Automatic Control (MSc in Engineering Physics)
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Given and family names .........................SOLUTIONS .........................

Student ID number ......................... Signature  .........................

- Duration: 120 minutes
- Number of exercises/questions: 5
- Maximum score: 32 pts.
- All solutions must be written (either in English or Italian) within the available blank space (and not on additional paper sheets)
- It is forbidden to use any electronic device, apart from a non programmable and non graphic calculator
- Books and lecture material are not allowed
- It is not allowed to leave the class room within the first 30 minutes
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Question 1

Consider a linear time invariant closed-loop system with negative feedback and loop transfer function $L(s)$ (which satisfies the applicability of the Bode criterion). Provide a definition of phase margin $\psi_m$ and elaborate on how a small, yet positive, phase margin, e.g. $\psi_m < 30\,\text{deg}$ is responsible for an oscillating closed-loop response. Provide (without necessarily proving it) an estimate of the corresponding damping factor $\xi$.

Solution: Let $\omega_c$ the sole frequency such that $|L(j\omega_c)| = 1$, define $\psi_c = \angle L(j\omega_c)$, then $\psi_m = \pi - |\psi_c|$.

We need to compute $F|\omega_c| = \frac{1}{|1 + L(j\omega_c)|} = \frac{1}{2\sin\left(\frac{\psi_m}{2}\right)} = \frac{1}{2\xi}$. 
Exercise 2

Draw the root locus and discuss the stability of transfer function \( F(s) = \frac{L(s)}{1 + L(s)} \), where

\[
L(s) = \frac{s - 1}{(s + 1)^2}
\]

depending on the parameter \( \rho \).

Solution: For \( \rho = 0 \), the (open-loop) system is asymptotically stable.

For \( \rho > 0 \), we can use the direct root locus. The angle is: \( \theta_a = 180 \text{ deg} \).

Evaluating the root locus \( s = 0 \), i.e. \( \rho = \frac{\Pi_i |p_i|}{\Pi_j |z_j|} = 1 \) we can conclude that for \( \rho > 0 \), asymptotic stability is for \( \rho < 1 \).

For \( \rho < 0 \), we can use the inverse root locus. The angle is: \( \theta_a = 0 \text{ deg} \).

Since it is not possible to apply the centre of mass rule, stability of the closed-loop system is checked with the Routh criterion. The closed-loop characteristic polynomial is \( \Pi(s) = s^2 + (2 + \rho) s + 1 - \rho \), asymptotic stability then occurs for \( -2 < \rho < 1 \).
Exercise 3

Consider the following closed-loop system, where \( G(s) = \frac{1 - 0.1s}{(1 + 0.1s)(1 + 3s)} e^{-s\tau} \).

For \( \tau = 0 \), design the transfer function \( R(s) \) in order to achieve the following requirements:

1. a steady state error for a ramp disturbance \( d \) and a unit step reference \( w \) less than 0.1, i.e. \( |e_{\infty}| \leq 0.1 \) when \( d(t) = \text{ramp}(t) \) and \( w(t) = \text{step}(t) \);
2. a minimum phase margin of 60 degrees, i.e. \( \psi_m \geq 60 \text{ deg} \);
3. a crossover frequency of at least 0.1 rad/s, i.e. \( \omega_c \geq 0.1 \text{ rad/s} \).

Determine the maximum delay \( \tau > 0 \) the closed-loop system can tolerate before becoming unstable and finally compute the expected minimum attenuation of disturbance \( d \) on the output \( y \) in the bandwidth \( (0, 0.01] \text{ rad/s} \).

**Solution:** Apart from the sign, the transfer functions from the disturbance \( d \) to the error \( e \) and from the reference \( w \) to the error are equal to the sensitivity function. Therefore, assuming stability

\[
e_{\infty} = \lim_{s \to 0} s \left( \frac{1}{1 + \frac{\mu_R s^2 + 1}{s \mu_R}} \right) = \lim_{s \to 0} s \frac{s^{g_R}}{s^{g_R} + \mu_R} \left( \frac{1}{s + \frac{1}{s^2}} \right) = \lim_{s \to 0} \frac{s^{g_R-1}}{s^{g_R} + \mu_R}
\]

Then for \( g_R \geq 2 \), the steady state error would be zero, alternatively for \( g_R = 1 \), the steady state error would be \( 1/\mu_R \). In order to meet the requirement we can select \( R_1(s) = 10/s \).
In order to meet all the other requirement we need to decrease the crossover frequency, by introducing the following lag controller: 

\[ R_2(s) = \frac{1 + 100s}{1 + 10^4s} \]

The final loop transfer function is as follows:

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}
Exercise 4

Given the following linear time invariant continuous time system

\[
A = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = [0 \ 1]
\]

Verify whether it is possible to design a LQR controller which minimises the following cost functional

\[
J = \int_0^\infty (u^2 + y_{LP}^2) \, dt
\]

where \( y_{LP} \) represents the low-pass component of \( y = Cx \) up to frequency 1 rad/s.

Finally, design a block diagram of the corresponding controller.

**Solution:** We need to augment the state of the system in order to consider the low pass filter from \( y \) to \( y_{LP} \). Therefore, we add the following equation:

\[
\dot{z} = -z + y = -z + Cx, \quad y_{LP} = z, \quad \text{i.e. } Y_{LP}(s) = \frac{1}{s+1} Y(s)
\]

The extended system, in terms of state vector \( x_E = [x^T \ z]^T \) is as follows:

\[
A_E = \begin{bmatrix} A & 0 \\ C & -1 \end{bmatrix}, \quad B_E = \begin{bmatrix} B \\ 0 \end{bmatrix}
\]

We need first to check the reachability of the pair \((A_E, B_E)\), i.e. evaluating the rank of

\[
K_R = [B_E \ A_E B_E \ A_E^2 B_E], \quad \text{rank} (K_R) = 3
\]

Then, according to the new state vector \( x_E \) the cost functional can be rewritten as follows:

\[
J = \int_0^\infty (u^2 + x_E^T Q x_E) \, dt
\]

where \( Q = C_E^T C_E \) and \( C_E = [0 \ 0 \ 1] \). Finally, we need first to check the observability of the pair \((A_E, C_E)\), i.e. evaluating the rank of

\[
K_O = \begin{bmatrix} C_E^T & A_E^T C_E & A_E^2 C_E^T \end{bmatrix}, \quad \text{rank} (K_O) = 3
\]

The corresponding control law is \( u = K_E x_E = K_E x + K_{LP} y_{LP} \). Block diagram directly follows from it.
Exercise 5

Consider the following closed loop system, where \( G(s) = R(s) \frac{1}{(s + 1)^2} \) and \( N \) represents a static relay\(^1\).

For \( R(s) = 1 \), draw the polar diagram of \( G(j\omega) \) and discuss whether the closed-loop system is expected to oscillate. Then, design the controller \( R(s) \) in order to establish a persistent oscillation with amplitude \( A = 1 \) and compute the corresponding frequency.

Solution: Since the polar diagram of \( G(j\omega) \) when \( R(s) = 1 \) intersects the negative real axis only in the origin, no persistent oscillation is expected.

We can introduce the following controller \( R(s) = K/(s + 1) \), the corresponding transfer function would be \( G(s) = K/(s + 1)^2 \) which is clearly a low-pass one.

In order to find the intersection with the locus \( \Lambda(A) = -1/\phi(A) \) we can solve the following equation

\[
\angle G(j\omega_\pi) = -\pi, \omega_\pi = \tan(60) = \sqrt{3}
\]

Then, by solving \( \pi A/4 = |G(j\omega_\pi)| = K/8 \), we finally obtain \( K = 2\pi \).

\(^1\)Its describing function being \( \phi(A) = 4/(\pi A) \).