \sin \left(\frac{\psi_m}{2}\right)}$$

$$\psi_m = \pi + \psi_c \quad \psi_m > 0$$

The behaviour of the complementary sensitivity in the neighbourhood of the crossover frequency depends on the phase margin.
Complementary sensitivity – cont’d

**Examples:** consider $L_1(s) = \frac{0.5}{s}$ and $L_2(s) = \frac{0.55099}{s(s + 1)}$ for both $\omega_c = 0.5$

$\psi_c = \angle L_1(j0.5) = -90^\circ, \psi_m = 90^\circ > 0$

$\psi_c = \angle L_2(j0.5) = -168.7^\circ, \psi_m = 11.3^\circ > 0$
Complementary sensitivity – cont’d

A good approximation of the complementary sensitivity should account for the phase margin.

Therefore we are facing two options:
- for “big” phase margin (usually > 75°):

\[ F'(s) \approx \frac{1}{1 + \frac{s}{\omega_c}} \]

- for “small” (still positive) phase margin the complementary sensitivity looks like a second order transfer function with complex/conjugate poles like:

\[ F'(s) \approx \frac{\omega_c^2}{s^2 + 2\xi \omega_c s + \omega_c^2} \]

\[
\left| \frac{\omega_c^2}{-\omega^2 + j2\xi \omega_c \omega + \omega_c^2} \right|_{\omega = \omega_c} = \frac{1}{2\xi} = \frac{1}{2 \sin (\psi_m/2)} \Rightarrow \xi = \sin (\psi_m/2)
\]
Complementary sensitivity – cont’d

*Examples:* let’s compare the step responses in the previous two examples.

\[ L_1(s) = \frac{0.5}{s} \]

\[ L_2(s) = \frac{0.55.099}{s(s + 1)} \]
Sensitivity

Another possible requirement concerns the steady state behaviour of the closed-loop system and, in particular, the behaviour of the tracking error

\[ e = y^0 - y \]

In absence of disturbances, the tracking error can be computed as

\[ E(s) = \frac{1}{1 + L(s)} Y^0(s) = S(s) Y^0(s) = [1 - F(s)] Y^0(s) \]

Notice that \( S(s) + F(s) = 1 \). From this property, the name “complementary sensitivity”.

Notice that \( S(s) + F(s) = 1 \). From this property, the name “complementary sensitivity”. 
Sensitivity – cont’d

As $|L(j\omega)|$ is “big” before the crossover frequency and “small” after, the sensitivity function can be approximated as

$$|S(j\omega)| = \frac{1}{|1 + L(j\omega)|} \approx \begin{cases} |L(j\omega)|^{-1}, & \omega << \omega_c \\ 1, & \omega >> \omega_c \end{cases}$$
Sensitivity – cont’d

As the sensitivity function relates the error to the reference value, we are interesting on understanding the steady-state property.

\[ \lim_{s \to 0} S(s) - \frac{s^g}{\mu} = 0 \]

Applying the theorem of the final value, we obtain

\[ e_\infty = \lim_{s \to 0} sS(s) Y^0(s) = \lim_{s \to 0} sS(s) \frac{1}{s^j} = \lim_{s \to 0} S(s) \frac{1}{s^{j-1}} = \lim_{s \to 0} \frac{s^{g-j+1}}{s^g + \mu} \]

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<thead>
<tr>
<th>( g )</th>
<th>( y^0(t) = \text{step}(t) )</th>
<th>( y^0(t) = \text{ramp}(t) )</th>
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Control sensitivity

We should have now understood that for each requirement there is a sensitivity function (a tool) to properly address it.

How about the control effort related to the reference value?

\[ U(s) = \frac{R(s)}{1 + L(s)} Y_0(s) = Q(s) Y_0(s) \]

As for the approximation, we have

\[ |Q(j\omega)| = \frac{|R(j\omega)|}{|1 + L(j\omega)|} \approx |G(j\omega)|^{-1}, \omega << \omega_c \]

\[ |R(j\omega)|, \omega >> \omega_c \]
Disturbances

How can we predict the effect of disturbances on the output based on the shape of the loop-transfer function?

Consider the first disturbance

\[ Y(s) = \frac{1}{1 + L(s)} D(s) = S(s) D(s) \]

Assuming the closed-loop system asymptotically stable, as a consequence of the frequency response we can conclude that the output is affected by the disturbance, approximately only in the bandwidth where

\[ |S(j\omega)| > 1 \]
Disturbances – cont’d

As for the second disturbance, we have

\[ Y(s) = -\frac{L(s)}{1 + L(s)}N(s) = -F(s)N(s) \]

Therefore, still assuming the closed-loop system to be asymptotically stable, the output is affected by the disturbance, approximately only in the bandwidth where

\[ |F(j\omega)| > 1 \]
Disturbances – cont’d

By observing how the two sensitivity transfer functions are related to the loop transfer function

We can conclude that:

- having \(|L(j\omega)|\) “big” at low frequencies helps in attenuating the disturbance \(d(t)\)
- having \(|L(j\omega)|\) “small” at high frequencies helps in attenuating the disturbance \(n(t)\)
- the crossover frequency \(\omega_c\) should be selected accordingly
Disturbances – cont’d

Example: consider \( L(s) = \frac{100}{s + 1} \) and determine the bandwidth where disturbances are attenuated (on the output) of at least 10. (20 dB).

First, plot the Bode diagrams, we have \( \omega_c \approx 100, \psi_m \approx 90^\circ > 0 \)
Disturbances – cont’d

Bode Diagram

- Attenuation of $d(t) > 20$ dB
- $|L(j\omega)|$
- $|F(j\omega)|$
- $|S(j\omega)|$
- Att. of $n(t)$
Bode integrals

A natural question is the following one: can we attain (a good) disturbance rejection at any frequency? There is a result which says:

**Bode’s integral formula:** if the closed-loop control systems is asymptotically stable, $P = 0$, and the loop transfer function has at least a relative degree of two, then

$$
\int_{0}^{+\infty} 20 \log |S(j\omega)| \, d\omega = 0
$$

This conservation law shows that to get lower sensitivity in one frequency range, we must get higher sensitivity in some other region.

In other words, that we cannot attenuate disturbances at any frequency. Inevitably disturbances at some frequency will be amplified.
**Example:** consider the transfer functions \( L(s) = \frac{k}{(ks + 1)(1 + 0.001s)} \)

- For \( k = 10 \)
- For \( k = 100 \)

The diagram shows:
- **Same crossover frequency**
- **One more pole not visible**
- **Same phase margin**
Bode integrals – cont’d

Bode plots of the corresponding sensitivity functions

![Bode Diagram](image)
Bode integrals – cont’d

Figure 3. Sensitivity reduction at low frequency unavoidably leads to sensitivity increase at higher frequencies.

From G. Stein “Respect the Unstable”, IEEE Control System Magazine, August 2003.
Performance

Few definitions:

- we call **static performance** of a control system everything that can be addressed at steady-state
  - steady-state error (through final value)
  - disturbance attenuation (through frequency response)

- in turn, we call **dynamic performance** everything related to the transients:
  - damping and overshoot,
  - promptness of the response (dominant time-constant)
Homework & take home message

Designing a controller for an LTI system requires to address a series of (typically conflicting) requirements.

As we have understood which tools to be used for each requirement, we need a systematic way to account for all of them.

Designing a control system usually has different feasible solutions, we might look for the best one, according to the available tools.

The ability to analyse the performance of closed-loop system corresponds to being (almost) able to design it!
Control synthesis

Having understood the analysis of a closed-loop control system, we can now start to learn how to **design a controller**.

We will make intense use of the Bode diagrams and of the corresponding stability criterion (hence we assume $P = 0$).

The focus will be on a system like the following one:
Control synthesis – cont’d

We should design the controller (a transfer function) so that the closed-loop system has some required properties.

**Stability**: we clearly want the closed-loop system to be asymptotically stable, this happens whenever

\[ \psi_m > 0 \]

provided that there are no cancellation in the right half-plane.

**Stability margin and damped transients**: we want to avoid oscillations and to take some margin with respect the critical situation, hence

\[ \psi_m > \bar{\psi}_m > 0 \]
Control synthesis – cont’d

Quick transients: we have seen that the complementary sensitivity can be approximated by either a first order or a second order system

\[
F(s) \approx \frac{1}{1 + s/\omega_c} \quad F(s) \approx \frac{\omega_c^2}{s^2 + 2\xi \omega_c s + \omega_c^2} \quad \xi = \sin(\psi_m/2)
\]

The settling time of those transfer functions depends on the crossover frequency (and on the phase margin). In both cases we want:

\[ \omega_c > \bar{\omega}_c \]

Static performance: they tipically ask for the steady-state error to be small when a particular input is applied, e.g.

\[
|e_{\infty}| < \bar{e}_{\infty} \quad y^0(t) = \text{step}(t) \quad d(t) = \text{ramp}(t)
\]
Control synthesis – cont’d

Static performance (disturbances): we may also want to attenuate the effect of disturbances on the output, e.g.

\[ d(t) = \sin(\bar{\omega}t), \bar{\omega} \in [\bar{\omega}_{\text{min}}, \bar{\omega}^{\text{max}}] \]

\[ y(t) = k \sin(\bar{\omega}t), k < \bar{k} \]

Other specifications: strongly depend on the application, we may want the order of the transfer function to be limited, and others…
Control synthesis – cont’d

As we discussed what we should obtain at the end of the design process, let’s try to understand what we are looking for.

The transfer function we are looking for can be written as follows

\[ R(s) = \frac{\mu_R \prod_i 1 + s\tau_i}{s^{g_R} \prod_j 1 + sT_j} = R_1(s) R_2(s), T_j > 0 \]

where we have divided two different contributions:

- **static part**: gain and type (poles/zeros in the origin)
- **dynamic part**: other real poles and/or zeros

Notice that we are looking for a transfer function with poles in the closed left half-plane, hence with \( P = 0 \) (to apply the Bode criterion).
Control synthesis – cont’d

As we divided the structure of the controller into two parts (static and dynamic), we can design them separately. We will then talk about static and dynamic design.

Static design:
In this part we account for the following static performance

\[ |e_\infty| < \bar{e}_\infty \quad y^0(t) = \text{step}(t) \]
\[ d(t) = \text{ramp}(t) \]

as well as steady-state attenuation of sine disturbances, e.g.

\[ d(t) = \sin(\bar{\omega}t), \bar{\omega} \in [\bar{\omega}_{min}, \bar{\omega}_{max}] \]
\[ y(t) = k \sin(\bar{\omega}t), k < \bar{k} \]
Control synthesis – cont’d

**Type and gain:** usually, we try to select the minimum type for the controller which satisfies the constraint.

Notice that

\[ \mu_L = \mu_R \mu_G \quad g^L = g^R + g^G \]

therefore

\[ L_1(s) = R_1(s) G(s) = \frac{\mu_R}{s g_R} G(s) \]
Control synthesis – cont’d

Example: consider $G(s) = \frac{10}{1 + s}$ and define the constraints on the controller such that

$|e_\infty| \leq 0.1, y^0(t) = \text{step}(t)$

The transfer function between the reference and the error is

$S(s) = \frac{1}{1 + R(s)G(s)}$

Assuming we will be able to stabilize the system, we can apply the final value theorem:

$|e_\infty| = \lim_{s \to 0} sS(s)Y^0(s) = \lim_{s \to 0} S(s) = \lim_{s \to 0} \frac{1}{1 + \frac{\mu_R \mu G}{sg_R + g_G}}$

$\mu_R \geq 0.9, g_R = 0 \quad \forall \mu_R, g_R \geq 1$

$0, g_R \geq 1$
Control synthesis – cont’d

Once type and/or gain of the controller has been selected, we can address the problem of disturbance attenuation, still assuming we will be able to stabilize the closed-loop system.

**Disturbance attenuation:** this requirement will be translated in regions to be avoided in the Bode plot.

Let’s focus on the disturbance $d(t)$, the treatment of $n(t)$ is analogous.
Control synthesis – cont’d

Assuming we will be able to satisfy the Bode’s criterion, we can use the frequency response to analyse the steady-state behaviour of the output.

The attenuation requires $|S(j\omega)|$ to be “small”, in the interested bandwidth, or equivalently requires $|L(j\omega)|$ to be “big” in the same region.
Control synthesis – cont’d

*Example:* assume we want to achieve the following disturbance attenuation

\[ d(t) = A \sin(\bar{\omega} t), \bar{\omega} \in [1, 10], y(t) = 0.1A \sin(\bar{\omega}) \]

The transfer function between the disturbance and the output is

\[ S(s) = \frac{1}{1 + L(s)} \]

\[ |S(j\omega)|^{-1} \approx |L(j\omega)| \]

\[ |L(j\omega)| \geq 10, \omega \in [1, 10] \]
Control synthesis – cont’d

Dynamic design: the output of the static design consists of type and gain of the regulator, i.e. the transfer function

\[ R_1 (s) = \frac{\mu R}{sg_R} \]

as well as some regions to be avoided by the loop transfer function.

In other words we now have

\[ L_1 (s) = R_1 (s) G (s) \]

Usually, this preliminary loop transfer function has the following characteristics

\[ |L_1 (j\omega)| \geq |G (j\omega)| \quad \angle L_1 (j\omega) \leq \angle G (j\omega) \]

which might compromise the stability of the closed-loop system.
Control synthesis – cont’d

Within the dynamic design, we are aiming at achieving dynamic properties (e.g. settling time and damping) while ensuring stability of the closed-loop system by restoring a “good” behaviour of the final loop transfer function.

Notice that since $R(s) = R_1(s) \cdot R_2(s)$ we have

$$L(s) = L_1(s) \cdot R_2(s) = L_1(s) \frac{\Pi_i 1 + s\tau_i}{\Pi_j 1 + sT_j}$$

Within the dynamic design, we then have to insert poles and/or zeros in the regulator in order to achieve the desired behaviour of the closed-loop system (e.g. desired crossover frequency and good phase margin).
Control synthesis – cont’d

Notice that:

- at low frequencies $L(s) = L_1(s)$ since $R_2(s) = 1$ (has type zero and unit gain)
- at high frequency the slope of $|L(j\omega)|$ should not be bigger than the slope of $|L_1(j\omega)|$
- as a rule of thumb, the crossing slope (at crossover frequency) should be equal to -1
Control synthesis – cont’d

Once the final loop transfer function $L(s)$ has been shaped, the dynamic part of the controller can be computed as

$$R_2(s) = \frac{L(s)}{L_1(s)}$$

while the final controller is $R(s) = R_1(s) \cdot R_2(s)$.

Rules of thumb for a good design:
1. crossover with slope -1 to guarantee a good phase margin
2. same slope of $L_1$ and $L$ at low frequencies
3. if a constraint on the gain applies, then at low frequencies $|L| \geq |L_1|$
4. at high frequency slope of $L$ lower than the slope of $L_1$
5. at high frequencies $|L| \leq |L_1|$ to reduce control effort
Control synthesis – cont’d

A complete example: given the following transfer function

\[ G(s) = \frac{1 + 10s}{(1 + s)^2} \]

design a controller such that

- nullify steady-state error with respect to a step reference
- provides an attenuation of 20 dB of \( d(t) \) on the output in the bandwidth \(< 0.1 \text{ rad/s} \)
- provides a phase margin of 70 deg
- is such that the output has a settling time of 5 seconds in response to a step reference
Control synthesis – cont’d

Static design:

\[ R_1 (s) = \frac{\mu_R}{s g_R} \quad G (s) = \frac{1 + 10s}{(1 + s)^2} \]

In order to nullify the steady state error for a step reference, we consider

\[ E (s) = S (s) Y^0 (s) \]

Assuming we will be able to stabilize the closed-loop system we have

\[ e_\infty = \lim_{s \to 0} s S (s) Y^0 (s) = \lim_{s \to 0} S (s) = \lim_{s \to 0} \frac{s g_L}{1 + \mu_L} = \lim_{s \to 0} \frac{s g_R}{1 + \mu_R} \]

\[ g_R \geq 1, \forall \mu_R \quad g_R = 1, \forall \mu_R \]
Control synthesis – cont’d

As for attenuation of the disturbance, the transfer function to consider is

\[ Y(s) = S(s)D(s) \approx L(s)^{-1}D(s) \]

Assuming the closed-loop system stable, from the frequency response we know that

\[ |L(j\omega)| > 10, \omega \leq 0.1 \]

\[ \psi_m \approx 35^\circ < 75^\circ \]

\[ T_{\text{steady}} = 5 \Rightarrow \omega_c > 1 \]

\[ \omega_c \approx 3 > 1 \]
Control synthesis – cont’d

Dynamic design:
the phase margin is too small, on the other hand the crossover frequency of 3 rad/s guarantees the promptness of the response.

\[ L(s) = \frac{1}{s(1 + 0.1s)} \]

\[ R_2(s) = \frac{L(s)}{L_1(s)} = \frac{(s + 1)^2}{(1 + 0.1s)(1 + 10s)} \]

\[ R(s) = R_1(s)R_2(s) = \frac{(s + 1)^2}{s(1 + 0.1s)(1 + 10s)} \]

\[ \omega_c \approx 1 \]

\[ \psi_m \approx 90^\circ \]
Control synthesis – cont’d
Control synthesis – cont’d

Assume we neglect to restore the high frequency behaviour $|L| \leq |L_1|$. We still have the same properties (in terms of both crossover frequency and phase margin).

Let’s see what happens to the controller.

$$L(s) = \frac{1}{s} \quad R(s) = \frac{1}{s} \frac{(1+s)^2}{1+10s}$$
Control synthesis – cont’d

Control sensitivity \( Q(s) = \frac{R(s)}{1 + L(s)} \)

We expect more high frequency components of the control variable in the second case.
Control synthesis – cont’d

Behaviour of control variable due to measurement noise.
Design limitations

We introduced a systematic methodology to design a controller for a given system (with no poles on the right half-plane).

Before introducing other design methods (e.g. to address exponentially unstable systems, i.e. $P > 0$), we want to see whether the only methodology we have is suitable for any type of problem or presents some limitations (e.g. in obtaining prescribed performance).

Those limitations might be either due to the methodology or intrinsic of the system.

Let’s have a look…
Design limitations – cont’d

Consider the following transfer function

\[ G(s) = \tilde{G}(s) e^{-\tau s}, \tau > 0 \]

in which the first part is assumed to be a minimum phase transfer function.

The methodology we have introduced so far is based on the magnitude Bode plot, and we know that

\[ |G(j\omega)| = |\tilde{G}(j\omega)|, |e^{-j\omega\tau}| = 1, \forall \omega, \tau \]

Therefore the presence of a delay does not apparently influence the design of the loop transfer function.
Design limitations – cont’d

However, we know that

\[ \angle e^{-j \omega \tau} = -\omega \tau \]

Therefore, for a given controller we can write the loop transfer function

\[ L(s) = R(s) \tilde{G}(s) e^{-\omega \tau s} \]

Assume we have computed the crossover frequency, i.e.

\[ |L(j \omega_c)| = \left| R(j \omega_c) \tilde{G}(j \omega_c) \right| = 1 \]

which has unitary negative slope in its neighbourhood.

Then, without the delay we might expect a reasonably “good” phase margin, e.g.

\[ \psi_m^{\tau=0} \approx 90^\circ \]
Design limitations – cont’d

However, due to the presence of the delay we have

\[ \psi_c = \angle L(j\omega_c) = \angle R(j\omega_c) \tilde{G}(j\omega_c) - \omega_c \tau = -90^\circ - \frac{180}{\pi} \omega_c \tau \]

As for the phase margin we have

\[ \psi_m = \psi_m^{\tau=0} - \omega_c \tau = 90^\circ - \frac{180}{\pi} \omega_c \tau \]

In order to ensure stability, we must ensure

\[ \omega_c < \frac{\pi}{2\tau} \]

which represents the stability limit of the closed-loop system (for a given delay, higher crossover frequency will make the system unstable!). More in general

\[ \psi_m \geq \bar{\psi}_m \Rightarrow \omega_c < \frac{\pi}{2\tau} - \bar{\psi}_m \frac{\pi}{180\tau} \]
Design limitations – cont’d
Design limitations – cont’d
Design limitations – cont’d

Another problem is represented by non-minimum phase zeros, consider for example

\[ G(s) = \tilde{G}(s) (1 - \tau s), \tau > 0 \]

in which the first part is assumed to be a minimum phase strictly proper transfer function.

Differently from the previous case, this time the non-minimum phase part does modify the magnitude Bode diagram.

Note: a non-minimum phase zero in the system under controlled cannot be canceled with a unstable pole in the controller. If this happens, the closed loop system will inevitably result unstable.
Design limitations – cont’d

For a given controller we can write the loop transfer function as

\[ L(s) = R(s) \tilde{G}(s) (1 - s\tau) \]

assume the Bode criterion to be applicable and

\[ |L(j\omega_c)| = 1 \]

We are interested to understand whether any practical limitation on the value of the crossover frequency exists.

\[ \psi_m = 180^\circ - |\angle L(j\omega_c)| = 180^\circ + \angle R(j\omega_c) \tilde{G}(j\omega_c) - \tan^{-1}(\tau\omega_c) \]

Stability requires the phase margin to be positive, therefore

\[ \tan^{-1}(\tau\omega_c) < 180^\circ + \angle R(j\omega_c) \tilde{G}(j\omega_c) \]
Design limitations – cont’d

A good practice is to cross the 0 dB axis with slope equals to -1, hence

- \( \tau \omega_c > 1 \) the frequency of the non-minimum phase zero is lower than the crossover frequency, in this case we have
  
  \[ 45^\circ \leq \text{atan}(\tau \omega_c) < 90^\circ \]

  moreover, we need to have at least two low frequency poles

  \[ \angle R(j\omega_c) \tilde{G}(j\omega_c) < -90^\circ \Rightarrow \psi_m < 45^\circ \]

- \( \tau \omega_c < 1 \), i.e. the frequency of the non-minimum phase zero is greater than the crossover frequency, hence
  
  \[ 0 < \text{atan}(\tau \omega_c) \leq 45^\circ \]

  in this case we just need one low frequency pole to guarantee the correct crossover slope, hence

  \[ \angle R(j\omega_c) \tilde{G}(j\omega_c) > -45^\circ \Rightarrow \psi_m > 45^\circ \]
Design limitations – cont’d

We have seen that when the frequency of a non-minimum phase zero is lower the crossover frequency, it is difficult to achieve a reasonable phase margin.

Practically, with the proposed design methodology, its frequency represents an upper bound of any attainable crossover frequency.

*Example*: consider the transfer function

\[ G(s) = 100 \frac{1 - s}{(1 + 10s)(1 + 0.1s)} \]

and the very simple controller

\[ R(s) = K \]
Design limitations – cont’d
Design limitations – cont’d
Design limitations – cont’d

The design procedure we have seen usually returns a controller which poles/zeros cancelled out corresponding zeros/poles in the transfer function to be controlled. In our example, we obtained

\[ G(s) = \frac{1 + 10s}{(1 + s)^2} \quad R(s) = \frac{1}{s} \frac{(1 + s)^2}{1 + 10s} \]

We want to better understand the effect of this cancellations. This time, however, we consider a slightly different control loop.
Design limitations – cont’d

We focus on one example, consider

\[
G(s) = \frac{b}{s + a} \quad R(s) = k \frac{s + a}{s}
\]

The loop transfer function results \( L(s) = \frac{kb}{s} \)

By acting on the controller gain we can achieve any crossover frequency, still guaranteeing a phase margin of 90°!!

The complementary sensitivity results as follows

\[
F(s) = \frac{L(s)}{1 + L(s)} = \frac{kb}{s + kb}
\]

which apparently seems to be a very good controller design.
Design limitations – cont’d

Closed-loop response of the output with respect to a step reference with $a = b = 1$, $k = 100$. 

![Step Response Diagram]
Design limitations – cont’d

Let’s investigate about the behaviour of the error in response to a step disturbance:

\[
V(s) = \frac{G(s)}{1 + L(s)} = \frac{bs}{(s + kb)(s + a)} \approx \frac{s}{k(s + a)}
\]

This transfer function presents a zero in the origin (disturbance will be perfectly compensated at steady state) a high frequency pole (due to the large gain of the controller) and…

… a low frequency pole (which remained untouched due to the cancellation).
Design limitations – cont’d

Closed-loop response of the error with respect to a step disturbance with $a = b = 1, k = 100$.

Slow response due to the low frequency pole (cancelled)
Design limitations – cont’d

We cancelled out the “slow” pole of the system with a zero in the controller. The final crossover frequency was way bigger of the frequency where the cancellation happened.

The effect of this cancellation was not observable in the complementary sensitivity function, however it could be observed on other sensitivity functions (in our example the one from the disturbance to the error).

We learned not to cancel out non-minimum phase zeros, now we can extend this rule (of thumb) by saying that cancelling out “slow” frequency behaviour can be inconvenient, even though differently from the case of non-minimum phase zeros, they do not compromise stability.
Design limitations – cont’d

In the light of the previous discussion, let’s try to do better with our example.

\[
G(s) = \frac{b}{s + a} \quad R(s) = k \frac{s + z}{s}
\]

The following question arises: can we obtain the same performance (crossover frequency and phase margin) while avoiding the slow convergence of the error due to a step disturbance?

We then want to achieve the same crossover frequency (of 100 rad/s). Let’s put the zero of the regulator one decade before such a frequency, hence

\[
z = 10 \quad L(s) = k \frac{1}{1 + s} \frac{s + 10}{s}
\]
Design limitations – cont’d

We obtain the following loop transfer function, with $k = 1$.

With $k = 100$, we will obtain the desired crossover frequency and good phase margin.
Design limitations – cont’d

The step response looks similar due to the same crossover frequency. Overshoot is due to the slightly reduced phase margin.
Design limitations – cont’d

On the other hand, in response to a step disturbance the second controller performs definitely better, at which cost?
Design limitations – cont’d

At no cost! as the two control sensitivity transfer function are approximately the same (design change at low frequencies).
Building blocks

So far, we introduced the problem of controlling a plant with a regulator with arbitrary structure (number of zeros/poles and their positions).

For a technological view, there are a number of typical controller structures which cover most of practical SISO control problem.

In the next we are going to address the following structures and transfer functions:

- Lead-lag compensators
- Notch filters
- PID controllers
- Feedforward actions
- Disturbance compensators
- Cascaded control loops
Building blocks – cont’d

Lead-lag compensators are a family of transfer functions with exactly one pole and one zero and a certain gain.

\[ R(s) = \frac{1 + s\tau}{1 + sT} \]

When \( \tau > T \) we refer to a lag compensator (when the frequency of the pole anticipates the one of the zero), or to a lead compensator vice versa.)
Building blocks – cont’d

A lead compensator is used to increase the crossover frequency and the phase margin (whilst possibly decreasing the gain margin).

\[ G(s) = \frac{100}{s(1 + 0.1s)} \quad \text{and} \quad R(s) = \frac{1 + 0.1s}{1 + 0.001s} \]

\[ \omega_c = 31, \psi_m = 18^\circ \quad \text{and} \quad \omega_c = 100, \psi_m = 84^\circ \]
Building blocks – cont’d

A **lag compensator** is used to decrease the crossover frequency and to increase the gain margin (whilst possibly decreasing the phase margin).

\[
G(s) = \frac{3}{(1 + 0.01s)^3}
\]

\[
R(s) = \frac{1 + 0.1s}{1 + s}
\]

\[
\omega_c = 100, \ k_m = 8.5 \ dB
\]

\[
\omega_c = 3, \ k_m = 27.7 \ dB
\]
Building blocks – cont’d

A notch filter is a transfer function with a pair of complex poles and a pair of complex zeros at the same frequency

\[ C(s) = \frac{s^2 + 2\xi_z \omega_n s + \omega_n^2}{s^2 + 2\xi_p \omega_n s + \omega_n^2} \]

\[ 0 < \xi_z, \xi_p < 1 \]

\[ |C(j\omega_n)| = \frac{\xi_z}{\xi_p} \]

\[ \xi_z < \xi_p \]

\[ \xi_z > \xi_p \]
Building blocks – cont’d

A notch filter is typically adopted to cancelled out resonant poles in the process to be controlled.

*Example:* consider a mass-spring-damper system with a resonant behaviour

\[
G(s) = \frac{1}{ms^2 + ds + k} \quad \omega_n = \sqrt{\frac{k}{m}} \quad \xi = \frac{d}{2\sqrt{mk}}
\]

assuming \( m = 1 \), \( d = 0.1 \) and \( k = 1 \), let’s try to design a controller for such a system, e.g.

\[
R(s) = \frac{0.1}{s(1 + s)}
\]
Building blocks – cont’d

The step response of the closed-loop system is the following one

\[ F(s) \approx \frac{\omega_c}{s + \omega_c} \]
Building blocks – cont’d

Let’s adopt the same controller together with the following notch filter

\[ C(s) = \frac{s^2 + 0.4\xi\omega_n s + \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \]
Building blocks – cont’d

The so-called **PID controller** is a controller transfer function which became popular as it can be adopted in the 95% of the (“industrial”) control problems.

The control action is composed as the sum of three terms

\[
  u(t) = K_P e(t) + K_I \int_0^t e(\tau) d\tau + K_D \dot{e}(t)
\]

- **proportional (P) action**
- **integral (I) action**
- **derivative (D) action**

Alternative (equivalent) formulation

\[
  u(t) = K_P \left( e(t) + \frac{1}{T_I} \int_0^t e(\tau) d\tau + T_D \dot{e}(t) \right)
\]
Building blocks – cont’d

Not all actions are necessarily present, we might have variations (such as PI, PD, P, etc. controllers).

The transfer function of the PID controller is like the following one

\[ R(s) = \frac{K_D s^2 + K_P s + K_I}{s} \]

We should immediately notice that the transfer function contains more zeros than poles. Therefore, it is not realizable (due to the “ideal” derivative action).

\[ K_D s \rightarrow \frac{K_D s}{1 + \frac{T_D}{N} s}, \quad N \in \{5, 10, 100, \ldots \} \]

High frequency pole
Building blocks – cont’d

The typical Bode plot of the module of a (real and ideal) PID

\[ |R(j\omega)| \]

The bigger \( N \), the better approximation. However, remind that

\[ |Q(j\omega)| = \frac{|R(j\omega)|}{|1 + L(j\omega)|} \approx \frac{|G(j\omega)|^{-1}}{|R(j\omega)|}, \omega << \omega_c \]

revealing that “bigger” \( N \) might result in excessive control actions. Moreover, the derivative action is usually problematic due to noisy signals (then, use it only if strictly necessary).
Building blocks – cont’d

A **feedforward action** on the reference signal consists in designing a transfer function acting (in open-loop) between the reference signal and the control signal.

\[
Y (s) = \left[ F (s) + \frac{C (s) G (s)}{1 + L (s)} \right] \quad Y^0 (s) = \frac{L (s) + C (s) G (s)}{1 + L (s)} Y^0 (s)
\]

The transfer function between the reference and the output signal becomes
Building blocks – cont’d

If, in principle, we are able to design properly the feedforward compensator we will obtain a unitary transfer function between the reference and the output, hence a perfect tracking!

\[ C(s)G(s) = 1 \Rightarrow Y(s) = \frac{L(s) + 1}{1 + L(s)} Y^0(s) = Y^0(s) \]

This is typically unfeasible as:

- the plant usually has more poles than zeros
- the high frequency behaviour of the plant might be affected by a lot of uncertainties
- in case the plant is a non-minimum phase system (positive zeros and/or delays), the feedforward compensator might contain an anticipatory action or even turn out to be unstable!
Building blocks – cont’d

A compromise solution is to obtain a feedforward compensator which approximate the ideal solution at least for the magnitude and within a certain bandwidth (higher than the crossover frequency), i.e.

$$|C(j\omega)G(j\omega)| \approx 1, \omega < \bar{\omega}$$

**Example**: consider the following transfer functions

$$G(s) = 10 \frac{1 - 0.1s}{s(1 + s)}$$

$$R(s) = 0.1 \frac{1 + s}{1 + 0.1s}$$

The ideal feedforward compensation should look like the following

$$C^{id}(s) = 0.1 \frac{s(1 + s)}{1 - 0.1s}$$
Building blocks – cont’d

\[ C(s) = 0.1 \frac{s(1+s)}{(1 + 0.1s)^2} \]

\[ C^{id}(s) = 0.1 \frac{s(1+s)}{1 - 0.1s} \]
Building blocks – cont’d

Compare our solution with $C(s) = 0$: $Y(s) = \frac{L(s) + C(s)G(s)}{1 + L(s)}Y^0(s)$

\[ C(s) = 0.1 \frac{s(1 + s)}{(1 + 0.1s)^2} \]

\[ C(s) = 0 \]
Building blocks – cont’d

While in terms of step response, we have

\[ C(s) = 0.1 \frac{s(1 + s)}{(1 + 0.1s)^2} \]

\[ C(s) = 0 \]
Building blocks – cont’d

A **disturbance compensator** consists in designing a transfer function acting (in open-loop) between a measurable disturbance and the control signal.

The transfer function between the disturbance and the output signal becomes

\[
Y(s) = [H(s)S(s) + C(s)G(s)S(s)]D(s) = \frac{H(s) + C(s)G(s)}{1 + L(s)}D(s)
\]
Building blocks – cont’d

If, in principle, we are able to design properly the disturbance compensator we will obtain a zero transfer function between the disturbance and the output, hence a perfect disturbance rejection!

\[ C(s)G(s) + H(s) = 0 \Rightarrow Y(s) = 0 \]

Again, this is typically unfeasible with similar arguments to those in the previous case.

Once again, we can approximate the ideal compensator

\[ C^{id}(s) = -\frac{H(s)}{G(s)} \]

with a “similar” (but realizable) transfer function, at least in the bandwidth of the disturbance.
Building blocks – cont’d

*Example:* consider the following transfer functions

\[ G(s) = \frac{0.5}{(1 + s)^2 (1 + 0.1s)} \quad R(s) = 20 \frac{1 + s}{1 + 100s} \quad H(s) = \frac{2}{1 + 0.5s} \]

the ideal compensator would be

\[ C^{id}(s) = -4 \frac{(1 + s)^2 (1 + 0.1s)}{1 + 0.5s} \]

Let’s compare two solutions:

- static disturbance compensator \( C_1(s) = C^{id}(0) = -4 \)
- approximate compensator \( C_2(s) = -4 \frac{(1 + s)^2}{(1 + 0.5s)(1 + 0.01s)} \)
Building blocks – cont’d

Let’s compare the step responses

\[ C_0(s) = 0 \]

Exact compensation at steady-state, the dynamic compensator has better transient performance

\[ C_1(s) = -4 \]

\[ C_2(s) = -4 \frac{(1 + s)^2}{(1 + 0.5s)(1 + 0.01s)} \]
Cascaded (nested) control loops are common structures in practical problems. Such a control scheme is usually adopted when the system be controlled can be split into two parts in series and

- the inner part is “faster” than the “outer”
- the output of the former can be measured
Building blocks – cont’d

The design procedure looks like to following one

- a controller $R_1(s)$ is design for $G_1(s)$ regardless the outer part of the system with traditional methods
- assuming $\omega_{c,1}$ to be the crossover frequency of the inner loop, the outer controller $R_2(s)$ to be designed “sees” the system

$$\hat{G}_2(s) = F_1(s) G_2(s) = \frac{L_1(s)}{1 + L_1(s)} G_2(s)$$

- assuming the outer controller can be designed with a lower crossover frequency $\omega_{c,2}$ with respect to the inner one, i.e. $\omega_{c,2} \ll \omega_{c,1}$, then we can assume

$$F_1(s) = \frac{L_1(s)}{1 + L_1(s)} \approx 1$$

- the outer controller is then designed focusing on $G_2(s)$ only.
Building blocks – cont’d

Example: consider $G_1(s) = \frac{1}{1 + 0.1s}$ and $G_2(s) = \frac{1}{s}$ we can design

\[ R_1(s) = 10 \frac{1 + 0.1s}{s} \quad L_1(s) = \frac{10}{s} \quad \omega_{c,1} = 10 \quad F_1(s) = \frac{1}{1 + 0.1s} \]
Building blocks – cont’d

Assuming $\omega_{c,2} << 10$, typically $\omega_{c,2} = \alpha \omega_{c,1}, \alpha \in [0.1, 0.2]$ we can design

$$R_2(s) = 1 \quad L_2(s) \approx \frac{1}{s} \quad \omega_{c,2} = 1 \quad F_2(s) \approx \frac{1}{1 + s}$$
Root locus

So far, we have presented a design methodology based on Bode’s stability theorem, hence only applicable when $P = 0$. What can we do, if we face a SISO LTI system with $P > 0$?

*Example*: consider the inverted pendulum (upright position) $\bar{\theta} = \pi$

\[
\begin{align*}
\dot{\theta} &= \omega \\
\dot{\omega} &= -\frac{g}{L} \sin \theta - \frac{d}{mL^2} \omega + \frac{1}{mL^2} u \\
\delta \dot{\omega} &= \frac{g}{L} \delta \theta - \frac{d}{mL^2} \delta \omega + \frac{1}{mL^2} \delta u
\end{align*}
\]

\[
G(s) = \frac{1}{mL^2 s^2 + ds - mLg} \quad P = 1
\]
Root locus – cont’d

Let’s see whether the very trivial control law $u = -\delta \theta$ is able to stabilize the system.

$N = 0, P = 1$
Root locus – cont’d

Then, how about $u = -10 \delta \theta$ …?
Root locus – cont’d

We have seen that a simple proportional controller can be used to stabilize exponentially unstable systems. What what is (range of) stabilizing gains?

In other words, can we derive a systematic procedure to guide the selection of the controller gain?

Given the following closed-loop control system, with \( L(s) \) any transfer function (possibly unstable)

\[
\begin{align*}
    \dot{y}^0 & \rightarrow k \rightarrow u \rightarrow L(s) \rightarrow y \\

\end{align*}
\]

what can we say about the closed loop poles (roots) and their dependance on the parameter \( k \)?
Root locus – cont’d

Given

\[ L(s) = \frac{\prod (s + z_i)}{\prod (s + p_i)} \]

we define root locus a locus describing in the complex plane the path traversed by the closed-loop poles for varying \( k \).

Further, we will call
- direct locus, the one corresponding to \( k > 0 \)
- inverse locus, \( k < 0 \)
Root locus – cont’d

*Example:* consider \( L(s) = \frac{1}{(s + 1)(s + 2)} \)

The closed loop characteristic polynomial (whose roots are the closed-loop poles) is

\[
\Pi(s) = k + (s + 1)(s + 2)
\]

\[
s^2 + 3s + 2 + k = 0
\]

The corresponding roots are

\[
\Delta = 1 - 4k \quad s_{1,2} = \frac{-3 \pm \sqrt{\Delta}}{2}
\]

\[
\Re(s_{1,2}) = -\frac{3}{2}
\]

\[\Delta \geq 0\] real roots
Root locus – cont’d

Notice that, as we probably expect, for $k = 0$, the roots are open-loop poles, in fact

$$\Pi(s) = k + (s + 1)(s + 2)$$
Root locus – cont’d

Focusing on one example, we have been able to plot how closed-loop roots change as a function of the parameter $k$.

However, it is not always possible to explicitly compute closed-loop poles and how they change.

Moreover, we are also interested in understanding the range of values of $k$ such that the closed-loop system is asymptotically stable (i.e. roots are in the left hand-side plane).

Is there a systematic procedure?
Root locus – cont’d

There exists a series of plotting rules:

- the locus is always composed of \( n \) branches (where \( n \) is the number of open-loop poles, i.e. the order of the system)
- the locus is symmetric with respect to the Real axis (since poles are either real or complex and conjugate)
- each branch
  - “starts” \((k = 0)\) from the open-loop poles
  - “ends” \((|k| \to \infty)\) either on a zero of \( L(s) \) or has an asymptotic behaviour
- then, the locus has \( \nu \) asymptotes (being \( \nu \) is the relative degree)
Root locus – cont’d

• all asymptotes meet on the Real axis at coordinate

\[ x_a = \frac{\sum_i z_i - \sum_i p_i}{\nu} \]

• the angles between the asymptotes and the Real axis can be computed as

\[ \theta_a = \begin{cases} 
180^\circ + \frac{h360^\circ}{\nu}, & k > 0 \\
\frac{h360^\circ}{\nu}, & k < 0 
\end{cases} \quad i = 0, \ldots, \nu - 1 \]

• all points on the Real axis belong either to the D.L. or to the I.L., in particular
  • to the D.L. those leaving on their right an odd number of singularities (poles or zeros) of \( L(s) \)
Root locus – cont’d

*Example*: back to the transfer function

\[ L(s) = \frac{1}{(s + 1)(s + 2)} \]

\[ \nu = 2 \quad p_i = \{1, 2\} \quad z_i = \{\} \]

\[ x_a = \frac{-\left(1 + 2\right)}{2} = -\frac{3}{2} \]

\[ \theta_a^{D.L.} = \frac{180^\circ + h360^\circ}{2} = 90^\circ, 270^\circ \]

\[ \theta_a^{I.L.} = \frac{h360^\circ}{2} = 0^\circ, 180^\circ \]
Root locus – cont’d

*Example*: consider \( L(s) = \frac{s - 1}{(s + 1)(s + 2)} \)

- \( p_i = \{1, 2\} \quad z_i = \{-1\} \)
- \( \nu = 2 - 1 = 1 \)
- \( \theta^{I.L.}_a = \frac{h360^\circ}{1} = 0^\circ \)
- \( \theta^{D.L.}_a = \frac{180^\circ + h360^\circ}{1} = 180^\circ \)
Root locus – cont’d

In both the examples we have seen that the closed-loop system is asymptotically stable only when $k$ is within a certain region.

Can we compute this stability region?

Result: given a point on the locus $s$, the corresponding value can be computed as follows:

$$|k| = \frac{\prod_i |s + p_i|}{\prod_j |s + z_j|}$$

$$|k| = \prod_i |s + p_i| \quad z_i = \{\}$$
Root locus – cont’d

Example: back to the first example

\[ L(s) = \frac{1}{(s + 1)(s + 2)} \]

\[ s = 0 \]

\[ |k| = \prod_i |p_i| = 2 \]

Asymptotically stable \( k > -2 \)

Same result can be obtained with the Routh’s criterion

\[ s^2 + 3s + 2 + k = 0 \]
Root locus – cont’d

However, it is not always possible to compute such a point (as in the second example).

**Property:** if \( v \geq 2 \), then the sum of the real parts of the roots is invariant with respect to \( k \).

This property – when applicable – is extremely important to compute the stability region when closed-loop poles cross the imaginary axis at some point that we cannot “measure”.

This concept will be more clear on an example…
Root locus – cont’d

Example: consider the transfer function \[ L(s) = \frac{1}{(s + 1)(s + 2)(s + 3)} \]

\[ p_i = \{1, 2, 3\} \quad z_i = \{\} \]

\[ \nu = 3 \geq 2 \]

\[ x_a = \frac{0 - (1 + 2 + 3)}{3} = -2 \]

\[ \theta_{I.L.}^a = \frac{h360^\circ}{3} = 0^\circ, 120^\circ, 240^\circ \]

\[ \theta_{D.L.}^a = \frac{180^\circ + h360^\circ}{3} = 60^\circ, 180^\circ, 300^\circ \]

Asymptotic stability \( k_{\text{min}} < k < k_{\text{max}} \)
Root locus – cont’d

We have seen that by increasing $k$, the closed-loop system eventually becomes unstable. What is the limit value?

In this case we can clearly compute an approximation using the asymptotes. This is not always possible, however

$$\nu = 3 \geq 2$$

hence

$$\forall k: \sum_i \Re(\lambda_i) = -1 - 2 - 3 = -6$$

$$|k| = \Pi_i |6 + p_i| = 60$$

$$k^{max} = 60$$
Design using root locus

The root locus can be used to design a controller for a given system. However, there is not a systematic procedure as we have seen before.

*Example:* design a controller for the system \( G(s) = \frac{10}{s + 2} \) such that for a step reference \( y^0(t) = \text{step}(t) \)

- the steady state error \( e_\infty = 0 \)
- the closed-loop response has a 2nd order dominant behaviour with

\[
T_{a1} \leq 2 \quad S\% \leq 5\%
\]

We first notice that the controller must be of type 1, then

\[
R(s) = \frac{1}{s} \tilde{R}(s)
\]
Design using root locus – cont’d

Moreover, we want the closed-loop poles to look like the solutions of

\[ s^2 + 2\xi \omega_n s + \omega_n^2 = 0 \]

where

\[ S_\% = 100 \exp \left( - \frac{\xi \pi}{\sqrt{(1 - \xi^2)}} \right) \leq 5\% \quad T_{a1} = \frac{5}{\xi \omega_n} \leq 2 \]

Let’s choose

\[ \xi = 0.7 \quad (S_\% = 4.6) \quad \omega_n = 4 \quad (T_{a1} = 1.79) \]

\[ s_{1,2} = -2.8 \pm j2.8566 \]

and try with \( \tilde{R}(s) = \rho \)
Design using root locus – cont’d

The loop transfer function is

\[ L(s) = \frac{10\rho}{s(s + 2)} = \frac{k}{s(s + 2)} \]

We need to move the asymptote!!

\[ x_a = \frac{0 - (0 + 2)}{2} = -1 \]
Design using root locus – cont’d

In order to move the asymptote we can consider

\[ R(s) = \frac{\rho s + 2}{s(s + z)} \]

\[ L(s) = \frac{k}{s(s + z)} \]

\[ x_a = \frac{0 - (0 + z)}{2} = -\frac{z}{2} = -2.8 \]

\( \rho = 1.6 \)
Design using root locus – cont’d

With \( R(s) = \frac{1.6(s + 2)}{s(s + 5.8)} \), we obtain the following closed-loop response.
Homework & take home message

So far, the root locus is the only method we know to design a controller for an unstable system.

Try to design a controller for $G(s) = \frac{s - 1}{(s - 2)(s + 1)}$ such that the closed-loop system is stable.